Statistics on Dyck paths

January 15, 2006

Toufik Mansour
Department of Mathematics, University of Haifa, 31905 Haifa, Israel.
Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China.
toufik@math.haifa.ac.il

Abstract

In this paper we consider several statistics on the set of Dyck paths. Enumeration of Dyck paths according to length and various other parameters has been studied in several papers. However, the statistic "number of *udu*'s" has been considered only recently. We generalize this statistic and derive an explicit formula for the number of Dyck paths of length $2n$ according to the statistic "number of $uu\dots udu$'s" ("number of $udud\dots udu$'s"). As a consequence, we derive several known results, as well as many new results.

1 Introduction

A *lattice path* of length n is a sequence of points $P_1, P_2, \dots, P_n$ with $n \geq 1$ such that each point P_i belongs to the plane integer lattice and consecutive points P_i and P_{i+1} are connected by a line segment. We will consider lattice paths in $\mathbb{Z}^2$ whose permitted step types are up-steps $u = (1, 1)$, down-steps $d = (1, -1)$, and horizontal steps $h = (1, 0)$. We will focus on paths that start from the origin and return to the x-axis, and that never pass below the x-axis. Let $\mathcal{D}_n$ denote the set of such paths, *Dyck paths*, of length $2n$ when only up-steps and down-steps are allowed, and let $\mathcal{M}_n$ denote the set of such paths, *Motzkin paths*, of length n when all the three types are allowed. It is well known that $|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see [8, A000108]), having ordinary generating function $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ which satisfies the relation

$$xC^2(x) - C(x) - 1 = 0, \tag{1}$$

and $|\mathcal{M}_n| = M_n$, the n-th Motzkin number (see [8, A001006]), having ordinary generating function $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, which satisfies the relation

$$x^2M^2(x) + (x-1)M(x) + 1 = 0. \tag{2}$$

Let D be any Dyck path and p be any word on alphabet $\{u, d\}$. We say that the Dyck path D occurs p at *low level* if D can be decomposed as $D = D'pR$ such that D' is a Dyck path. Otherwise we say that the Dyck path D occurs p at *high level*.

The enumeration of Dyck paths according to length and various other parameters has been studied by several authors [1]-[7]. However, the statistic "number of udu's" has been considered only recently [7, 9].

The paper is organized as follows. In Section 2 we consider the statistic "the number of $uu\ldots udu$'s" and the statistic "the number of $udud\ldots du$'s" in the set of Dyck paths. In Section 3 we consider these statistics at low and high level in the set of Dyck paths.

2 Enumeration of Dyck paths according to number of $uu\ldots udu$'s or number of $udud\ldots udu$'s

In this section we enumerate the number of Dyck paths according to the length and either the number of $uu\ldots udu$'s or the number of $udud\ldots udu$'s.

2.1 Enumeration of Dyck paths according to number of $uu\ldots udu$'s

Let $\alpha_i(D)$ be the number of occurrences of $\underbrace{uu\ldots u}_{i}du$ in the Dyck path D, see Table 1.

$n\backslash\alpha_1$	0	1	2	3	4	5	6	$n\backslash\alpha_2$	0	1	2	3	4	5	6
1	1							1	1						
2	1	1						2	2						
3	2	2	1					3	4	1					
4	4	6	3	1				4	10	3	1				
5	9	16	12	4	1			5	26	12	3	1			
6	21	45	40	20	5	1		6	72	41	15	3	1		
7	51	126	135	80	30	6	1	7	206	143	58	18	3	1	

Table 1: Number of Dyck paths of length $2n$ according to the statistics α_1 and α_2.

Define the ordinary generating function for the number of Dyck paths of length $2n$ according to the statistics $\alpha_1, \alpha_2, \ldots$ as $F(x; q_1, q_2, \ldots) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n \prod_{j\geq 1} q_j^{\alpha_j(D)}$. To study the above ordinary generating function we need the following notation

$$A_k(x; q_1, q_2, \ldots) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n \prod_{j\geq 1} q_j^{\alpha_j^k(D)},$$

where $\alpha_j^k(D) = \alpha_j\left(\underbrace{uu\ldots u}_{k} D \underbrace{dd\ldots d}_{k}\right)$. These ordinary generating functions satisfy the following recurrence relations.

Proposition 2.1 *For all $k \geq 0$,*

$$\begin{aligned} A_k(x;q_1,q_2,\ldots) &= 1 + x(1-q_1q_2\cdots q_{k+1}) - x(1-q_1q_2\cdots q_{k+1})A_0(x;q_1,q_2,\ldots) \\ &+ xA_{k+1}(x;q_1,q_2,\ldots)A_0(x;q_1,q_2,\ldots). \end{aligned}$$

Proof An equation for the generating function $A_k(x;q_1,q_2,\ldots)$ is obtained from the "first return decomposition" of a Dyck paths D: $D = uPdQ$, where P, Q are Dyck paths. Thus, the four possibilities of P and Q either being empty or nonempty Dyck paths (see Figure 1) give contribution x, $x(A_{k+1}(x;q_1,q_2,\ldots)-1)$, $xq_1q_2\cdots q_{k+1}(A_0(x;q_{1,2},\ldots)-1)$, and

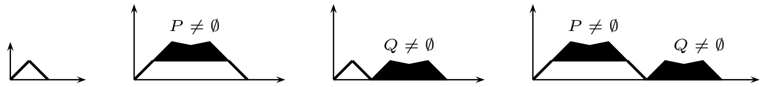

Figure 1: First return decomposition of a Dyck path with nonempty Dyck paths.

$x(A_{k+1}(x;q_1,q_2,\ldots)-1)(A_0(x;q_1,q_2,\ldots)-1)$, respectively. Hence, we have the following recurrence relation

$$\begin{aligned} A_k(x;q_1,q_2,\ldots) &= 1 + x(1-q_1q_2\cdots q_{k+1}) - x(1-q_1q_2\cdots q_{k+1})A_0(x;q_1,q_2,\ldots) \\ &+ xA_{k+1}(x;q_1,q_2,\ldots)A_0(x;q_1,q_2,\ldots), \end{aligned}$$

for all $k \geq 0$, as required. □

Theorem 2.2 *The generating function $F(x;q_1,q_2,\ldots)$ satisfies the following equation*

$$F(x;q_1,q_2,\ldots) = \sum_{m\geq 0} x^m F^m(x;q_1,q_2,\ldots)(1 + x(1-F(x;q_1,q_2,\ldots))(1-q_1q_2\ldots q_{m+1})).$$

Proof We simply use the fact $A_0(x;q_1,q_2,\ldots) = F(x;q_1,q_2,\ldots)$ and apply (see Proposition 2.1) the recurrence relation

$$\begin{aligned} A_k(x;q_1,q_2,\ldots) &= 1 + x(1-q_1q_2\cdots q_{k+1})(1-A_0(x;q_1,q_2,\ldots)) \\ &\quad + xA_{k+1}(x;q_1,q_2,\ldots)A_0(x;q_1,q_2,\ldots). \end{aligned}$$

an infinite number of times and in each step we perform some rather tedious algebraic manipulations. □

Example 2.3 *Define $q_1 = q$, $q_2 = q^{-1}$, and $q_i = 1$ for all $i \geq 3$. Then Theorem 2.2 gives that the ordinary generating function $f(x;q) = F(x;q,q^{-1},1,1,\ldots)$ satisfies*

$$f(x;q) = \frac{1}{1-xf(x;q)} + x(1-f(x;q))(1-q).$$

Solving the above equation we get that the ordinary generating function

$$f(x;q)=\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\alpha_1(D)-\alpha_2(D)}$$

is given by

$$\frac{1+x(1+x)(1-q)-\sqrt{(x(1+x)(1-q)+1)^2-4x(1+x(1-q))^2}}{2x(1+x(1-q))}.$$

In particular, the number of Dyck paths D of length $2n$ such that $\alpha_1(D)=\alpha_2(D)$ is given by (see [8, A078481])

$$\frac{1+x+x^2-\sqrt{(1+x+x^2)^2-4x(1+x)^2}}{2x(1+x)}.$$

Example 2.4 *Define $q_{2i-1}=q$ and $q_{2i}=q^{-1}$ for all $i\geq 1$. Then Theorem 2.2 obtains that*

$$F(x;q,1/q,q,1/q,\ldots)=1+x+xq(F(x;q,1/q,q,1/q,\ldots)-1)+x^2F^3(x;q,1/q,q,\ldots).$$

Then the Lagrange inversion formula (see [10]) gives that

$$\sum_{n\geq 1}\sum_{D\in\mathcal{D}_n} x^n q^{\alpha_1(D)-\alpha_2(D)+\alpha_3(D)-\cdots}=\sum_{n\geq 1}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{i}{j}\binom{3i}{n-1+i-j}q^{j-i}\frac{x^{n+i}}{n}.$$

In particular, the number of Dyck paths D of length $2n$ with $\sum_{i\geq 0}\alpha_{2i+1}(D)=\sum_{i\geq 0}\alpha_{2i}(D)$ is given by

$$\sum_{j=0}^{n-1}\frac{1}{n-j}\binom{n-j}{j}\binom{3j}{n-1-j}.$$

Example 2.5 *Define $q_i=q$ for all $i\geq 1$. Then Theorem 2.2 gives that*

$$S(x;q)=\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\alpha_1(D)+\alpha_2(D)+\cdots}=\sum_{m\geq 0} x^m S^m(x;q)(1+x(1-S(x;q))(1-q^{m+1})),$$

which is equivalent to $S(x;q)=1+\frac{x}{1-xS(x;q)}-\frac{xq(1-S(x;q))}{1-xqS(x;q)}$. Differentiating the above equation with respect to q and substituting $q=1$ with using the fact that $S(x;1)=C(x)$, the generating function for the catalan numbers, gives that

$$\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n}(\alpha_1(D)+\alpha_2(D)+\cdots)x^n=\frac{x^2C^4(x)}{1-xC^2(x)}=\sum_{n\geq 2}\binom{2n-1}{n-2}x^2.$$

In other words, the statistic $\sum_{D\in\mathcal{D}_n}\alpha_1(D)+\alpha_2(D)+\cdots$ equals $\binom{2n-1}{n-2}$, for all $n\geq 2$ (see [8, A002054]).

Denote the ordinary generating function $F(x;q_1,q_2,\ldots)$ with $q_k=q$ and $q_i=1$ for all $i\neq k$ by $F_k(x;q)$. Fix $k\geq 1$, $q_k=q$ and $q_i=1$ for all $i\neq k$. From the definitions it is easy to see that $F_k(x;q)=F_{k-1}(x;q)$. Thus, Proposition 2.1 gives that

$$F_{k-1}(x;q)=1+x(1-q)-x(1-q)F_0(x;q)+xF_{k-1}(x;q)F_0(x;q),$$

and

$$F_{j+1}(x;q)=\frac{F_j(x;q)-1}{xF_j(x;q)},$$

for all $j=0,1,\ldots,k-2$. To give an explicit formula for $F_k(x;q)$ we need the following lemma.

Lemma 2.6 *For all* $j=0,1,\ldots,k-1$,

$$F_j(x;q)=\frac{\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[\sqrt{x}U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-2}\left(\frac{1}{2\sqrt{x}}\right)\right]},$$

where $U_j(t)$ *is the* j*-th Chebyshev polynomial of the second kind.*

Proof We proceed by induction on j. The result is true for $j=0$. For $j\geq 0$ we have that $F_{j+1}(x;q)=\frac{F_j(x;q)-1}{xF_j(x;q)}$. Thus, by the induction hypothesis for j we obtain that the ordinary generating function $F_{j+1}(x;q)$ equals

$$\frac{\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)-\sqrt{x}\left[\sqrt{x}U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-2}\left(\frac{1}{2\sqrt{x}}\right)\right]}{x\left[\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)\right]},$$

which is

$$\frac{\sqrt{x}\left[\frac{1}{\sqrt{x}}U_j\left(\frac{1}{2\sqrt{x}}\right)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)\right]F_0(x;q)-\left[\frac{1}{\sqrt{x}}U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)-U_{j-2}\left(\frac{1}{2\sqrt{x}}\right)\right]}{\sqrt{x}\left[\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

Using the recurrence relation $U_m(t)=2tU_{m-1}(t)-U_{m-2}(t)$ twice we get

$$\frac{\sqrt{x}U_{j+1}\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_j\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[\sqrt{x}U_j\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q)-U_{j-1}\left(\frac{1}{2\sqrt{x}}\right)\right]},$$

as claimed. □

Now we are ready to give an explicit formula for the ordinary generating function $F_k(x;q)$.

Theorem 2.7 *Let*

$$\alpha(x;q) = x^2(1-q)U_{k-2}\left(\tfrac{1}{2\sqrt{x}}\right) - x\sqrt{x}U_{k-1}\left(\tfrac{1}{2\sqrt{x}}\right),$$
$$\beta(x;q) = x\sqrt{x}(1-q)\left[U_{k-3}\left(\tfrac{1}{2\sqrt{x}}\right) + \sqrt{x}U_{k-2}\left(\tfrac{1}{2\sqrt{x}}\right)\right] - \sqrt{x}U_{k-1}\left(\tfrac{1}{2\sqrt{x}}\right),$$
$$\gamma(x;q) = \sqrt{x}(1+x(1-q))U_{k-3}\left(\tfrac{1}{2\sqrt{x}}\right) - U_{k-2}\left(\tfrac{1}{2\sqrt{x}}\right).$$

Then the ordinary generating function $F_k(x;q)$ is given by

$$\frac{\gamma(x;q)}{\beta(x;q)}C\left(\frac{\alpha(x;q)\gamma(x;q)}{\beta^2(x;q)}\right),$$

where $U_j(t)$ is the j-th Chebyshev polynomial of the second kind and $C(x)$ is the ordinary generating function for the Catalan numbers.

Proof The generating function $F_0(x;q)$ satisfies the following equation

$$F_{k-1}(x;q) = 1 + x(1-q) - x(1-q)F_0(x;q) + xF_{k-1}(x;q)F_0(x;q).$$

Using Lemma 2.6 for $j = k-1$ we get

$$F_{k-1}(x;q) = \frac{\sqrt{x}U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q) - U_{k-2}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[\sqrt{x}U_{k-2}\left(\frac{1}{2\sqrt{x}}\right)F_0(x;q) - U_{k-3}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

Substituting the expression for $F_{k-1}(x;q)$ in the above equation and using the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ we obtain

$$\alpha(x;q)F_0^2(x;q) - \beta(x;q)F_0(x;q) + \gamma(x;q) = 0.$$

Hence, by solving the above equation we get the desired result. □

For example, Theorem 2.7 for $k=1$ gives $F_1(x;q) = C\left(\frac{x}{1+x(1-q)}\right)$ (see [9, Equations 2.2 and 2.3]). In particular, $F_1(x;0) = C(x/(1+x))$. This result can be generalized by using Theorem 2.7 with $q=0$.

Corollary 2.8 *The ordinary generating function for the number of Dyck paths of length $2n$ that avoid $\underbrace{uu\dots u}_{k}du$ is given by*

$$\frac{U_k\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{(1+x)U_k\left(\frac{1}{2\sqrt{x}}\right)}C\left(\frac{xU_k\left(\frac{1}{2\sqrt{x}}\right) + x\sqrt{x}U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{(1+x)^2U_k\left(\frac{1}{2\sqrt{x}}\right)}\right),$$

where $U_j(t)$ is the j-th Chebyshev polynomial of the second kind and $C(x)$ is the ordinary generating function for the Catalan numbers.

For instance, the ordinary generating function for the number of Dyck paths of length $2n$ that avoid $uudu$ is given by $\frac{1}{1-x^2}C\left(\frac{x}{(1-x)(1+x)^2}\right)$ (see [8, A105633]).

2.2 Enumeration of Dyck paths according to number of $ud\ldots udu$'s

Let $\beta_i(D)$ be the number of occurrences of $\underbrace{udud\ldots ud}_{i}u$ in the Dyck path D, see Table 2. Clearly, $\beta_1(D) = \alpha_1(D)$. Define the ordinary generating function for the number of Dyck

$n\backslash\beta_2$	0	1	2	3	4	5	6
1	1						
2	2						
3	4	1					
4	11	2	1				
5	31	8	2	1			
6	92	28	9	2	1		
7	283	99	34	10	2	1	

Table 2: Number of Dyck paths of length $2n$ according the statistic β_2.

paths of length $2n$ according to the statistics $\beta_1, \beta_2, \ldots$ as

$$G(x; q_1, q_2, \ldots) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n \prod_{j\geq 1} q_j^{\beta_j(D)}.$$

To study the above ordinary generating function we need the following notation

$$B_k(x; q_1, q_2, \ldots) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n \prod_{j\geq 1} q_j^{\beta_j^k(D)},$$

where $\beta_j^k(D) = \beta_j\left(\underbrace{ud\ldots ud}_{k}D\right)$. These ordinary generating functions satisfy the following recurrence relations.

Proposition 2.9 *For all $k \geq 0$,*

$$B_k(x; q_1, q_2, \ldots) = \prod_{i=1}^{k} q_i^{k-i} + x\prod_{i=1}^{k+1} q_i^{k+1-i}(B_0(x; q_1, q_2, \ldots) - 1)B_0(x; q_1, q_2, \ldots) \\ + xB_{k+1}(x; q_1, q_2, \ldots).$$

Proof An equation for the generating function $B_k(x; q_1, q_2, \ldots)$ is obtained from the "first return decomposition" of a Dyck paths D: $D = uPdQ$, where P and Q are Dyck paths. Then each Dyck path D that starts with $\underbrace{udud\ldots ud}_{k}$ can be decomposed as either $D = \underbrace{udud\ldots ud}_{k}$, $D = \underbrace{udud\ldots ud}_{k}uPdQ$ with P a nonempty Dyck path, or $D = \underbrace{udud\ldots ud}_{k+1}Q$ (see Figure 2 for $k = 2$). Thus, the ordinary generating function $B_k(x; q_1, q_2, \ldots)$ is given by

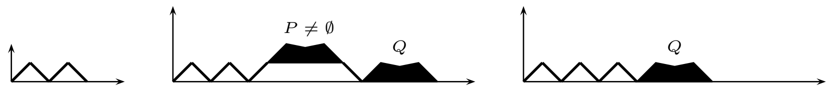

Figure 2: First return decomposition of a Dyck path according the statistic *ududu*.

$$\prod_{i=1}^{k} q_i^{k-i} + x\prod_{i=1}^{k+1} q_i^{k+1-i}(B_0(x;q_1,q_2,\ldots)-1)B_0(x;q_1,q_2,\ldots) + xB_{k+1}(x;q_1,q_2,\ldots),$$

for all $k \geq 0$, as required. □

Theorem 2.10 *The generating function $G(x;q_1,q_2,\ldots)$ is given by*

$$G(x;q_1,q_2,\ldots) = C\left(\frac{x\sum_{m\geq 0} x^m \prod_{i=1}^{m+1} q_i^{m+1-i}}{1+x\sum_{m\geq 0} x^m \prod_{i=1}^{m+1} q_i^{m+1-i}}\right),$$

where $C(x)$ is the generating function for the Catalan numbers.

Proof We simply use the fact that $B_0(x;q_1,q_2,\ldots) = G(x;q_1,q_2,\ldots)$ and apply the recurrence relation

$$\begin{aligned} B_k(x;q_1,q_2,\ldots) &= \prod_{i=1}^{k} q_i^{k-i} + x\prod_{i=1}^{k+1} q_i^{k+1-i}(B_0(x;q_1,q_2,\ldots)-1)B_0(x;q_1,q_2,\ldots) \\ &\quad + xB_{k+1}(x;q_1,q_2,\ldots) \end{aligned}$$

an infinite number of times and in each step we perform some rather tedious algebraic manipulations. □

Example 2.11 *Let $q_{2i-1} = q$ and $q_{2i} = q^{-1}$ for all $i \geq 1$. Then Theorem 2.10 gives that*

$$\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\beta_1(D)-\beta_2(D)+\cdots} = C\left(\frac{x(1+xq)}{1+x}\right)$$

which is equivalent to

$$\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\beta_1(D)-\beta_2(D)+\cdots} = 1 + \sum_{n\geq 1} x^n \sum_{i=0}^{n+1}\sum_{j=0}^{i+1} C_i \binom{i+1}{j}\binom{n-1-j}{i} q^j.$$

In particular, the ordinary generating function for the number of Dyck paths D of length $2n$ with

$$\sum_{i\geq 1} \beta_{2i-1}(D) = \sum_{i\geq 1} \beta_{2i}(D) \qquad (3)$$

is given by $C(x/(1+x)) = M(x)$, that is, the number of Dyck paths D of length $2n$ with (3) is given by M_n, the n-th Mozkin number.

Example 2.12 *Let $q_i = q$ for all $i \geq 1$. Then, similar to Example 2.5, we obtain from Theorem 2.10 that*

$$\sum_{D\in\mathcal{D}_n} (\beta_1(D) + \beta_2(D) + \cdots) = \sum_{j=0}^{n-2} \binom{2j+2}{j},$$

for all $n \geq 2$.

Denote the ordinary generating function $G(x; q_1, q_2, \ldots)$ with $q_k = q$ and $q_i = 1$ for all $i \neq k$ by $G_k(x; q)$. For example, if $q_1 = q$ and $q_i = 1$, $i = 2, 3, \ldots$, then Theorem 2.10 gives $G_1(x; q) = C\left(\frac{x}{1+x(1-q)}\right)$ (see [9, Equations 2.2 and 2.3]). In general, if $q_k = q$ and $q_i = 1$ for all $i \neq k$, then Theorem 2.7 gives an explicit formula for the ordinary generating function $G_k(x; q)$.

Corollary 2.13 *The ordinary generating function for the Dyck paths of length $2n$ according to the statistic β_k is given by*

$$G_k(x; q) = \sum_{n\geq 0} \sum_{D\in\mathcal{D}_n} x^n q^{\beta_k} = C\left(\frac{x(1 - xq - x^k(1-q))}{1 - xq - x^{k+1}(1-q)}\right),$$

where $C(x)$ is the generating function for the Catalan numbers.

For instance, Corollary 2.13 for $k = 2$ and $q = 0$ gives that the ordinary generating function for the number of Dyck paths of length $2n$ that avoid $UDUDU$ is given by

$$C\left(\frac{x(1-x^2)}{1-x^3}\right) = 1 + \sum_{n\geq 1} x^n \left(\sum_{i=1}^{n} \sum_{j=0}^{(n-m)/2} (-1)^{\frac{n-m+j}{3}} \binom{m}{j} \binom{\frac{n+2m-2j}{3} - 1}{m-1} C_m\right).$$

3 Enumeration of Dyck paths according to number of $uu\ldots udu$'s or number of $udud\ldots udu$'s at low and high levels

In this section we consider the following statistics on Dyck paths: number of low (high) level $uu\ldots udu$'s or number of low (high) level $udud\ldots udu$'s (for the case $k = 1$, see [9, Section 3]).

3.1 Enumeration of Dyck paths according to number of $uu\ldots udu$'s at low and high levels

Let γ_k (resp. γ'_k) be the number of occurrences of $\underbrace{uu\ldots u}_{k}du$ at low (resp. high) level. Denote the ordinary generating function for the number of Dyck paths of length $2n$ according to the statistics γ_k (resp. γ'_k) by

$$L_k(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\gamma_k(D)} \quad \left(\text{resp. } H_k(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\gamma'_k(D)}\right).$$

Theorem 3.1 *Let $k \geq 2$. Then the number of Dyck paths of length $2n$ with $\gamma_k = m$ is given by*

$$\sum_{j\geq 0}(-1)^j \frac{(m+j)(k+1)+1}{2n+1-(m+j)(k+1)}\binom{m+j}{j}\binom{2n+1-(m+j)(k+1)}{n+1}.$$

Proof Consider any Dyck path $D = uPdQ$, where P and Q are Dyck paths. So D contains $a_k := \underbrace{uu\ldots u}_{k}du$ if either P starts with a_{k-1} or Q contains a_k (see Figure 3). Thus,

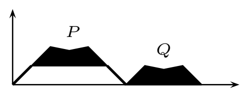

Figure 3: First return decomposition of a Dyck path.

$L_k(x;q) = 1 + xt_1(x;q)L_k(x;q)$, where $t_j(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\gamma_k\left(\underbrace{uu\ldots u}_{j} D \underbrace{dd\ldots d}_{j}\right)}$ for all $j = 1, 2, \ldots k-1$. Again, using the first return decomposition of a Dyck path we obtain that the ordinary generating function $t_j(x;q)$ equals $1 + xt_{j+1}(x;q)C(x)$, where $C(x)$ is the ordinary generating function for the Catalan numbers and $j = 1, 2, \ldots, k-2$. In addition, $t_{k-1}(x;q) = 1 + x + xC(x)(C(x)-1) + xq(C(x)-1)$. Therefore, it is easy to check that

$$t_1(x;q) = (xC(x))^{k-2}t_{k-1}(x;q) + \sum_{j=0}^{k-3}(xC(x))^j.$$

This implies that the ordinary generating function $L_k(x;q)$ is given by

$$L_k(x;q) = \frac{1}{1 - x\left(\frac{1-(xC(x))^{k-2}}{1-xC(x)} + (xC(x))^{k-2}t_1(x;q)\right)}.$$

By using the identity $C(x) = \frac{1}{1-xC(x)}$ (see (1)) we arrive at

$$L_k(x;q) = \frac{C(x)}{1+(1-q)(xC(x))^{k+1}} = \sum_{m\geq 0}\frac{x^{m(k+1)}C^{m(k+1)+1}(x)}{(1+(xC(x))^{k+1})^{m+1}},$$

which is equivalent to

$$L_k(x;q) = \sum_{m\geq 0}\sum_{j\geq 0}(-1)^j\binom{m+j}{j}x^{(m+j)(k+1)}C^{(m+j)(k+1)+1}(x).$$

On other hand, by applying the Lagrange inversion formula in [10], we get that

$$C^k(x) = \sum_{n\geq 0} \frac{k}{2n+k}\binom{2n+k}{n}x^n.$$

Hence, the $x^n q^m$ coefficient in the ordinary generating function $L_k(x;q)$ is given by

$$\sum_{j\geq 0}(-1)^j \frac{(m+j)(k+1)+1}{2n+1-(m+j)(k+1)}\binom{m+j}{j}\binom{2n+1-(m+j)(k+1)}{n+1},$$

as required. □ For example, Theorem 3.1 for $k=2$ (for the case $k=1$ see [9, Theorem 3.1]) gives that the number of Dyck paths of length $2n$ having $\gamma_2 = m$ is given by

$$\sum_{j\geq 0}(-1)^j \frac{3(m+j)+1}{2n+1-3(m+j)}\binom{m+j}{j}\binom{2n+1-3(m+j)}{n+1}.$$

Theorem 3.2 *The ordinary generating function for the number of Dyck paths of length $2n$ according to the statistic γ'_k is given by*

$$H_k(x;q) = \frac{1}{1 - x\frac{1-x^k(1-q)F_k^{k-1}(x;q)(F_k(x;q)-1)}{1-xF_k(x;q)}},$$

where the explicit formula for the ordinary generating function $F_k(x;q)$ is given by Theorem 2.7.

Proof Again (see Theorem 3.1), by using the first return decomposition of a Dyck path D ($D = uPdQ$ where P and Q are Dyck paths, see Figure 3), we obtain that the ordinary generating function $H_k(x;q)$ satisfies $H_k(x;q) = 1 + xt_1(x;q)H_k(x;q)$, where $t_j(x;q)$ satisfies the recurrence relation $t_j(x;q) = 1 + xt_{j+1}(x;q)F_k(x;q)$, for $j = 1,2,\dots k$, and $t_k(x;q) = 1 + x + x(t_k(x;q)-1)F_k(x;q) + xq(F_k(x;q)-1)$. Solving the above system of equations we get the desired result. □

Theorems 3.1 and 3.2 give the following result. For $\gamma'_1 = 0$, we obtain that the number of Dyck paths of length $2n$ is given by the n-th Motzkin number (see [9, Equation 3.6]). For $\gamma'_2 = 0$, we obtain that the ordinary generating function for the number of Dyck paths of length $2n$ is given by

$$\frac{2x^3+x^2-5x+3-(2x+1)\sqrt{x^4-2x^3-5x^2-2x+1}}{2(x^5+x^4+4x^3+5x^2-4x+1)}.$$

3.2 Enumeration of Dyck paths according to number of $ud\dots udu$'s at low and high levels

Let δ_k (resp. δ'_k) be the number of occurrences of $\underbrace{udud\dots ud}_{k}u$ at low (resp. high) level. Denote the ordinary generating function for the number of Dyck paths of length $2n$ according to the statistics δ_k (resp. δ'_k) by

$$L'_k(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\delta_k(D)} \quad \left(\text{resp. } H'_k(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\delta'_k(D)}\right).$$

Using the same techniques as in the previous subsection we obtain the following results.

Theorem 3.3 *Let $k \geq 1$. Then the ordinary generating function for the number of Dyck paths of length $2n$ according to the statistic δ_k is given by*

$$L'_k(x;q) = \frac{1-x^{k+1}-xq(1-x^k)}{1-x^{k+1}-x(1-x^k)C(x)-xq(1-x^k-x(1-x^{k-1})C(x))}.$$

Moreover, the ordinary generating function for the number of Dyck paths of length $2n$ with $\delta_k = m$ is

$$\frac{x^{m+k}(1-x)^2C(x)(1-x^k-x(1-x^{k-1})C(x))^{m-1}}{(1-x^{k+1}-x(1-x^k)C(x))^{m+1}}$$

when $m \geq 1$, and $\frac{1-x^{k+1}}{1-x^{k+1}-x(1-x^k)C(x)}$ when $m = 0$.

For example, the number of Dyck paths of length $2n$ having $\delta_1 = 0$ is given by $\frac{1+x}{1+x-xC(x)}$ (see [9, Page 5]), and having $\delta_2 = 0$ is given by $\frac{1+x+x^2}{1+x+x^2-x(1+x)C(x)}$.

To enumerate the number of Dyck paths of length $2n$ according to the statistic δ'_k we use the first return decomposition of a Dyck path and Corollary 2.13. Thus we obtain the following result.

Theorem 3.4 *Let $k \geq 1$. Then the ordinary generating function for the number of Dyck paths of length $2n$ according to the statistic δ'_k is given by*

$$H'_k(x;q) = \frac{1}{1-xG_k(x;q)},$$

where

$$G_k(x;q) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^n q^{\beta_k} = C\left(\frac{x(1-xq-x^k(1-q))}{1-xq-x^{k+1}(1-q)}\right).$$

For example, the ordinary generating function for the number of Dyck paths of length $2n$ having $\delta'_1 = 0$ is given by $M(x)$ (see [9, Equation 3.6]).

References

[1] E. Deutsch, Dyck paths Enumeration, *Discr. Math.* **204** (1999), 167–202.

[2] G. kreweras, Joint Distribution of three descriptive paramters of bridges, *Combinatoire Énumérative, Proc. (Monntreal, Que., 1985/Quebec. Que., 1985), Lec. Notes in Math.* **1234** (1986), 177–191.

[3] G. kreweras and P. Moszkowski, A new enumerative property of the Narayana numbers, *J. Statist. Plann. and Infer.* **14** (1986), 63–67.

[4] G. kreweras and Y. Poupard, Subdivision des nombers de Narayana Suivant deux paramétres supplémentaires, *Europ. J. Combin.* **7** (1986), 141–149.

[5] T. Mansour, Counting peaks at height k in a Dyck path, *Journal Integer of Sequences* **5** (2002), Article 02.1.1 (10pp).

[6] D. Merlini, D.G. Rogers, R. Sprugnoli, and M.C. Verri, On some alternative characterizations of Riordan arrays, *Canad. J. Math.* **49:2** (1997), 301–320.

[7] D. Merlini, R. Sprugnoli, and M.C. Verri, Some statistics on Dyck paths, *J. Statist. Plann. and Infer.* **101** (2002), 211–227.

[8] N.J.A. Sloane and S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, New York (1995).

[9] Y. Sun, The statistic "number of *udu*'s" in Dyck paths, *Discr. Math.* **287** (2004), 177–186.

[10] H. Wilf, *Generatingfunctionology*, Academic Press, New York, 1990.

2000 *Mathematics Subject Classification*: 05A05, 05A15, 42C05.

Keywords: Chebyshev polynomials, Dyck paths, Generating functions.

(Concerned with sequences A000108, A001006, A002054, A078481, and A105633)