Factors of the Gaussian Coefficients

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Abstract. We present some simple observations on factors of the q-binomial coefficients, the q-Catalan numbers, and the q-multinomial coefficients. Writing the Gaussian coefficient with numerator n and denominator k in a form such that $2k \leq n$ by the symmetry in k, we show that this coefficient has at least k factors. We also deduce that the Gaussian coefficents have no multiple roots. Some divisibility results of Andrews, Brunetti and Del Lungo are also discussed.

Keywords: q-multinomial coefficient, Gaussian coefficient, q-Catalan number, cyclotomic polynomial.

AMS Classification: 05A10, 33D05, 12D05.

Suggested Running Title: Factors of the Gaussian Coefficients

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The *q-multinomial coefficients* are defined by

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_r}},$$

where $n_1 + n_2 + \cdots + n_r = n$ and

$$(q;q)_m = (1-q)(1-q^2)\cdots(1-q^m).$$

For $r = 2$, they are usually called the *q-binomial coefficients* or the *Gaussian coefficients* and are written as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)}. \quad (1)$$

The factorization of q-binomial coefficients plays an important role in the study of divisibility properties of generalized Euler numbers [2, 4, 7, 11].

There are many reasons for the Gaussian coefficients to be polynomials. From the point of view of cyclotomic polynomials, the divisibility for the Gaussian coefficients turns out to be a rather natural fact.

Let $\Phi_n(x)$ be the n-th cyclotomic polynomial defined by

$$\Phi_n(x) = \prod_{\substack{1\le j\le n\\ \gcd(j,n)=1}} (x-\zeta_n^j),$$

where $\zeta_n = e^{2\pi\sqrt{-1}/n}$ is the n-th root of unity and $\gcd(j, n)$ denotes the great common divisor of j and n. It is well-known that $\Phi_n(x) \in \mathbb{Z}[x]$ is the irreducible polynomial for ζ_n (see, for example, [12]). The polynomial $x^n - 1$ has the following factorization into irreducible polynomials over $\mathbb{Z}$:

$$x^n - 1 = \prod_{j\,|\,n} \Phi_j(x). \tag{2}$$

Knuth and Wilf [8] provided the factorization of q-binomial coefficients. In the same manner, one may get the following factorization of q-multinomial coefficients, where the notation $\lfloor x\rfloor$ stands for the largest integer less than or equal to x.

Lemma 1 *The q-multinomial cofficients $\begin{bmatrix} n \\ n_1,n_2,\ldots,n_r\end{bmatrix}$ are polynomials in q and can be factored as*

$$\prod_{i=1}^{n} \left(\Phi_i(q)\right)^{\lfloor \frac{n}{i}\rfloor - \lfloor \frac{n_1}{i}\rfloor - \lfloor \frac{n_2}{i}\rfloor - \cdots - \lfloor \frac{n_r}{i}\rfloor}. \tag{3}$$

Proof. By Equation (2), we have

$$(-1)^m (q;q)_m = \prod_{j=1}^{m}\prod_{i\,|\,j} \Phi_i(q) = \prod_{i=1}^{m} \Phi_i^{\lfloor \frac{m}{i}\rfloor}(q) = \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{m}{i}\rfloor}(q).$$

Therefore,

$$\begin{aligned}\begin{bmatrix} n \\ n_1,n_2,\ldots,n_r\end{bmatrix} &= \frac{\prod_{i=1}^{n} \Phi_i^{\lfloor \frac{n}{i}\rfloor}(q)}{\prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_1}{i}\rfloor}(q)\cdot \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_2}{i}\rfloor}(q)\cdots \prod_{i=1}^{\infty} \Phi_i^{\lfloor \frac{n_r}{i}\rfloor}(q)} \\ &= \prod_{i=1}^{\infty} \left(\Phi_i(q)\right)^{\lfloor \frac{n}{i}\rfloor - \lfloor \frac{n_1}{i}\rfloor - \lfloor \frac{n_2}{i}\rfloor - \cdots - \lfloor \frac{n_r}{i}\rfloor}.\end{aligned}$$

Since $\sum_{j=1}^{r} n_j = n$ and $\lfloor a \rfloor + \lfloor b \rfloor \le \lfloor a+b \rfloor$, all the power indices in (3) are nonnegative, which implies that the q-multinomial coefficients are polynomials in q. ∎

Here is an observation.

Theorem 2 *The Gaussian coefficients* $\left[{n \atop k}\right]$ *have no multiple roots.*

Proof. It is sufficient to prove that the factorization of $\left[{n \atop k}\right]$ into irreducible factors contains no repeated factors. Using the following inequality for real numbers a and b

$$\lfloor a \rfloor + \lfloor b \rfloor + 1 \ge \lfloor a+b \rfloor ,$$

we derive that the power of $\Phi_i(q)$ in the factorization (3) equals

$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{k}{i} \right\rfloor - \left\lfloor \frac{n-k}{i} \right\rfloor \le 1, \quad \text{for} \quad 1 \le i \le n.$$

Since $\Phi_1(q), \Phi_2(2), \ldots, \Phi_n(q)$ are pair-wise relatively prime, it follows that $\left[{n \atop k}\right]$ have no multiple roots. ∎

Combining Lemma 1 and Theorem 2, we may compute the number of irreducible factors of $\left[{n \atop k}\right]$. Here we are interested in the following bounds:

Theorem 3 *The Gaussian coefficient* $\left[{n \atop k}\right]$ *has at most* $n-1$ *irreducible factors. It has at least* k *irreducible factors if* $n \ge 2k$.

Proof. It is obvious that $\Phi_1(q) = q-1$ is not a divisor of $\left[{n \atop k}\right]$. From (3) it follows that $\left[{n \atop k}\right]$ has at most $n-1$ irreducible factors. Assume that $n \ge 2k$ and $n-k+1 \le i \le n$. Then we have $2i \ge 2n-n+2 > n$, $i \ge 2k-k+1 = k+1$ and $i \ge n-k+1$. Hence,

$$\left\lfloor \frac{n}{i} \right\rfloor = 1 \quad \text{and} \quad \left\lfloor \frac{k}{i} \right\rfloor = \left\lfloor \frac{n-k}{i} \right\rfloor = 0,$$

which implies that $\Phi_i(q)$ is an irreducible factor of $\left[{n \atop k}\right]$. Therefore, $\left[{n \atop k}\right]$ has at least k irreducible factors: $\Phi_{n-k+1}, \Phi_{n-k+2}, \ldots, \Phi_n$. ∎

Remark 1. Theorem 3 implies that the Gaussian coefficient can be written as a product of exactly k nontrivial factors if one carries out the divisibility computation without further factorization. That is, applying the command `simplify` to

$$\frac{(1-q^n)\cdots(1-q^{n-k+1})}{(1-q)\cdots(1-q^k)}$$

in Maple, we get a product of exactly k factors. In fact, we may factorize $\begin{bmatrix} n \\ k \end{bmatrix}$ into k factors by the following procedure. Let $S_i = \{j \colon j \text{ divides } n-i+1\}, i = 1, \ldots, k$. Then

$$(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n) = (-1)^k \prod_{i=1}^{k} \prod_{j \in S_i} \Phi_j(q).$$

Similarly,

$$(1-q)(1-q^2)\cdots(1-q^k) = (-1)^k \prod_{i=1}^{k} \prod_{j \in T_i} \Phi_j(q),$$

where $T_i = \{j \colon j \text{ divides } i\}, i = 1, \ldots, k$. Cancelling the common elements in S_i and T_j, we get the subsets R_i of S_i such that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{k} \prod_{j \in R_i} \Phi_j(q).$$

Note that $n-i+1 \in S_i$, but it does not belong to any T_j. It follows that $n-i+1 \in R_i$, which implies that $\prod_{j \in R_i} \Phi_j(q)$ are not constant polynomials in q.

Remark 2. Let $A(n,k)$ be the number of irreducible factors of $\begin{bmatrix} n \\ k \end{bmatrix}$. Let $B(k)$ be the minimum number $A(n,k)$ for $n \geq 2k$. As pointed by one of the referees, it seems that $\lim_{k\to\infty} B(k)/k$ exists and equals approximately 1.3.

The irreducible factors of $\begin{bmatrix} n \\ k \end{bmatrix}$ can be characterized as follows, where $\{x\}$ denotes the fractional part of x, namely, $\{x\} = x - \lfloor x \rfloor$.

Theorem 4 *$\Phi_i(q)$ is a factor of $\begin{bmatrix} n \\ k \end{bmatrix}$ if and only if $\left\{\frac{k}{i}\right\} > \left\{\frac{n}{i}\right\}$.*

Proof. From Lemma 1, we see that $\Phi_i(q)$ is a factor of $\begin{bmatrix} n \\ k \end{bmatrix}$ if and only if

$$\begin{aligned} & \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{k}{i} \right\rfloor - \left\lfloor \frac{n-k}{i} \right\rfloor = 1 \\ \Longleftrightarrow & \left\{ \frac{n}{i} \right\} - \left\{ \frac{k}{i} \right\} - \left\{ \frac{n-k}{i} \right\} = -1 \\ \Longleftrightarrow & \left\{ \frac{k}{i} \right\} = \left\{ \frac{n}{i} \right\} + 1 - \left\{ \frac{n-k}{i} \right\} > \left\{ \frac{n}{i} \right\}. \end{aligned}$$ ∎

Let us consider the value of $\Phi_n(q)$ at $q = 1$. It is easy to see that $\Phi_1(1) = 0$. For $n > 1$, we have

$$\Phi_n(1) = \begin{cases} p, & \text{if } n = p^m \text{ for some prime number } p, \\ 1, & \text{otherwise,} \end{cases}$$

which follows immediately from the construction of $\Phi_n(x)$:

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} \quad \text{and} \quad \Phi_{np}(x) = \begin{cases} \frac{\Phi_n(x^p)}{\Phi_n(x)}, & \text{if } p \nmid n, \\ \Phi_n(x^p), & \text{if } p \mid n, \end{cases}$$

where p is a prime number (see [10]). Based on this evaluation and Theorem 4, we obtain Kummer's theorem.

Corollary 5 (Kummer's Theorem) *The power of prime p dividing $\binom{n}{m}$ is given by the number of integers $j > 0$ for which $\{m/p^j\} > \{n/p^j\}$.*

Remark 3. Let $[n]! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. From the following factorization

$$[n]! = \prod_{i=2}^{n} (\Phi_i(q))^{\lfloor \frac{n}{i} \rfloor},$$

one obtains the power of a prime p dividing $n!$ by taking $q = 1$ (see [6]):

$$\epsilon_p(n) = \sum_{r \geq 0} \left\lfloor \frac{n}{p^r} \right\rfloor.$$

As a q-generalization of the Catalan numbers, the q-Catalan numbers have been extensively investigated (see [5, 9]). Based on Theorem 3, we derive the following divisibility properties of the q-Catalan numbers.

Corollary 6 *The q-Catalan numbers $\frac{1-q}{1-q^{n+1}}\begin{bmatrix} 2n \\ n \end{bmatrix}$ are polynomials in q and have at least $n-1$ irreducible factors.*

Proof. Since $\Phi_{n+2}, \Phi_{n+3}, \ldots, \Phi_{2n}$ are irreducible factors of $\begin{bmatrix} 2n \\ n \end{bmatrix}$ and are coprime with $1-q^{n+1}$, they are also irreducible factors of the q-Catalan number. For each factor Φ_i $(i \geq 2)$ of $1-q^{n+1}$, we have $i \mid n+1$ and

$$\left\{\frac{n}{i}\right\} = \frac{i-1}{i} > \left\{\frac{2n}{i}\right\} = \frac{i-2}{i}.$$

From Theorem 3, it follows that Φ_i is a factor of $\begin{bmatrix} 2n \\ n \end{bmatrix}$. ∎

As a generalization of Theorem 3, we have

Theorem 7 *Let $M = \max\{n_1, n_2, \ldots, n_r\}$. Then $\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix}$ has at least $n - M$ irreducible factors.*

Proof. For any $M+1 \le i \le n$, we have

$$\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n_1}{i} \right\rfloor - \left\lfloor \frac{n_2}{i} \right\rfloor - \cdots - \left\lfloor \frac{n_r}{i} \right\rfloor = \left\lfloor \frac{n}{i} \right\rfloor \ge 1.$$

Thus $\Phi_i(q)$ is an irreducible factor of $\left[{n \atop n_1, n_2, \ldots, n_r}\right]$. ∎

Remark 4. As in the above theorem, the lower bound $n-M$ can be reached only for the following cases:

$$\begin{bmatrix} p \\ 1 \end{bmatrix} (p \text{ is prime}), \ \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \ \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \ \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \ \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \ \begin{bmatrix} 11 \\ 5 \end{bmatrix}, \ \begin{bmatrix} 3 \\ 1,1,1 \end{bmatrix}, \ \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}.$$

An interesting factor of the q-multinomial coefficient $\left[{n \atop n_1, \ldots, n_r}\right]$ is

$$(q^n - 1)/(q^d - 1) = 1 + q^d + q^{2d} + \cdots + q^{n-d}.$$

where $d = \gcd(n, n_1, \ldots, n_r)$. Andrews [1] proved the existence of this factor for the q-binomial coefficient $\left[{n \atop k}\right]$ with n and k being relatively prime. Brunetti and Del Lungo [3] extended this result to q-multinomial coefficients. We note that this divisibility property easily follows from Lemma 1.

Theorem 8 (Brunetti and Del Lungo) *Let $n_1, n_2, \ldots, n_r$ be nonnegative integers such that $n = n_1 + \cdots + n_r$. If $d = \gcd(n, n_1, \ldots, n_r)$, then*

$$f(q) = \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix} \frac{q^d - 1}{q^n - 1} = \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix} \Big/ \ (1 + q^d + q^{2d} + \cdots + q^{n-d})$$

is a polynomial in q with nonnegative coefficients. Moreover, $f(q)$ can be written as a product of $n - M - 1$ nonconstant polynomials, where $M = \max\{n_1, \ldots, n_r\}$.

Proof. Firstly, we prove that $f(q)$ is a polynomial in q. Since $q^n - 1 = \prod_{j|n} \Phi_j(q)$ has no multiple roots, it suffices to show that for any $j \mid n$, $\Phi_j(q)$ is a factor of $\left[{n \atop n_1, \ldots, n_r}\right](q^d - 1)$. In fact, if $j \nmid n_t$ for some $1 \le t \le r$, then we have $\lfloor n_t/j \rfloor < n_t/j$ and

$$\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n_1}{j} \right\rfloor - \cdots - \left\lfloor \frac{n_r}{i} \right\rfloor > \frac{n}{j} - \frac{n_1 + n_2 + \cdots + n_r}{j} = 0.$$

By Lemma 1, $\Phi_j(q)$ is a factor of $\left[{n \atop n_1, \ldots, n_r}\right]$. Otherwise, we have $j \mid n_t$ for any $1 \le t \le r$. Thus $j \mid \gcd(n, n_1, \ldots, n_r)$ and $\Phi_j(q)$ is a factor of $q^d - 1$.

Since

$$f(q) = \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix} \frac{1 + q + \cdots + q^{d-1}}{1 + q + \cdots + q^{n-1}},$$

the nonnegativity of the coefficients of $f(q)$ follows from the the unimodal property of

$$\begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix} (1 + q + \cdots + q^{d-1}).$$

Without loss of generality, we may assume $n_1 = \max\{n_1, \ldots, n_r\}$. Then

$$f(q) = \frac{(1-q^{n_1+1})(1-q^{n_1+2})\cdots(1-q^{n-1})}{\left(\prod_{k=2}^{r}\prod_{j=1}^{n_k}(1-q^j)\right)/(1-q^d)}. \tag{4}$$

Since n_1 is maximal, for $n_1 + 1 \le j \le n-1$, $\Phi_j(q)$ is a factor of $1 - q^j$ but is not a factor of the polynomial

$$\left(\prod_{k=2}^{r}\prod_{j=1}^{n_k}(1-q^j)\right)/(1-q^d).$$

Thus, after cancelling the common factors of the numerator and denominator of (4), $1 - q^{n_1+1}, \ldots, 1 - q^{n-1}$ become $n - n_1 - 1$ nontrivial factors of $f(q)$. ∎

Acknowledgments. The authors would like to thank the referees for valuable comments. This work was done under the auspices of 973 Project on Mathematical Mechanization, the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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