DYCK PATHS AND RESTRICTED PERMUTATIONS

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Abstract

This paper is devoted to characterize permutations with forbidden patterns by using canonical reduced decompositions, which leads to bijections between Dyck paths and $S_n(321)$ and $S_n(231)$ respectively. We also discuss permutations in S_n avoiding two patterns, one of length 3 and the other of length k. These permutations produce a kind of discrete continuity between the Motzkin and the Catalan numbers.

Keywords: Dyck path, restricted permutation, canonical reduced decomposition.

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1. Introduction

1.1. **Pattern avoidance.** Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$. For a permutation σ of k positive integers, the *pattern* (or *type*) of σ is defined as a permutation τ on $[k]$ obtained from σ by substituting the minimum element by 1, the second smallest element by 2, ..., and the maximum element by k. Sometimes we say that a permutation is *order equivalent* to its pattern. For a permutation $\tau \in S_k$ and a permutation $\pi \in S_n$, $k \leq n$, we say that π is τ-*avoiding* if there is no subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ $(i_1 < i_2 < \cdots < i_k)$ whose pattern is τ. We write $S_n(\tau)$ for the set of τ-avoiding permutations of $[n]$.

A *barred permutation* $\bar{\tau}$ of $[k]$ is a permutation of S_k with a bar over exactly one of its elements. Let τ be the permutation on $[k]$ obtained by unbarring $\bar{\tau}$, and $\hat{\tau}$ the pattern of the permutation obtained from τ by removing the barred element. A permutation $\pi \in S_n$ contains a subsequence ω of type $\bar{\tau}$ if and only if ω is of type $\hat{\tau}$ and it is not contained in any subsequence of type τ. In other words, a subsequence ω of π is of type $\bar{\tau}$ if it is of type $\hat{\tau}$ and it cannot be extended to a subsequence of type τ. Equivalently, for a permutation $\pi \in S_n$, if every subsequence of type $\hat{\tau}$ can be extended to a subsequence of type τ, then we say that π avoids the barred pattern $\bar{\tau}$. We denote by $S_n(\bar{\tau})$ the set of permutations of S_n avoiding the pattern $\bar{\tau}$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids every $\tau \in T$, the corresponding subsets of S_n is denoted by $S_n(T)$.

Restricted permutations have been extensively studied over last decade. The first paper devoted entirely to the study of permutations avoiding certain patterns appeared in 1985 (see [15]). Currently there exist more than two hundred papers on this subject. While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [15]) and for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [16]). Several recent papers [5, 6, 9, 10, 11, 12] deal with the case $\tau_1 \in S_3, \tau_2 \in S_k$ for various pairs τ_1 and τ_2.

In this paper, we also investigate the case of multiple pattern avoidance permutations. The tools used in this paper include canonical reduced decompositions, continued fractions, Chebyshev polynomials, and Dyck paths.

1.2. **Canonical reduced decomposition.** For any $1 \leq i \leq n-1$, define the map s_i: $S_n \to S_n$, such that s_i acts on a permutation π by interchanging the elements in positions i and $i+1$. We call s_i a *simple transposition*, and write the action of s_i on the right of the permutation as πs_i. Therefore one has $\pi(s_i s_j) = (\pi s_i) s_j$.

For any permutation $\pi \in S_n$, the *canonical reduced decomposition* of π has the following form:

$$\pi = (1\ 2\ \cdots\ n)\sigma = (1\ 2\ \cdots\ n)\sigma_1\sigma_2\cdots\sigma_k, \tag{1}$$

where

$$\sigma_i = s_{h_i}s_{h_i-1}\cdots s_{t_i}, \quad h_i \geq t_i \quad (1 \leq i \leq k) \quad \text{and} \quad 1 \leq h_1 < h_2 < \cdots < h_k \leq n-1.$$

Note that if one writes π in the two row notation (as a permutation in the symmetric group), then from (1) one has the relation $\pi = \sigma_1\sigma_2\cdots\sigma_k$. We call h_i the *head* and t_i the *tail* of σ_i. For example, for $\pi = 315264 \in S_6$, the canonical reduced decomposition is $(s_2s_1)(s_4s_3)(s_5)$.

It is well-known that the above canonical reduced decomposition is unique. In fact, we have the following algorithm to generate the canonical reduced decomposition based on the recursive construction of a permutation on $[n]$ by inserting the element n into a permutation on $[n-1]$. From this point of view, the idea of the canonical reduced decompositions falls into the general framework of the ECO methodology [2, 3].

Algorithm: Observe that the product $s_js_{j-1}\cdots s_i$ is equivalent to the action of the cyclic permutation on the segment from position i to position $j+1$. For the permutation 1, the canonical reduced decomposition is the identity. Suppose that we have constructed the canonical reduced decomposition for the permutation $\pi\backslash n$, which is obtained from π by deleting the element n. Assume that n is in position i in π. If $i = n$, the canonical reduced decomposition of π is the same as that of $\pi\backslash n$. For $i \neq n$, the action of $s_{n-1}s_{n-2}\cdots s_i$ would bring the element n to the proper position and shift other relevant elements to the positions on their right. This gives the canonical reduced decomposition of π.

The canonical reduced decomposition has the following property [1, 7]:

Lemma 1. *If σ is the canonical reduced decomposition of $\pi \in S_n$, then π has k inversions if and only if σ has exactly k simple transpositions.*

Chen, Deng and Yang were the first to use the canonical reduced decomposition to study permutations with forbidden patterns (see [4] and references therein).

1.3. **Chebyshev polynomials of the section kind.** *Chebyshev polynomials of the second kind* [13] (in what follows just Chebyshev polynomials) are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. Clearly, $U_r(t)$ is a polynomial of degree r in t with integer coefficients, and the following recurrence holds:

$$U_0(t) = 1,\ U_1(t) = 2t,\ \text{and } U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for all } r \geq 2. \tag{2}$$

The same recurrence is used to define $U_r(t)$ for $r < 0$ (for example, $U_{-1}(t) = 0$ and $U_{-2}(t) = -1$). Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [13]).

Apparently, first the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West [5], and later by Mansour and Vainshtein [9, 10, 11, 12], and by Krattenthaler [6].

1.4. **Dyck paths.** A *Dyck path* of semilength n is a path on the plane from the origin $(0,0)$ to $(2n,0)$ consisting of up steps and down steps such that the path does not go across the x-axis. We will use u and d to represent the up and down steps, respectively. An up step followed by down step, ud, is called a *peak*. The *height* of a step, peak, of a Dyck path is defined as its largest y-axis coordinate. The set of Dyck paths of semilength n is denoted by D_n, and the cardinality of D_n is the well-known Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

1.5. **Discrete continuity.** Let $H_k(x) = \sum_{n\geq 0} a_k(n)x^n$, $k \geq 1$, be a family of generating functions. We say $H_k(x)$ yields a *discrete continuity* between $H_1(x)$ and $f(x) = \sum_{n\geq 0} f(n)x^n$ (or a discrete continuity between the numbers $a_1(n)$ and the values of $f(n)$) if $\lim_{k\to\infty} H_k(x) = f(x)$ and $a_{k-1}(n) \leq a_k(n) \leq f(n)$ for all $n \geq 0$ and $k \geq 1$.

Recently, Barucci, Del Lungo, Pergola, and Pinzani [3] characterized the permutations avoiding 321 and $(k+2)\bar{1}(k+3)23\cdots(k+1)$. The enumeration of the corresponding permutations for $k=1$ gives the Motzkin numbers, when k goes to infinity, it gives the Catalan numbers, and for $1 < k < \infty$ it gives sequences that lie between the Motzkin numbers and the Catalan numbers. By the above definition this enumeration is a discrete continuity between the Motzkin numbers and the Catalan numbers. In [3], the authors posed the question of giving a combinatorial descriptions of these number sequences.

In this paper, we first give bijections between $S_n(321)$, $S_n(231)$ and Dyck paths respectively, in terms of the canonical reduced decomposition. Then we give combinatorial descriptions for the sequences mentioned in [3]. We study the generating functions of the number of permutations in $S_n(\tau_1,\tau_2)$, where $\tau_1 \in S_3$, and τ_2 is a barred pattern of length k ($34\cdots(k-1)1\bar{k}2$ or $34\cdots k\bar{1}2$). In several interesting cases the generating function depends only of k and is expressed via Chebyshev polynomials, the generating function of the Motzkin numbers $M(x)$, and the generating function of the Catalan numbers $C(x)$. In particular, we present many classes which produce a discrete continuity between the Motzkin and the Catalan numbers.

2. Reduced decompositions for $S_n(321)$ and the zigzag decompositions for Dyck paths

2.1. **A bijection between $S_n(321)$ and D_n.** In this subsection, we present a bijection between $S_n(321)$ and the set of Dyck paths of semilength n based on the algorithm given in the preceding section. First we characterize the permutations in $S_n(321)$ by the following canonical reduced decompositions.

Theorem 2. *For $\pi \in S_n(321)$, then $\sigma = \sigma_1\cdots\sigma_k$ is the canonical reduced decomposition of π, where $\sigma_i = s_{h_i}s_{h_i-1}\cdots s_{t_i}$ for $1 \leq i \leq k$ if and only if the set of parameters $\{(h_i,t_i)|1\leq i\leq k\}$ satisfies*

$$1 \leq h_1 < h_2 < \cdots < h_k \leq n-1, \tag{3}$$
$$h_i \geq t_i \qquad (1 \leq i \leq k), \tag{4}$$
$$t_i \geq t_{i-1}+1 \qquad (2 \leq i \leq k). \tag{5}$$

Proof. We only need to show that $t_i \geq t_{i-1}+1$, $(2 \leq i \leq k)$ since inequalities (3) and (4) are required by the definition of canonical reduced decomposition.

"$\Longrightarrow$". We use induction on n. Clearly, the statement is true for $n = 1,2$. Suppose it is true for $n-1$. Assume that n is in position i in π. If $i = n$, then the assertion is automatically true because the canonical reduced decomposition of $\pi\setminus n$ is the same as that of π. When $i \neq n$, the canonical reduced decomposition of π has one more factor $\sigma_k = s_{n-1}s_{n-2}\cdots s_i$. In other words, $h_k = n-1$ and $t_k = i$. We aim to show that $i \geq t_{k-1}+1$. Let

$$\begin{aligned} \pi\setminus n &= \beta_1\,\beta_2\,\cdots\,\beta_{i-1}\,\beta_i\,\cdots\,\beta_{n-1}, \\ \pi &= \beta_1\,\beta_2\,\cdots\,\beta_{i-1}\,n\,\beta_i\,\cdots\,\beta_{n-1}. \end{aligned}$$

By the inductive hypothesis, $\pi\backslash n \in S_{n-1}(321)$. Assume that $t_{k-1} \geq i$. From the recursive construction of the canonical reduced decomposition, one sees that $\beta_{t_{k-1}} > \beta_{t_{k-1}+1}$. Thus $n\,\beta_{t_{k-1}}\,\beta_{t_{k-1}+1}$ has pattern 321, which is a contradiction. Therefore, we conclude that $i \geq t_{k-1}+1$.

"$\Longleftarrow$". We use induction on n. Clearly, the statement is true for $n = 1, 2$. Suppose that it is true for $n-1$. Assume that n is in position i in π. When $i = n$, the canonical reduced decomposition of π is the same as that of $\pi \setminus n$. Then we have $\pi \in S_n(321)$ since $\pi \setminus n \in S_{n-1}(321)$. When $i \neq n$, the canonical reduced decomposition of π has one more factor $\sigma_k = s_{n-1}s_{n-2}\cdots s_i$. In other words, $h_k = n-1$ and $t_k = i$. Notice that we have the condition $i \geq t_{k-1}+1$. Let $h_{k-1} = n-m-1$ with $m \geq 1$, and let

$$\pi \setminus n = \beta_1\beta_2\cdots\beta_{n-1} = (1\ 2\ \cdots\ n-1)\sigma_1\sigma_2\cdots\sigma_{k-1}.$$

Then we have

$$\pi \setminus n \quad = \quad \beta_1\ \cdots\ \beta_{t_{k-1}-1}\,(n-m)\,\beta_{t_{k-1}+1}\ \cdots\ \beta_{n-m}\,(n-m+1)\ \cdots\ (n-1),$$

where $\beta_{t_{k-1}} = n-m$. By the inductive hypothesis, $\pi\backslash n \in S_{n-1}(321)$, thus the subsequence $\beta_{t_{k-1}+1}\ \cdots\ \beta_{n-m}$ is increasing. Since $i \geq t_{k-1}+1$, we have that $n-m$ precedes β_{i-1} in $\pi\backslash n$. Therefore, we obtain

$$\pi = \beta_1\ \cdots\ (n-m)\ \cdots\ \beta_{i-1}\,n\,\beta_i\ \cdots\ \beta_{n-m}\ \cdots\beta_{n-1}.$$

Assume that there exists a 321-pattern in π which contains n, namely, a subsequence $n\,\beta_j\,\beta_k$ for some $i \leq j < k$. One sees that this can be possible only for $j < k \leq n-m$. However, $(n-m)\,\beta_j\,\beta_k$ would form a 321-pattern in $\pi \setminus n$, which leads to a contradiction. $\square$

For a Dyck path P, we define the $(x+y)$-*labelling* of P as follows: Each cell in the region enclosed by P and the x-axis, whose corner points are (i,j), $(i+1,j-1)$, $(i+2,j)$ and $(i+1,j+1)$ is labelled by $(i+j)/2$. If $((i-1,j-1),(i,j))$ and $((i,j),(i+1,j+1))$ are two successive up steps in P, we call this cell an *essential cell*, and the step $((i-1,j-1),(i,j))$ its *left arm*.

Now we can define the *zigzag strip* of P as follows:

- If there is no essential cell in P, then the zigzag strip is simply the empty set.
- Otherwise, we define the zigzag strip of P as the border strip that begins at the rightmost essential cell.

As an example, Figure 1 illustrates the $(x+y)$-labelling of a Dyck path, whose zigzag strip is the light gray stip with labels 11, 12, 12.

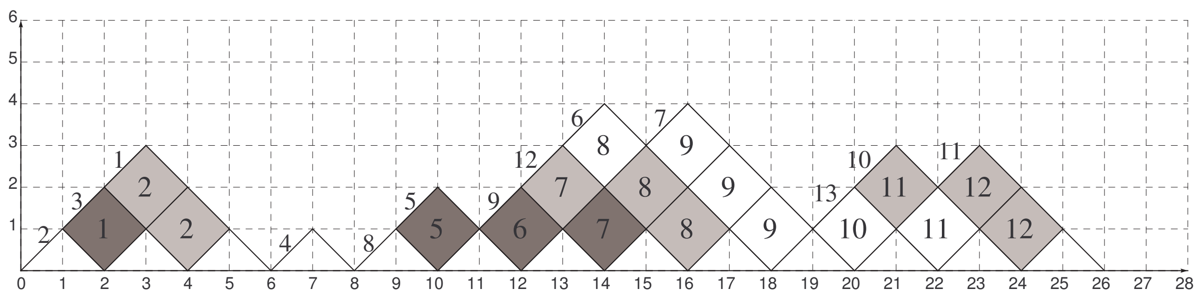

FIGURE 1. The $(x+y)$-labelling and zigzag decomposition for Dyck path.

Suppose $P_{n,k}$ is a Dyck path of semilength n that contains k essential cells, then we can obtain the *zigzag decomposition* of $P_{n,k}$ through the following procedure:

1. If $k = 0$, then the zigzag decomposition of $P_{n,0}$ is the empty set.
2. If $k = 1$, then the zigzag decomposition is the zigzag strip.

3. If $k \geq 2$, then decompose $P_{n,k}$ into $P_{n,k} = P_{n,k-1}Q$, where Q is the zigzag strip of $P_{n,k}$, and $P_{n,k-1}$ is the Dyck path obtained from P by deleting Q. If we read the labels of Q from left to right, we will get a sequence of numbers at the form $\{i, i+1, \ldots, j-1, j\}$, and we associate Q with a sequence of simple decompositions $\sigma_k = s_j\, s_{j-1}\, \cdots\, s_i$.
4. For $P_{n,i}$, $i \leq k-1$, repeat the above procedure, we will get $\sigma_{k-1}, \cdots, \sigma_1$.
5. The zigzag decomposition of $P_{n,k}$ is then given by $\sigma = \sigma_1\, \sigma_2\, \cdots\, \sigma_k$.

From the zigzag decomposition we can obtain a unique permutation $\varphi(P_{n,k}) = (12\cdots n)\sigma$ for the Dyck path $P_{n,k}$. Next we want to show that σ is indeed the canonical reduced decomposition of $\varphi(P_{n,k})$. For example, for the Dyck path $P_{13,6} = uuudddduduuduuudud dddduududdd$ shown in Figure 1, in which cells in the same zigzag strip are labelled with the same color. From the zigzag decomposition of this path, we get $\sigma = \sigma_1 \cdots \sigma_6 = (s_1)(s_2)(s_7 s_6 s_5)(s_8 s_7)(s_{11} s_{10} s_9 s_8)(s_{12} s_{11})$, the corresponding permutation is $\varphi(P_{13,6}) = 2\ 3\ 1\ 4\ 8\ 5\ 9\ 12\ 6\ 7\ 13\ 10\ 11$, and one can check that σ is the canonical reduced decomposition of $\varphi(P_{13,6})$.

From the construction of zigzag decomposition, it is clear that conditions (3)–(5) are ensured. That is, the zigzag decomposition of Dyck paths satisfies

$$\begin{gathered} 1 \leq h_1 < h_2 < \cdots < h_k \leq n-1, \\ h_i \geq t_i \qquad (1 \leq i \leq k), \\ t_i \geq t_{i-1} + 1 \qquad (2 \leq i \leq k). \end{gathered}$$

Therefore, we have that $\varphi(P_{13,6}) \in S_{13}(321)$. Moreover, by reversing zigzag decomposition, we can see that such a map φ is a bijection.

Theorem 3. *φ is a bijection between D_n and $S_n(321)$.*

2.2. **Some statistics of $S_n(321)$.** Here we show some applications of the bijection φ which give generating functions for several statistics of 321-avoiding permutations. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ recall that i is called a *fixed point* of π if $\pi_i = i$, and π_i is called a *right-to-left-minimum* of π if there exists no j such that $j > i$ and $\pi_j < \pi_i$. We will use $\mathrm{fp}(\pi)$ and $\mathrm{rlm}(\pi)$, respectively, to denote the number of fixed points and the number of right-to-left-minima of π, and use $\mathrm{lis}_1(\pi)$ to denote the length of the longest subsequence of π with the pattern $23\ldots k1$. The following corollary follows from the definition of φ.

Corollary 4. *Let $\pi \in S_n(321)$ and suppose D is a Dyck path of semilength n such that $\varphi(D) = \pi$, then*

(i) $\mathrm{fp}(\pi) = \#\{$*peaks in D of height* $1\}$;
(ii) $\mathrm{rlm}(\pi) = \#\{$*peaks in D*$\}$;
(iii) $\mathrm{lis}_1(\pi) =$ *height of D;*
(iv) *π avoids the pattern $(3\ \cdots\ (k-1)\ 1\ \bar{k}\ 2)$ iff each udu that begins at height $k-2$ is contained in a uduud;*
(v) *π avoids the pattern $(3\ \cdots\ k\ \bar{1}\ 2)$ iff uududd appears before every mountain of height less than $k-1$.*

Proof. (i) If ud with starting point $(2j, 0)$ is a peak of height 1, then s_j and s_{j+1} will not appear in σ, hence $j+1$ is a fixed point. Conversely, if $\pi(j) = j$ is a fixed point, since $\pi \in S_n(321)$, it is easy to check that neither s_{j-1} nor s_j appears in the canonical reduced decomposition of π, that is, there exists i satisfying $h_i + 2 < t_{i+1}$, therefore there is a peak ud which starts at $(2j-2, 0)$.

(ii) If we label the up steps of D from left to right with the elements of the permutation π (as shown in Figure 1), then for each $\sigma_i = s_{h_i} s_{h_i - 1} \cdots s_{t_i}$, the label of the left arm of its essential cell is exactly $h_i + 1$, (in fact the action of σ_i is to bring the element $h_i + 1$ leftward to the proper position and shift other relevant elements to the positions on the right), hence the labels of the up steps of the peaks of D are the right-to-left minima of π.

(iii) From (ii) we know that any peak in a Dyck path corresponds to a right-to-left minimum, say m, in the permutation. Furthermore, the height of the peak is equal to the number of elements to the left of m that are larger than m. These elements are in increasing order, thus these elements together with m form a subsequence of the pattern $2\ 3\ \cdots\ k\ 1$, which cannot be longer under the assumption that m is the smallest element in the occurrence of the pattern.

Using (iii), the statements (iv) and (v) can be obtained easily. □

Example 5. (see [14]) *From Corollary 4(i), we see that the number of derangements avoiding 321 of length n is the n-th Fine number.*

Theorem 6. *The generating function for* 321*-avoiding permutations with respect to the number of fixed points and the number of right-to-left-minima is given by*

$$A(x,p,q) \quad = \sum_{n\geq 0}\sum_{\pi\in S_n(321)} x^n p^{\mathrm{fp}(\pi)} q^{\mathrm{rlm}(\pi)} = \frac{2}{1+x(1+q-2qp)+\sqrt{(1-x(1+q))^2-4x^2q}}$$

Proof. By Corollary 4, we can express A as

$$A(x,p,q) = \sum_{n\geq 0}\sum_{D\in D_n} x^n p^{\#\{\text{peaks in } D \text{ of height } 1\}} q^{\#\{\text{peaks in } D\}}.$$

Indeed, any nonempty Dyck path D can be written uniquely in one of the following two forms: (1) $D = udD''$ or (2) $D = uD'dD''$, where D' is an arbitrary nonempty Dyck path and D'' is an arbitrary Dyck path. Using these cases, we obtain the following equation for the generating function A:

$$A(x,p,q) = 1 + xpqA(x,p,q) + x(A(x,1,q)-1)A(x,p,q).$$

The rest is easy to check. □

Example 7. (see [14]) *Theorem 6 gives that the generating function for the number of* 321*-avoiding permutations with respect to the number of fixed points is given by*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(321)} x^n p^{\mathrm{fp}(\pi)} = \frac{2}{1+2x(1-p)+\sqrt{1-4x}}.$$

Example 8. *Theorem 6 gives that the generating function for the number of* 321*-avoiding permutations with respect to the number of right-to-left-minima is given by*

$$\sum_{n\geq 0}\sum_{\pi\in S_n(321)} x^n q^{\mathrm{rlm}(\pi)} = \frac{1+x(1-q)-\sqrt{(1-x(1+q))^2-4x^2q}}{2x}.$$

2.3. **Avoiding 321 and another pattern.**

Theorem 9. *For $k > 0$, let $B_k(x,p,q) := \sum_{n\geq 0}\sum_{\pi\in S_n(321,23\ldots(k+1)1)} x^n p^{\mathrm{fp}(\pi)} q^{\mathrm{rlm}(\pi)}$ be the generating function for* 321*-avoiding permutations which avoid $23\ldots(k+1)1$ with respect to the number of fixed points and the number of right-to-left-minima. Then we have the recurrence*

$$B_k(x,p,q) = \frac{1}{1+x(1-pq)-xB_{k-1}(x,1,q)},$$

with $B_0(x,p,q) = 1$. Thus, B_k can be expressed as a continued fraction of the form

$$B_k(x,p,q) = \cfrac{1}{1+x(1-pq)-\cfrac{x}{\cfrac{\ddots}{1+x(1-q)-\cfrac{x}{1+x(1-q)-x}}}},$$

where the fraction has k levels, or in terms of Chebyshev polynomials of the second kind, as

$$B_k(x,p,q) = \frac{U_{k-1}(t) - \sqrt{x}U_{k-2}(t)}{\sqrt{x}\left[U_k(t) - \sqrt{x}(1-q(1-p))U_{k-1}(t) - xq(1-p)U_{k-2}(t)\right]},$$

where $t = \frac{1+x(1-q)}{2\sqrt{x}}$.

Proof. By Corollary 4, permutations in $S_n(321, 23\dots(k+1)1)$ are mapped by φ^{-1} to Dyck paths of height less than $k+1$. Thus, we can express B_k as

$$B_k(x,p,q) = \sum_{n\geq 0} \sum_{D\in D_n \text{ of height}<k+1} x^n p^{\#\{\text{peaks in } D \text{ of height } 1\}} q^{\#\{\text{peaks in } D\}}.$$

For $k>1$, we use again the standard decomposition of Dyck paths, and obtain the equation

$$B_k(x,p,q) = 1 + x(B_{k-1}(x,1,q)-1)B_k(x,p,q) + xpqB_k(x,p,q).$$

For $k=1$, the path can have only peaks of height one, so we get $B_1(x,p,q) = \frac{1}{1-xpq}$. Now, using the above recurrence and Equation (2) we get the desired result. □

Example 10. *Theorem 9 for $p=q=1$ together with Equation (2) gives that the generating function for the number of permutations in $S_n(321, 23\dots(k+1)1)$ is given by (see* [11, 12, 6]*)*

$$\frac{U_k\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_{k+1}\left(\frac{1}{2\sqrt{x}}\right)}.$$

More generally, the generating function for the number of 321-avoiding permutations which avoid $23\dots(k+1)1$ with respect to the number fixed points is given by

$$\frac{U_k\left(\frac{1}{2\sqrt{x}}\right)}{U_k\left(\frac{1}{2\sqrt{x}}\right) - p\sqrt{x}U_{k-1}\left(\frac{1}{2\sqrt{x}}\right) - x(1-p)U_{k-2}\left(\frac{1}{2\sqrt{x}}\right)},$$

and the generating function for the number of 321-avoiding permutations which avoid $23\dots(k+1)1$ with respect to the number of right-to-left-minima is given by

$$\frac{U_{k-1}\left(\frac{1+x(1-q)}{2\sqrt{x}}\right) - \sqrt{x}U_{k-2}\left(\frac{1+x(1-q)}{2\sqrt{x}}\right)}{\sqrt{x}\left[U_k\left(\frac{1+x(1-q)}{2\sqrt{x}}\right) - \sqrt{x}U_{k-1}\left(\frac{1+x(1-q)}{2\sqrt{x}}\right)\right]}.$$

Theorem 11. *The generating function for the number of permutations avoiding both 321 and $34\dots(k-1)1\bar{k}2$ can be expressed as*

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{\ddots\, 1-\cfrac{\ddots}{1-xM(x)}}}},$$

where the continued fraction has $k-4$ levels and $M(x)$ is the generating function for the Motzkin numbers, that is, $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, or in terms of Chebyshev polynomials of the second kind, as

$$\frac{U_{k-5}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}M(x)U_{k-4}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}\left[U_{k-4}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}M(x)U_{k-3}\left(\frac{1}{2\sqrt{x}}\right)\right]}.$$

Proof. Using the standard decomposition of Dyck paths, we obtain the following equation for the generating function $A_k(x) = \sum_{n\geq 0}\sum_{\pi\in S_n(321,34\dots(k-1)1\bar{k}2)} x^n$: $A_k(x) = 1 + xA_{k-1}(x)A_k(x)$, where $k \geq 5$, and $A_4(x) = 1 + xA_4(x) + x^2A_4(x)^2$. Now, using the above recurrences and Equation (2) we get the desired result. □

Theorem 12. *The generating function for the number of permutations avoiding both* 321 *and* $3\dots k\bar{1}2$ *is given by*

$$\frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}.$$

Proof. Using the standard decomposition of Dyck paths, we obtain the following equation for the generating function $D_k(x) = \sum_{n\geq 0}\sum_{\pi\in S_n(321,34\dots k\bar{1}2)} x^n$: $D_k(x) = 1 + xD_{k-1}(x)D_k(x)$, where $k \geq 2$, and $D_1(x) = 1$. Now, using the above recurrence and Equation (2) we get the desired result. □

3. Reduced decompositions for $S_n(231)$ and the Trapezoidal Decomposition

3.1. **A bijection between $S_n(231)$ and D_n.** The recursive construction of $S_n(231)$ can be described as follows.

Theorem 13. *For* $n \geq 2$*, a permutation* π *in* $S_n(231)$ *can be constructed by the following recursive procedure: for* $\pi_1 \in S_i(231)$ *and* $\pi_2 \in S_{n-i-1}(231)$*, and* $0 \leq i \leq n-1$*, set*

$$\pi = \pi_1 \; n \; \widetilde{\pi}_2,$$

where $\widetilde{\pi}_2$ *denotes the sequence obtained from* π_2 *by adding* i *to every entry. Conversely, given any permutation* $\pi \in S_n(231)$*, one can uniquely decompose it into a pair of shorter permutations.*

In this section, we will introduce the *trapezoidal decomposition* of Dyck paths, which is based on another labelling for Dyck paths: the $(x-y)$-labelling.

For a Dyck path P, each cell in the region enclosed by P and the x-axis, whose corner points (i,j), $(i+1,j-1)$, $(i+2,j)$, and $(i+1,j+1)$ is labelled by $(i-j+2)/2$, such a labelling is called an $(x-y)$-*labelling.* Now we can define the *trapezoidal strip* of P:

- If there is no essential cell in P, then the trapezoidal strip is simply the empty set.
- Otherwise, we define the trapezoidal strip of P as the horizontal strip that touches the x-axis and starts at the rightmost essential cell.

For example, the trapezoidal strip of the Dyck path in Figure 2 is the dark gray strip with labels 5,6,7,8,9,10,11,12.

Suppose $P_{n,k}$ is a Dyck path of semilength n that contains k essential cells, we define the *trapezoidal decomposition* of $P_{n,k}$ as follows:

1. If $k = 0$, then the trapezoidal decomposition of $P_{n,0}$ is the empty set.
2. If $k = 1$, then the trapezoidal decomposition is the trapezoidal strip.
3. If $k \geq 2$, then decompose $P_{n,k}$ into $P_{n,k} = Q_1uQ_2d$, where u is the left arm of the rightmost essential cell that touches the x-axis, and d is the last down step of $P_{n,k}$, and Q_1 and Q_2 carry the labels in $P_{n,k}$. If we read the labels of the trapezoidal strip $P_{n,k}$ from left to right, we will get a sequence of the form $\{i, i+1, \dots, j\}$; let $\sigma_k = s_j\, s_{j-1} \cdots s_i$.
4. Repeat the above procedure for Q_1 and Q_2. Suppose the trapezoidal decomposition of Q_1 and Q_2 is σ' and σ'', then the trapezoidal decomposition of $P_{n,k}$ is $\sigma = \sigma'\sigma''\sigma_k$.

From the trapezoidal decomposition we can obtain a permutation $\psi(P_{n,k}) = (12\cdots n)\sigma$ from $P_{n,k}$. For example, for the Dyck path $P_{13,6} = uuudddduduuduuududddduududdd$ shown in Figure 2, in which cells in the same trapezoidal strip are labelled with the same color. From the trapezoidal decomposition of $P_{13,6}$ we get

$$\sigma = \sigma_1 \cdots \sigma_6 = (s_1)(s_2s_1)(s_7s_6)(s_8s_7s_6)(s_{11}s_{10})(s_{12}s_{11}s_{10}s_9s_8s_7s_6s_5),$$

the corresponding permutation is $\psi(P_{13,6}) = (12\cdots n)\sigma = 3\ 2\ 1\ 4\ 13\ 5\ 9\ 8\ 6\ 7\ 12\ 10\ 11$.

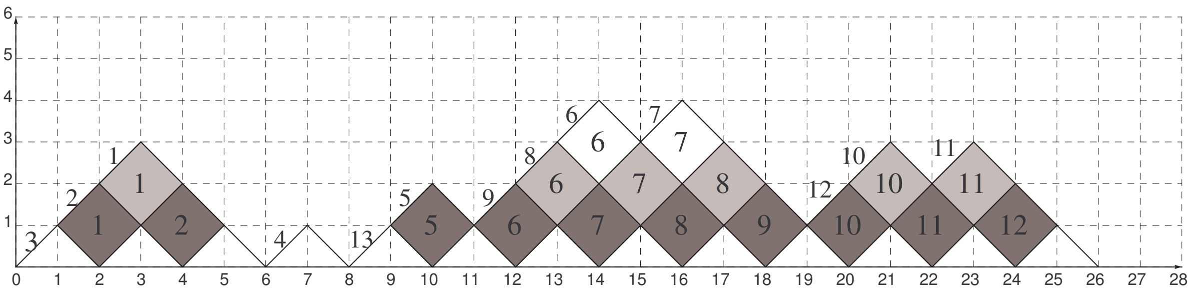

FIGURE 2. The $(x-y)$-labelling and trapezoidal decomposition for Dyck path.

From the construction of trapezoidal decomposition, we have the following assertion:

Theorem 14. *Let P be a Dyck path of semilength n, and $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ the trapezoidal decomposition of P. Then we have $\psi(P) = (12\cdots n)\sigma \in S_n(231)$.*

Proof. We use induction on n. For $n = 1$, it is clear that $\psi(P) = 1$. Assume that the statement is true for Dyck paths of semilength less than n. If P ends with a peak of height 1, then there exists a unique Dyck path $Q \in D_{n-1}$ such that $P = Qdu$. Observing that s_{n-1} does not appear in $\sigma_1\sigma_2\cdots\sigma_k$, we have

$$\psi(Qud) = (12\cdots n)\sigma = \psi(Q)n.$$

By induction, $\psi(Q) \in S_{n-1}(231)$. It follows that $\pi = \psi(Q)n \in S_n(231)$. If P does not end with a peak of height 1, we may have the decomposition $P = Q_1uQ_2d$, where Q_1 is a Dyck path of semilength i. From the construction of the trapezoidal decomposition of P, we see that $\sigma_k = s_{n-1}\cdots s_{i+1}$. Let $\sigma_1\cdots\sigma_{k-j}$ denote the trapezoidal decomposition corresponding to Q_1. Let s_r be any factor of $\sigma_1\sigma_2\cdots\sigma_{k-j}$, then we have $r \le i-1$. Thus, they act only on the first i elements. Let $\sigma_{k-j+1}\cdots\sigma_{k-1}$ denote the trapezoidal decomposition corresponding to Q_2, and s_r be any factor in $\sigma_{k-j+1}\sigma_{k-j+2}\cdots\sigma_{k-1}$, then we have $i < r \le n-2$. Therefore, they act only on the elements whose positions are bigger than i and smaller than $n-1$. It follows that

$$\begin{aligned}
\psi(P) &= (1\ 2\ \cdots\ n)\sigma_1\sigma_2\cdots\sigma_k \\
&= (1\ 2\ \cdots\ n)(\sigma_1\cdots\sigma_{k-j})(\sigma_{k-j+1}\cdots\sigma_{k-1})(s_{n-1}\cdots s_{i+1}) \\
&= ((1\cdots i)\sigma_1\cdots\sigma_{k-j})\ (((i+1\cdots n-1)\sigma_{k-j+1}\cdots\sigma_{k-1})\ n)(s_{n-1}\cdots s_{i+1}) \\
&= \psi(Q_1)\ n\ \widetilde{\psi(Q_2)},
\end{aligned}$$

From the induction hypothesis, we know that $\psi(Q_1)$ and $\psi(Q_2)$ are both in $S_n(231)$. From the above equation and Theorem 13 we can see that $\pi = \psi(P) \in S_n(231)$, as required. $\square$

We should point out that for $\pi \in S_n(231)$, the decomposition in Theorem 14 is indeed a canonical reduced decomposition since it satisfies (1). So we have the following theorem.

Theorem 15. *Let $\sigma = \sigma_1 \cdots \sigma_k$ be the canonical reduced decomposition of $\pi \in S_n$, where $\sigma_i = s_{h_i} s_{h_i - 1} \cdots s_{t_i}$ for $1 \le i \le k$, then $\pi \in S_n(231)$ if and only if*

$$\begin{gathered} 1 \le h_1 < h_2 < \cdots < h_k \le n-1, \\ h_i \ge t_i \qquad (1 \le i \le k), \\ t_i \le t_{i-1}, \quad \textit{or} \quad t_i \ge h_{i-j} + 2, \qquad (2 \le i \le k, 1 \le j \le i-1). \end{gathered} \tag{6}$$

But we prefer the recursive structure of π, which is easier and direct.

3.2. **Some statistics about $S_n(231)$.** Here we show some applications of the bijection ψ to give generating functions for several statistics of 231-avoiding permutations. For any permutation π, the number of inversions, descents, rises and the length of the longest decreasing subsequence in π are denoted by $\mathrm{inv}(\pi)$, $\mathrm{des}(\pi)$, $\mathrm{rise}(\pi)$, and $\mathrm{lds}(\pi)$, respectively. The following corollary follows from the definition of ψ.

Corollary 16. *Let $\pi \in S_n(231)$, and suppose D is a Dyck path of semilength n such that $\psi(D) = \pi$. Then*

(i) $\mathrm{inv}(\pi) = \#\{$*cells in D*$\}$.
(ii) $\mathrm{rlm}(\pi) = \#\{$*peaks in D*$\}$.
(iii) $\mathrm{rise}(\pi) = \#\{$*valleys in D*$\} = \#\{$*peaks in D*$\} - 1$.
(iv) $\mathrm{des}(\pi) = \#\{$*enssential cells in D*$\}$.
(v) $\mathrm{lds}(\pi) =$ *height of D*.
(vi) *π avoids the pattern $(k \ \cdots \ 3 \ \bar{1} \ 2)$ iff uududd appears before every mountain of height less than $k-1$.*
(vii) *π avoids the pattern $(k \ \bar{1} \ (k-1) \ \cdots \ 2)$ iff each udu that begins at height $k-2$ is contained in a uduu.*
(viii) *π avoids the pattern $((k-1) \ \cdots \ 1 \ \bar{k})$ iff the height of the last mountain is less than $k-1$.*

Proof. (i) It can be obtained from Lemma 1 immediately.

(ii) This property is the same as Corollary 4 (ii). That is, if we label the up steps of D from left to right with the elements of the permutation π, then for each $\sigma_i = s_{h_i} s_{h_i-1} \cdots s_{t_i}$, the label of the left arm of its essential cell is exactly $h_i + 1$, hence the labels of the up steps of the peaks of D are the right-to-left minima of π.

(iii) From (ii), each right-to-left minimum and the element next to it are both rises in π, except for the last element, and the number of valleys is equal to the number of peaks plus one.

(iv) The action of σ_i is to bring the element $h_i + 1$ leftward, which will create a new descent since π is 231-avoiding.

(v) From (ii), the labels of the consecutive up steps appear in decreasing order since π is 231-avoiding, thus the height of the mountain is the length of the decreasing subsequences.

using (v), the properties (vi), (vii) and (viii) can be obtained easily. □

3.3. **Avoiding 231 and another pattern.**

Theorem 17. *Let $A(x; a, b, c) = \sum_{n \ge 0} \sum_{\pi \in S_n(231)} x^n a^{\mathrm{inv}(\pi)} b^{\mathrm{rlm}(\pi)} c^{\mathrm{rise}(\pi)}$. Then the generating function $A(x; a, b, c)$ is given by*

$$1 - \frac{1}{c} + \cfrac{1/c}{1 + x(1-bc) - \cfrac{x}{1 + ax(1-bc) - \cfrac{ax}{1 + a^2x(1-bc) - \cfrac{a^2x}{\ddots}}}}.$$

Proof. Let $B(x;a,b) = \sum_{n\geq 0}\sum_{\pi\in S_n(231)} x^n a^{\mathrm{inv}(\pi)+n} b^{\mathrm{rlm}(\pi)}$. By Lemma 16, we can express B as

$$B(x;a,b) = \sum_{n\geq 0}\sum_{D\in D_{2n}} x^n a^{n+\#\{\text{cells in } D\}} b^{\#\{\text{peaks in } D\}}.$$

Using the standard decomposition of Dyck paths, we obtain the following

$$B(x;a,b) = 1 + xabB(x;a,b) + ax(B(ax;a,b)-1)B(x;a,b). \tag{7}$$

Indeed, any nonempty Dyck path D can be written uniquely in one of the following two forms: (1) $D = udD''$ or (2) $D = uD'dD''$, where D' is an arbitrary nonempty Dyck path and D'' is an arbitrary Dyck path. Let us write an equation for $B(x;a,b)$. The first and second of the block decompositions above contributes as $xabB(x;a,b)$ and $ax(B(ax;a,b)-1)B(x;a,b)$, respectively. Therefore,

$$B(x;a,b) = \cfrac{1}{1+ax(1-b) - \cfrac{ax}{1+a^2x(1-b) - \cfrac{a^2x}{\ddots}}}.$$

Hence, by using the fact that $A(x;a,b,c) = 1 + \frac{1}{c}(B(x/a;a,bc)-1)$, we get the desired expression. □

For example, Theorem 17 gives

$$\sum_{n\geq 0}\sum_{\pi\in S_n(231)} \mathrm{sign}(\pi) b^{\mathrm{rlm}(\pi)} x^n = \frac{1-2x-x^2(1-b)^2 - \sqrt{(1+x^2(1-b^2))^2 + 4bx^2(1-x^2(1-b)^2)}}{2x(x(b-1)-1},$$

where $\mathrm{sign}(\pi) = (-1)^{\mathrm{inv}(\pi)}$. Thus, the generating function $\sum_{n\geq 0}\sum_{\pi\in S_n(231)} \mathrm{sign}(\pi)x^n$ is given by

$$\frac{-1+2x+\sqrt{1+4x^2}}{2x} = 1 + xC(-x^2),$$

where $C(x)$ is the generating function for the Catalan numbers (see [8] for the case $S_n(132)$).

Theorem 18. *For $k > 0$, let $A_k(x;a,b,c) := \sum_{n\geq 0}\sum_{\pi\in S_n(231,k\ldots 21)} x^n a^{\mathrm{inv}(\pi)} b^{\mathrm{rlm}(\pi)} c^{\mathrm{rise}(\pi)}$ be the generating function for permutations avoiding both 231 and $k\ldots 21$ with respect to the number of rises, right-to-left-minima, and the number of inversions. Then*

$$A_k(x;a,b,c) = 1 - \frac{1}{c} + \frac{1}{c}(B_k(x/a;a,bc)-1),$$

where

$$B_k(x;a,b) = \sum_{n\geq 0}\sum_{\pi\in S_n(231,k\ldots 21)} x^n a^{n+\mathrm{inv}(\pi)} b^{\mathrm{rlm}(\pi)} = \frac{1}{1+ax(1-b) - axB_{k-1}(ax;a,b)},$$

with $B_1(x;a,b) = 1$. Thus, A_k can be expressed as

$$A_k(x;a,b,c) = 1 - \frac{1}{c} + \cfrac{1/c}{1+x(1-bc) - \cfrac{x}{1+ax(1-bc) - \cfrac{ax}{\ddots \; 1+a^2x(1-bc) - \cfrac{}{1+a^{k-2}x(1-bc) - a^{k-2}x}}}}.$$

Proof. The condition that π avoids $k\ldots 21$ is equivalent to the condition $\mathrm{lds}(\pi) \leq k$. By Lemma 16, permutations in $S_n(231)$ satisfying this condition are mapped to Dyck paths of height less than k. Thus, we can express A_k as

$$A_k(x;a,b,c) = 1 + \sum_{n\geq 1}\sum_{D\in D_n \text{ of height}<k} x^n a^{\#\{\text{cells in } D\}} b^{\#\{\text{peaks in } D\}} c^{\#\{\text{peaks in } D\}-1}.$$

For $k>2$, we use again the standard decomposition of Dyck paths $D=uD'dD''$. First, the height of udD'' is the same as the height of D''. Second, if the height of $uD'dD''$ is less than k, the height of D' has to be less than $k-1$. Therefore, we obtain the equation

$$B_k(x;a,b)=1+abxB_k(x;a,b)+ax(B_{k-1}(ax;a,b)-1)B_k(x;a,b).$$

For $k=2$, the path can have only peaks at height one, so we get $B_1(x;a,b)=1$. Now, using the above recurrence we get the desired result. $\square$

By applying Theorem 18 for $a=c=1$, $b=q$, and (2) we get the following result.

Corollary 19. *For $k>0$, let $A_k(x;q)=\sum_{n\geq 0}\sum_{\pi\in S_n(231,k(k-1)\ldots 1)}x^nq^{\mathrm{rlm}(\pi)}$ be the generating function for 231-avoiding permutations which avoid $k(k-1)\ldots 1$ with respect to the number of right-to-left-minima. Then we have the recurrence*

$$A_k(x;q)=\frac{1}{1+x(1-q)-xA_{k-1}(x;q)},$$

with $A_1(x;q)=1$. Thus, A_k can be expressed as a continued fraction of the form

$$A_k(x,q)=\cfrac{1}{1+x(1-q)-\cfrac{x}{\cfrac{\ddots}{1+x(1-q)-x}}},$$

where the fraction has $k-2$ levels, or in terms of Chebyshev polynomials of the second kind, as

$$A_k(x;q)=\frac{U_{k-2}(t)-\sqrt{x}U_{k-3}(t)}{\sqrt{x}\left[U_{k-1}(t)-\sqrt{x}U_{k-2}(t)\right]},$$

where $t=\frac{1+x(1-q)}{2\sqrt{x}}$.

For example, Corollary 19 for $q=0$ gives Theorem 3.1(iii) in [5] and Theorem 6 in [6], that is, the generating function for the number of permutations which avoid both 132 and $12\ldots k$ is given by $R_k(x)=\frac{2tU_{k-1}(t)}{U_k(t)}$, where $t=\frac{1}{2\sqrt{x}}$.

Another application of Theorem 17 with $a=-1$ and $b=c=1$ gives the following corollary (see [8]).

Corollary 20. *The generating function $A_k(x)=\sum_{n\geq 0}\sum_{\pi\in S_n(231,k\ldots 21)}\mathrm{sign}(\pi)x^n$ can be expressed as*

$$\cfrac{1}{1-\cfrac{x}{1+\cfrac{x}{1-\cfrac{x}{\ddots}}}},$$

where the fraction has $k-1$ levels, or in terms of Chebyshev polynomials of the second kind as

$$A_{2k+1}(x)=1+xR_k(-x^2)\ \text{and}\ A_{2k}(x)=\frac{R_k(-x^2)}{1-xR_k(-x^2)}.$$

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