ZERO-SUM PROBLEMS IN FINITE ABELIAN GROUPS: A SURVEY

WEIDONG GAO AND ALFRED GEROLDINGER

Abstract. We give an overview of zero-sum theory in finite abelian groups, a subfield of additive group theory and combinatorial number theory. In doing so we concentrate on the algebraic part of the theory and on the development since the appearance of the survey article by Y. Caro in 1996.

1. Introduction

Let G be an additive finite abelian group. In combinatorial number theory a finite sequence $S = (g_1, \ldots, g_l) = g_1 \cdot \ldots \cdot g_l$ of elements of G, where the repetition of elements is allowed and their order is disregarded, is simply called a sequence over G, and S is called a zero-sum sequence if $g_1 + \ldots + g_l = 0$. A typical direct zero-sum problem studies conditions which ensure that given sequences have non-empty zero-sum subsequences with prescribed properties. The associated inverse zero-sum problem studies the structure of extremal sequences which have no such zero-sum subsequences.

These investigations were initiated by a result of P. Erdős, A. Ginzburg and A. Ziv, who proved that $2n-1$ is the smallest integer $l \in \mathbb{N}$ such that every sequence S over a cyclic group of order n has a zero-sum subsequence of length n (see [47]). Some years later, P.C. Baayen, P. Erdős and H. Davenport (see [137], [45] and [142]) posed the problem to determine the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq l$ has a zero-sum subsequence. In subsequent literature that integer l has been called the Davenport constant of G. It is denoted by $\mathsf{D}(G)$, and its precise value - in terms of the group invariants of G - is still unknown in general.

These problems were the starting points for much research, as it turned out that questions of this type occur naturally in various branches of combinatorics, number theory and geometry. Conversely, zero-sum problems have greatly influenced the development of various subfields of these areas (among others, zero-sum Ramsey theory was initiated by the works of A. Bialostocki and P. Dierker). So there are intrinsic connections with graph theory, Ramsey theory and geometry (see [118], [4], [12, 13] for some classical papers and [11], [10], [104], [14], [107], [40], [122] for some recent papers). The following observation goes back to H. Davenport: If R is the ring of integers of some algebraic number field with ideal class group (isomorphic to) G, then $\mathsf{D}(G)$ is the maximal number of prime ideals (counted with or without multiplicity) which occur in the prime ideal decomposition of aR for irreducible elements $a \in R$. Indeed, in the theory of non-unique factorizations it has turned out that the monoid of all zero-sum sequences over G closely reflects the arithmetic of a Krull monoid which has class group G and every class contains a prime (see [96, Corollary 3.4.12]). On the other hand, it was factorization theory which promoted the investigation of inverse zero-sum problems, which appear naturally in that area. Apart from all that, zero-sum problems occur in various types of number theoretical topics (as Carmicheal numbers [1], Artin's conjecture on additive forms [19] or permutation matrices [134]).

Zero-sum problems are tackled with a huge variety of methods. First of all we mention methods from additive group theory including all types of addition theorems (see [135, 136], [141], [96], [111], [106, 108],

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[132], [123], [9]). Furthermore, group algebras ([74]), results from the covering area ([163, 131], [80]), from linear algebra ([32, 31]) and polynomial methods ([2, 3]) play crucial roles. Moreover, in the meantime zero-sum theory has already developed its own methods and a wealth of results which promote its further development.

The first survey article on zero-sum theory, written by Y. Caro, appeared ten years ago in 1996 (see [23] and [24]). The aim of the present article is to sketch the development in the last decade and to give an overview over the present state of the area under the following two restrictions. First, we do not outline the relationships to other areas, as graph theory, Ramsey theory or the theory of non-unique factorizations, but we restrict to what is sometimes called the algebraic part of zero-sum theory. Second, although since the 1960s zero-sum problems were studied also in the setting of non-abelian groups (see [36], [145, 149, 146, 147], [169], [63], [39], [171]) but we restrict to the case of abelian groups. Since Y. Caro's article has an extended bibliography on the literature until 1994, we also refer to his bibliography and concentrate ourselves on papers having appeared since that time. In Section 2 we fix our notations and terminology, and we give the definitions of the key invariants. Then in the subsequent sections we present the state of knowledge on these invariants and on the associated inverse problems.

Throughout this article, let G be an additive finite abelian group and let $G^\bullet = G \setminus \{0\}$

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of all prime numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$, and for $c \in \mathbb{N}$ let $\mathbb{N}_{\ge c} = \mathbb{N} \setminus [1, c-1]$. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to x, and by $\lceil x \rceil$ the smallest integer that is greater than or equal to x.

Throughout, all abelian groups will be written additively. For $n \in \mathbb{N}$, let C_n denote a cyclic group with n elements, and let $nG = \{ng \mid g \in G\}$. By the Fundamental Theorem of Finite Abelian Groups we have

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r} \cong C_{q_1} \oplus \ldots \oplus C_{q_s} \,,$$

where $r = \mathsf{r}(G) \in \mathbb{N}_0$ is the *rank* of G, $s = \mathsf{r}^*(G) \in \mathbb{N}_0$ is the *total rank* of G, $n_1, \ldots, n_r \in \mathbb{N}$ are integers with $1 < n_1 \mid \ldots \mid n_r$ and $q_1, \ldots q_s$ are prime powers. Moreover, $n_1, \ldots, n_r, q_1, \ldots, q_s$ are uniquely determined by G, and we set

$$\mathsf{d}^*(G) = \sum_{i=1}^{r} (n_i - 1) \quad \text{and} \quad \mathsf{k}^*(G) = \sum_{i=1}^{s} \frac{q_i - 1}{q_i} \,.$$

Clearly, $n_r = \exp(G)$ is the *exponent* of G, and if $|G| = 1$, then $\mathsf{r}(G) = \mathsf{d}^*(G) = \mathsf{k}^*(G) = 0$ and $\exp(G) = 1$.

Let $s \in \mathbb{N}$. An s-tuple $(e_1, \ldots, e_s)$ of elements of G is said to be *independent* if $e_i \ne 0$ for all $i \in [1, s]$ and, for every s-tuple $(m_1, \ldots, m_s) \in \mathbb{Z}^s$,

$$\sum_{i=1}^{s} m_i e_i = 0 \qquad \text{implies} \qquad m_1 e_1 = \ldots = m_s e_s = 0 \,.$$

An s-tuple $(e_1, \ldots, e_s)$ of elements of G is called a *basis* if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_s \rangle$.

We write sequences multiplicatively and consider them as elements of the free abelian monoid over G, a point of view which was put forward by the requirements of the theory of non-unique factorizations. Thus we have at our disposal all notions from elementary divisibility theory which provides a suitable framework when dealing with subsequences of given sequences, and we may apply algebraic concepts in a natural way.

Let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,, \quad \text{with} \quad \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all} \quad g \in G\,.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S *contains* g, if $\mathsf{v}_g(S) > 0$. S is called *squarefree* (in $\mathcal{F}(G)$) if $\mathsf{v}_g(S) \le 1$ for all $g \in G$. The unit element $1 \in \mathcal{F}(G)$ is called the *empty sequence*. A sequence S_1 is called a *subsequence* of S if $S_1 \,|\, S$ in $\mathcal{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$), and it is called a *proper subsequence* of S if it is a subsequence with $1 \ne S_1 \ne S$. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S \;=\; g_1 \cdot \ldots \cdot g_l \;=\; \prod_{g \in G} g^{\mathsf{v}_g(S)} \;\in \mathcal{F}(G)\,,$$

we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S\,,$$

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|] \qquad \text{the } \textit{maximum of the multiplicities} \text{ of } S\,,$$

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)} \in \mathbb{Q}_{\ge 0} \qquad \text{the } \textit{cross number} \text{ of } S\,,$$

$$\operatorname{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \qquad \text{the } \textit{support} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S\,,$$

$$\Sigma_k(S) = \Big\{ \sum_{i \in I} g_i \;\Big|\; I \subset [1,l] \text{ with } |I| = k \Big\} \text{ the } \textit{set of } k\textit{-term subsums} \text{ of } S\,, \text{ for all } k \in \mathbb{N}\,,$$

$$\Sigma_{\le k}(S) = \bigcup_{j \in [1,k]} \Sigma_j(S)\,, \qquad \Sigma_{\ge k}(S) = \bigcup_{j \ge k} \Sigma_j(S)\,,$$

and

$$\Sigma(S) = \Sigma_{\ge 1}(S) \text{ the } \textit{set of (all) subsums} \text{ of } S\,.$$

The sequence S is called

- *zero-sumfree* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a zero-sum sequence and every proper subsequence is zero-sumfree,
- a *short zero-sum sequence* if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

We denote by $\mathcal{B}(G)$ the set of all zero-sum sequences and by $\mathcal{A}(G)$ the set of all minimal zero-sum sequences. Then $\mathcal{B}(G) \subset \mathcal{F}(G)$ is a submonoid (also called the block monoid over G); it is a Krull monoid and $\mathcal{A}(G)$ is the set of atoms of $\mathcal{B}(G)$ (see [96, Proposition 2.5.6]). For any map of abelian groups $\varphi\colon G \to G'$, there exists a unique homomorphism $\overline{\varphi}\colon \mathcal{F}(G) \to \mathcal{F}(G')$ with $\overline{\varphi} \,|\, G = \varphi$. Usually we simply write φ instead of $\overline{\varphi}$. Explicitly, $\varphi\colon \mathcal{F}(G) \to \mathcal{F}(G')$ is given by $\varphi(g_1 \cdot \ldots \cdot g_l) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ for all $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. . If $S \in \mathcal{F}(G)$, then $|\varphi(S)| = |S|$ and $\operatorname{supp}\big(\varphi(S)\big) = \varphi\big(\operatorname{supp}(S)\big)$. If $\varphi\colon G \to G'$ is even a homomorphism, then $\sigma\big(\varphi(S)\big) = \varphi\big(\sigma(S)\big)$, $\Sigma\big(\varphi(S)\big) = \varphi\big(\Sigma(S)\big)$ and $\varphi\big(\mathcal{B}(G)\big) \subset \mathcal{B}(G')$. In particular, we use the inversion $(g \mapsto -g)$ and the translation $(g \mapsto g_0 + g)$, and for $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ we set

$$-S = (-g_1) \cdot \ldots \cdot (-g_l) \quad \text{and} \quad g_0 + S = (g_0 + g_1) \cdot \ldots \cdot (g_0 + g_l) \in \mathcal{F}(G)\,.$$

If $g \in G$ is a non-zero element and

$$S = (n_1 g) \cdot \ldots \cdot (n_l g), \quad \text{where } l \in \mathbb{N}_0 \quad \text{and} \quad n_1, \ldots, n_l \in [1, \mathrm{ord}(g)]\,,$$

then

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{\mathrm{ord}(g)}$$

is called the *g-norm* of S. If S is a zero-sum sequence for which $\{0\} \ne \langle \mathrm{supp}(S) \rangle \subset G$ is a finite cyclic group, then

$$\mathrm{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle \mathrm{supp}(S) \rangle = \langle g \rangle\} \in \mathbb{N}_0$$

is called the *index of* S. We set $\mathrm{ind}(1) = 0$, and if $\mathrm{supp}(S) = \{0\}$, then we set $\mathrm{ind}(S) = 1$.

Next we give the definition of the zero-sum invariants which we are going to discuss in the subsequent sections. We concentrate on invariants dealing with general sequences, as introduced in Definition 2.1. However, by an often used technique, problems on general sequences are reduced to problems on squarefree sequences, and thus we briefly deal also with invariants on squarefree sequences (or in other words, with sets), as introduced in Definition 2.2.

Definition 2.1. Let $\exp(G) = n$ and $k, m \in \mathbb{N}$ with $k \nmid \exp(G)$. We denote by

- $\mathsf{D}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a non-empty zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant* of G.
- $\mathsf{d}(G)$ the maximal length of a zero-sumfree sequence over G.
- $\eta(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a short zero-sum subsequence.
- $\mathsf{s}_{mn}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a zero-sum subsequence T of length $|T| = mn$. In particular, we set $\mathsf{s}(G) = \mathsf{s}_n(G)$.
- $\mathsf{s}_{n\mathbb{N}}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a non-empty zero-sum subsequence T of length $|T| \equiv 0 \mod n$.
- $\mathsf{E}_k(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a zero-sum subsequence T with $k \nmid |T|$.
- $\nu(G)$ the smallest integer $l \in \mathbb{N}_0$ with the following property:

  For every zero-sumfree sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ there exist a subgroup $H \subset G$ and an element $a \in G \setminus H$ such that $G^\bullet \setminus \Sigma(S) \subset a + H$.

A simple argument (see [96, Section 5.1] for details) shows that

$$\mathsf{d}(G) = \max\bigl\{|S| \bigm| S \in \mathcal{F}(G)\,,\ \Sigma(S) = G^\bullet\bigr\} \quad \text{and} \quad 1 + \mathsf{d}(G) = \mathsf{D}(G) = \max\bigl\{|S| \bigm| S \in \mathcal{A}(G)\bigr\}\,.$$

Definition 2.2. We denote by

- $\mathsf{Ol}(G)$ the smallest integer $l \in \mathbb{N}$ such that every squarefree sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a non-empty zero-sum subsequence. The invariant $\mathsf{Ol}(G)$ is called the *Olson constant* of G.
- $\mathsf{ol}(G)$ the maximal length of a squarefree zero-sumfree sequence $S \in \mathcal{F}(G)$.
- $\mathsf{cr}(G)$ the smallest integer $l \in \mathbb{N}$ such that every squarefree sequence $S \in \mathcal{F}(G^\bullet)$ of length $|S| \ge l$ satisfies $\Sigma(S) = G$. The invariant $\mathsf{cr}(G)$ is called the *critical number* of G.
- $\mathsf{g}(G)$ the smallest integer $l \in \mathbb{N}$ such that every squarefree sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a zero-sum subsequence T of length $|T| = \exp(G)$.

We use the convention that $\min(\emptyset) = \sup(\emptyset) = 0$. For a subset $G_0 \subset G$ and some integer $l \in \mathbb{N}$, R.B. Eggleton and P. Erdős (see [41]) introduced the f-invariant

$$\mathsf{f}(G_0, l) = \min\bigl\{|\Sigma(S)| \bigm| S \in \mathcal{F}(G_0),\ S \text{ squarefree and zero-sumfree},\ |S| = l\bigr\}\,.$$

The basic relationships between these invariants are summarized in Lemma 10.1.

3. On the Davenport Constant

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$, $r = \mathsf r(G)$ and let $(e_1, \ldots, e_r)$ be a basis of G with $\operatorname{ord}(e_i) = n_i$ for all $i \in [1, r]$. Then the sequence

$$S \;=\; \prod_{i=1}^{r} e_i^{n_i-1} \in \mathcal F(G)$$

is zero-sumfree whence we have the crucial inequality

$$\mathsf d(G) \ge \mathsf d^*(G)\,.$$

In the 1960s, D. Kruyswijk and J.E. Olson proved independently the following result (see [5, 44, 142, 143] and [96, Theorems 5.5.9 and 5.8.3]).

Theorem 3.1. *If G is a p-group or $\mathsf r(G) \le 2$, then $\mathsf d(G) = \mathsf d^*(G)$.*

We present two types of results implying that $\mathsf d(G) = \mathsf d^*(G)$. The first one is due to P. van Emde Boas et. al. (see [44, Theorems 3.9, 4.2], where more results of this flavor may be found) and the second is due to S.T. Chapman et. al. (see [25], and also the various conjectures in that paper).

Theorem 3.2. *Let $G = C_{2n_1} \oplus C_{2n_2} \oplus C_{2n_3}$ and $H = C_{n_1} \oplus C_{n_2} \oplus C_{n_3}$ with $1 \le n_1 \mid n_2 \mid n_3$. If $\nu(H) = \mathsf d^*(H) - 1$, then $\mathsf d(G) = \mathsf d^*(G)$.*

Theorem 3.3. *Let $G = H \oplus C_{km}$ where $k, m \in \mathbb N$ and $H \subset G$ is a subgroup with $\exp(H)|m$. If $\mathsf d(H \oplus C_m) = \mathsf d(H) + m - 1$ and $\eta(H \oplus C_m) \le \mathsf d(H) + 2m$, then $\mathsf d(G) = \mathsf d(H) + km - 1$. In particular (use Theorem 3.1 and* [96, Proposition 5.7.7]*), if m is a prime power and $\mathsf d(H) < m$, then $\mathsf d(G) = \mathsf d^*(G)$.*

These and similar results give rise to long lists of explicit groups satisfying $\mathsf d(G) = \mathsf d^*(G)$ (see [6], [44], [46], [35], [25]). The first example of a group G with $\mathsf d(G) > \mathsf d^*(G)$ is due to P.C. Baayen. In [44, Theorem 8.1] it is shown that

$$\mathsf d(G) > \mathsf d^*(G) \quad \text{for} \quad G = C_2^{4k} \oplus C_{4k+2} \quad \text{with} \quad k \in \mathbb N\,,$$

and more examples are given in [46]. Let $H \subset G$ be a subgroup. Then $\mathsf d(H) + \mathsf d(G/H) \le \mathsf d(G)$, and if G is as above, $I \subset [1, r]$ and

$$H = \bigoplus_{i \in I} C_{n_i}\,, \quad \text{then} \quad \mathsf d(H) > \mathsf d^*(H) \quad \text{implies} \quad \mathsf d(G) > \mathsf d^*(G)$$

(see [96, Proposition 5.1.11]). This shows that the interesting groups with $\mathsf d(G) > \mathsf d^*(G)$ are those with small rank. A. Geroldinger and R. Schneider showed that there are infinitely many G with $\mathsf r(G) = 4$ such that $\mathsf d(G) > \mathsf d^*(G)$. The following result may be found in [98] and [77, Theorem 3.3].

Theorem 3.4. *We have $\mathsf d(G) > \mathsf d^*(G)$ in each of the following cases:*

1. *$G = C_m \oplus C_n^2 \oplus C_{2n}$ where $m, n \in \mathbb N_{\ge 3}$ are odd and $m \mid n$.*
2. *$G = C_2^i \oplus C_{2n}^{5-i}$ where $n \in \mathbb N_{\ge 3}$ is odd and $i \in [2, 4]$.*

Let $G = C_2^r \oplus C_n$ where $r \in \mathbb N$ and $n \in \mathbb N_{\ge 3}$ is odd. Then $\mathsf d(G) = \mathsf d^*(G)$ if and only if $r \le 4$ (see [98, Corollary 2]). For some small $r \ge 5$ and $n \ge 3$ the precise value of $\mathsf d(G)$ was recently determined in [49]. The growth of $\mathsf d(G) - \mathsf d^*(G)$ is studied in [139].

We make the following conjecture.

Conjecture 3.5. *If $G = C_n^r$, where $n, r \in \mathbb N_{\ge 3}$, or $\mathsf r(G) = 3$, then $\mathsf d(G) = \mathsf d^*(G)$.*

For groups of rank three Conjecture 3.5 goes back to P. van Emde Boas (see [46]) and is supported by [69]. For groups of the form $G = C_n^r$ it is supported by [80, Theorem 6.6].

The next result provides upper bounds on $\mathsf{D}(G)$. The first one is due to P. van Emde Boas and D. Kruyswijk ([46, Theorem 7.1]) and is sharp for cyclic groups (for other approaches and related bounds see [8], [140]). The second bound is sharp for groups of rank 2 and with $H = pG$ for some prime divisor p of $\exp(G)$ (see [96, Theorem 5.5.5 and Proposition 5.7.11]).

Theorem 3.6. *Let* $\exp(G) = n \ge 2$ *and* $H \subset G$ *be a subgroup.*

1. $\mathsf{d}(G) \le (n-1) + n \log \frac{|G|}{n}$.
2. $\mathsf{d}(G) \le \mathsf{d}(H) \exp(G/H) + \max\big\{\mathsf{d}(G/H), \eta(G/H) - \exp(G/H) - 1\big\}$.

We end this section with a conjecture supported by [96, Theorem 6.2.8].

Conjecture 3.7. $\mathsf{D}(G) \le \mathsf{d}^*(G) + \mathsf{r}(G)$.

4. On the structure of long zero-sumfree sequences

Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence of length $|S| = \mathsf{d}(G)$. According to general philosophy in inverse additive number theory (see [141], [53, 54]), S should have some structure. Obviously, if G is cyclic of order $n \ge 2$, then $S = g^{n-1}$ for some $g \in \operatorname{supp}(S)$ with $\operatorname{ord}(g) = n$, and if S is an elementary 2-group of rank r, then $S = e_1 \cdot \ldots \cdot e_r$ for some basis $(e_1, \ldots, e_r)$ of G. Apart from these trivial cases very little is known up to now. The most modest questions one could ask are the following:

1. What is the order of elements in $\operatorname{supp}(S)$?
2. What is the multiplicity of elements in $\operatorname{supp}(S)$? What is a reasonable lower bound for $\mathsf{h}(S)$?
3. How large is $\operatorname{supp}(S)$?

Crucial in all investigations of zero-sumfree sequences is the following inequality of Moser-Scherk (see [96, Theorem 5.3.1]): Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence.

$$\text{If} \quad S = S_1 S_2\,, \quad \text{then} \quad |\Sigma(S)| \ge |\Sigma(S_1)| + |\Sigma(S_2)|\,.$$

By M. Freeze and W.W. Smith ([52, Theorem 2.5], [96, Proposition 5.3.5]) this implies that

$$|\Sigma(S)| \ge 2|S| - \mathsf{h}(S) \ge |S| + |\operatorname{supp}(S)| - 1\,.$$

We start with the following conjecture.

Conjecture 4.1. *Every zero-sumfree sequence* $S \in \mathcal{F}(G)$ *of length* $|S| = \mathsf{d}(G)$ *has some element* $g \in \operatorname{supp}(S)$ *with* $\operatorname{ord}(g) = \exp(G)$.

The conjecture is true for cyclic groups, p-groups (see [96, Corollary 5.1.13]), groups of the form $G = C_n \oplus C_n$ (see below) and for $G = C_2 \oplus C_{2n}$ (see [78]). As concerns the second question, the philosophy is that in groups where the exponent is large in comparison with the rank, $\mathsf{h}(S)$ should be large.

For cyclic groups, there are the following results going back to J.D. Bovey, P. Erdős, I. Niven, W. Gao, A. Geroldinger and Y. ould Hamidoune (see [17], [76], [97] and [96, Theorem 5.4.5]).

Theorem 4.2. *Let* G *be cyclic of order* $n \ge 3$, *and let* $S \in \mathcal{F}(G)$ *be a zero-sumfree sequence of length*

$$|S| \ge \frac{n+1}{2}\,.$$

1. *For all* $g \in \operatorname{supp}(S)$ *we have* $\operatorname{ord}(g) \ge 3$.
2. *There exists some* $g \in \operatorname{supp}(S)$ *with* $\mathsf v_g(S) \ge 2|S| - n + 1$.
3. *There exists some* $g \in \operatorname{supp}(S)$ *with* $\operatorname{ord}(g) = n$ *such that*

$$\mathsf v_g(S) \ge \frac{n+5}{6} \quad \text{if } n \text{ is odd}, \quad \text{and} \qquad \mathsf v_g(S) \ge 3 \quad \text{if } n \text{ is even}.$$

In cyclic groups long zero-sumfree sequences and long minimal zero-sum sequences can be completely characterized (see [71]).

Theorem 4.3. *Let* G *by cyclic of order* $n \ge 2$ *and let* $S \in \mathcal F(G)$ *a zero-sumfree sequence of length* $|S| = n - k$ *with* $k \in [1, \lfloor n/3 \rfloor] + 1]$. *Then there exists some* $g \in G$ *with* $\operatorname{ord}(g) = n$ *and* $x_1, \ldots, x_{k-1} \in [1, n-1]$ *such that*

$$S = g^{n-2k+1} \prod_{i=1}^{k-1} (x_i g) \quad \text{and} \quad \sum_{i=1}^{k-1} x_i \le 2k - 2\,.$$

In particular, every minimal zero-sum sequence $S \in \mathcal A(G)$ *of length* $|S| \ge n - \lfloor n/3 \rfloor$ *has* $\operatorname{ind}(S) = 1$.

The index of zero-sum sequences over cyclic groups is investigated in [26, 71, 29]. In [126] (page 344 with $d = n$) it is conjectured that every sequence $S \in \mathcal F(C_n)$ of length $|S| = n$ has a non-empty zero-sum subsequence T with $\operatorname{ind}(T) = 1$. Among others, the g-norm and the index of zero-sum sequences play a role in arithmetical investigations (see [96, Section 6.8]).

Next we discuss groups of the form $G = C_n \oplus C_n$ with $n \ge 2$ (see [77], [166], [79], [96, Section 5.8] and [130]).

Theorem 4.4. *Let* $G = C_n \oplus C_n$ *with* $n \ge 2$. *Then the following statements are equivalent*:

(a) *If* $S \in \mathcal F(G)$, $|S| = 3n - 3$ *and* S *has no zero-sum subsequence* T *of length* $|T| \ge n$, *then there exists some* $a \in G$ *such that* $0^{n-1} a^{n-2} \mid S$.

(b) *If* $S \in \mathcal F(G)$ *is zero-sumfree and* $|S| = \mathsf d(G)$, *then* $a^{n-2} \mid S$ *for some* $a \in G$.

(c) *If* $S \in \mathcal A(G)$ *and* $|S| = \mathsf D(G)$, *then* $a^{n-1} \mid S$ *for some* $a \in G$.

(d) *If* $S \in \mathcal A(G)$ *and* $|S| = \mathsf D(G)$, *then there exists a basis* (e_1, e_2) *of* G *and integers* $x_1, \ldots, x_n \in [0, n-1]$ *with* $x_1 + \ldots + x_n \equiv 1 \mod n$ *such that*

$$S = e_1^{n-1} \prod_{\nu=1}^{n} (x_\nu e_1 + e_2)\,.$$

Moreover, if $S \in \mathcal A(G)$ *and* $|S| = \mathsf D(G)$, *then* $\operatorname{ord}(g) = n$ *for every* $g \in \operatorname{supp}(S)$, *and if* n *is prime, then* $|\operatorname{supp}(S)| \in [3, n]$.

Conjecture 4.5. *For every* $n \ge 2$ *the four equivalent statements of Theorem 4.4 are satisfied.*

Conjecture 4.5 has been verified for $n \in [2, 7]$, and if it holds for some $n \ge 6$, then it holds for $2n$ (see [79, Theorem 8.1]). We continue with a result for non-cyclic groups having large exponent (see [79]).

Theorem 4.6.

1. *Let* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 < n_1 \mid n_2$ *and* $n_2 > n_1(n_1 + 1)$. *Let* $\varphi \colon G \to \overline{G} = C_{n_1} \oplus C_{n_1}$ *be the canonical epimorphism and* $S \in \mathcal A(G)$ *of length* $|S| = \mathsf D(G)$. *If* $\overline{g}^k | \varphi(S)$ *for some* $k > n_1$ *and some* $\overline{g} \in \overline{G}$, *then* $g^k | S$ *for some* $g \in \varphi^{-1}(\overline{g})$.

2. *Let* $G = H \oplus C_n$ *where* $\exp(G) = n = lm$, $H \subset G$ *a subgroup with* $\exp(H) \mid m$, $m \ge 2$ *and* $l \ge 4|H| > 4(m-2)$. *Let* $\varphi \colon G \to \overline{G} = H \oplus C_m$ *denote the canonical epimorphism and* $S \in \mathcal{F}(G)$ *a zero-sumfree sequence of length* $|S| = n$. *Then* S *has a subsequence* T *of length* $|T| \ge (l-2|H|+1)m$ *such that the following holds*: *If* $\overline{g}^k | \varphi(T)$ *for some* $k > m$ *and some* $\overline{g} \in \overline{G}$, *then* $g^k | T$ *for some* $g \in \varphi^{-1}(\overline{g})$.

For general finite abelian groups there is the following result (see [76], [96, Theorem 5.3.6]) which plays a key role in the proof of Theorem 10.4.2).

Theorem 4.7. *Let* $G_0 \subset G$ *be a subset,* $k \in \mathbb{N}$ *and* $k \ge 2$ *be such that* $\mathsf{f}(G_0, k) > 0$, *and let* $S \in \mathcal{F}(G_0)$ *be a zero-sumfree sequence of length*

$$|S| \ge \Big(\frac{|G| - k}{\mathsf{f}(G_0, k)} + 1 \Big) k \,.$$

Then there exists some $g \in G_0$ *such that*

$$\mathsf{v}_g(S) \ge \frac{|S|}{k-1} - \frac{|G| - k - 1}{(k-1)\mathsf{f}(G_0, k)} \,.$$

If the rank of the group is large in comparison with the exponent, there is in general no element with high multiplicity (see Theorem 10.4.1), but in case of elementary p-groups there is the following structural result (see [77, Theorem 10.3], [80, Corollary 6.3], [96, Corollary 5.6.9]).

Theorem 4.8. *Let* G *be a finite elementary* p*-group and* $S \in \mathcal{F}(G)$ *be a zero-sumfree sequence of length* $|S| = \mathsf{d}(G)$. *Then* (g, h) *is independent for any two distinct elements* $g, h \in \operatorname{supp}(S)$.

We continue with the following

Conjecture 4.9. *Let* $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ *with* $1 < n_1 | \cdots | n_r$, $k \in [1, n_1 - 1]$ *and* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| = k + \mathsf{d}(G)$. *If* S *has no zero-sum subsequence* S' *of length* $|S'| > k$, *then* $S = 0^k T$ *where* $T \in \mathcal{F}(G)$ *is zero-sumfree.*

The following example shows that in Conjecture 4.9 the restriction $k \in [1, n_1 - 1]$ is essential:

Let $T \in \mathcal{F}(G)$ be a zero-sumfree sequence of length $|T| = \mathsf{d}(G)$ such that $\mathsf{v}_g(S) = \operatorname{ord}(g) - 1$ for some $g \in G$. Then for every $l \in \mathbb{N}$ the sequence $S = g^{l\operatorname{ord}(g)} T$ has no zero-sum subsequence S' of length $|S'| > l\operatorname{ord}(g)$.

Next we discuss the invariant $\nu(G)$ which was introduced by P. van Emde Boas in connection with his investigations of the Davenport constant for groups of rank three (see [44, page 15] and [69]). An easy argument (see [96, Proposition 5.1.16]) shows that

$$\mathsf{d}(G) - 1 \le \nu(G) \le \mathsf{d}(G) \,,$$

and we make the following conjecture.

Conjecture 4.10. $\nu(G) = \mathsf{d}(G) - 1$.

The following result goes back to P. van Emde Boas, W. Gao and A. Geroldinger ([44, Theorem 2.8], [79, Theorem 5.3], [96, Theorems 5.5.9 and 5.8.10], for more see also [69, Theorem 5.2]).

Theorem 4.11. *Conjecture 4.10 holds true in each of the following cases*:

1. G *is cyclic.*
2. G *is a* p*-group.*
3. $G = C_n \oplus C_n$ *satisfies Conjecture 4.5.*

We end this section with a result (see [81]) showing that minimal zero-sum sequences are not additively closed (apart from some well-defined exceptions).

Theorem 4.12. *Let* $S \in \mathcal{F}(G^\bullet)$ *be a sequence of length* $|S| \ge 4$, *and let* $S = BC$ *with* $B, C \in \mathcal{F}(G)$ *such that* $|B| \ge |C|$. *If* $\sigma(T) \in \operatorname{supp}(S)$ *for all subsequences* T *of* B *with* $|T| = 2$ *and for all subsequences* T *of* C *with* $|T| = 2$, *then* S *has a proper zero-sum subsequence, apart from the following exceptions*:

1. $|C| = 1$, *and we are in one of the following cases*:
   (a) $B = g^k$ *and* $C = 2g$ *for some* $k \ge 3$ *and* $g \in G$ *with* $\operatorname{ord}(g) \ge k+2$.
   (b) $B = g^k(2g)$ *and* $C = 3g$ *for some* $k \ge 2$ *and* $g \in G$ *with* $\operatorname{ord}(g) \ge k+5$.
   (c) $B = g_1g_2(g_1{+}g_2)$ *and* $C = g_1{+}2g_2$ *for some* $g_1, g_2 \in G$ *with* $\operatorname{ord}(g_1) = 2$ *and* $\operatorname{ord}(g_2) \ge 5$.
2. $\{B, C\} = \{g(9g)(10g)\,,\,(11g)(3g)(14g)\}$ *for some* $g \in G$ *with* $\operatorname{ord}(g) = 16$.

If $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ such that $\operatorname{ord}(g_k) > k^k$ for all $k \in [1, l]$, then G. Harcos and I. Ruzsa showed that S allows a product decomposition $S = S_1S_2$ where S_1 and S_2 are both zero-sumfree (see [119]).

5. On generalizations of the Davenport constant

We discuss two generalizations of the Davenport constant in some detail (for yet another generalization, the barycentric Davenport constant, we refer to [34]). The first one was introduced by F. Halter-Koch in connection with the analytic theory of non-unique factorizations (see [109]).

Definition 5.1. Let $k \in \mathbb{N}$. We denote by

- $\mathsf{D}_k(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ is divisible by a product of k non-empty zero-sum sequences.
- $\mathsf{d}_k(G)$ the largest integer $l \in \mathbb{N}$ with the following property:
  There is a sequence $S \in \mathcal{F}(G)$ of length $|S| = l$ which is not divisible by a product of k non-empty zero-sum sequences.

Obviously, we have $\mathsf{D}_k(G) = 1 + \mathsf{d}_k(G)$, $\mathsf{d}_1(G) = \mathsf{d}(G)$ and $\mathsf{D}_1(G) = \mathsf{D}(G)$. We present one result on $\mathsf{d}_k(G)$ which, among others, may be found in [96, Section 6.1].

Theorem 5.2. *Let* $\exp(G) = n$ *and* $k \in \mathbb{N}$.

1. *Let* $G = H \oplus C_n$ *where* $H \subset G$ *is a subgroup. Then*
$$\mathsf{d}(H) + kn - 1 \le \mathsf{d}_k(G) \le (k-1)n + \max\{\mathsf{d}(G), \eta(G) - n - 1\}\,.$$
   In particular, if $\mathsf{d}(G) = \mathsf{d}(H) + n - 1$ *and* $\eta(G) \le \mathsf{d}(G) + n + 1$, *then* $\mathsf{d}_k(G) = \mathsf{d}(G) + (k-1)n$.
2. *If* $\mathsf{r}(G) \le 2$, *then* $\mathsf{d}_k(G) = \mathsf{d}(G) + (k-1)n$.
3. *If* G *a* p*-group and* $\mathsf{D}(G) \le 2n - 1$, *then* $\mathsf{d}_k(G) = \mathsf{d}(G) + (k-1)n$.

The following generalization of the Davenport constant was introduced by M. Skałba in connection with his investigations on binary quadratic forms (see [159], [160], [161]).

Definition 5.3. For every $g \in G$, let $\mathsf{D}_g(G)$ denote the largest integer $l \in \mathbb{N}$ with the following property:
There is a sequence $S \in \mathcal{F}(G)$ of length $|S| = l$ and sum $\sigma(S) = g$ such that every proper subsequence of S is zero-sumfree.

By definition, $\mathsf{D}_0(G) = \mathsf{D}(G)$, and if $g \ne 0$, then $\mathsf{D}_g(G) \le \mathsf{d}(G)$. The following result is due to M. Skałba (see [160, Theorem 2] and [161, Theorem 1])

Theorem 5.4. *Let* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$ *and* (e_1, e_2) *a basis of* G. *Let* $g = a_1e_1 + a_2e_2 \in G^\bullet$ *with* $a_1 \in [0, n_1 - 1]$, $a_2 \in [0, n_2 - 1]$ *and* $d = \gcd\big(\gcd(a_1, n_1), \gcd(a_2, n_2)\big)$. *Then*

$$\mathsf{D}_g(G) = \begin{cases} n_1 + n_2 - d - 1 & \text{if} \quad d \ne n_1, \\ n_1 + n_2 - \gcd(a_2, n_2) - 1 & \text{if} \quad d = n_1. \end{cases}$$

Lemma 5.5. *Let* $\exp(G) = n \ge 2$. *Then the following statements are equivalent*:

(a) *There exists some* $g \in G$ *with* $\operatorname{ord}(g) = n$ *such that* $\mathsf{D}_g(G) = \mathsf{d}(G)$.

(b) *For all* $g \in G$ *with* $\operatorname{ord}(g) = n$ *we have* $\mathsf{D}_g(G) = \mathsf{d}(G)$.

(c) *There exists a minimal zero-sum sequence* $S \in \mathcal{F}(G)$ *of length* $|S| = \mathsf{D}(G)$ *such that* $\max\{\operatorname{ord}(g) \mid g \in \operatorname{supp}(S)\} = n$.

Proof. (a) $\Rightarrow$ (b) Let $g, g^* \in G$ with $\operatorname{ord}(g) = \operatorname{ord}(g^*) = n$ and suppose that $\mathsf{D}_{g^*}(G) = \mathsf{d}(G)$. Then there exists a zero-sumfree sequence $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{d}(G)$ and $\sigma(S) = g^*$. If $\varphi \colon G \to G$ is a group automorphism with $\varphi(g^*) = g$, then $\varphi(S)$ is a zero-sumfree sequence of length $|\varphi(S)| = \mathsf{d}(G)$ and $\sigma(\varphi(S)) = \varphi(\sigma(S)) = g$ whence $\mathsf{D}_g(G) = \mathsf{d}(G)$.

(b) $\Rightarrow$ (c) Let $g \in G$ and $S \in \mathcal{F}(G)$ a zero-sumfree sequence with $\sigma(S) = g$ and $|S| = \mathsf{D}_g(G) = \mathsf{d}(G)$. Then the sequence $S^* = (-g)S$ has the required properties.

(c) $\Rightarrow$ (a) Assume to the contrary that for all $g \in G$ with $\operatorname{ord}(g) = n$ we have $\mathsf{D}_g(G) < \mathsf{d}(G)$. This means that for all zero-sumfree sequences $S \in \mathcal{F}(G)$ with $|S| = \mathsf{d}(G)$ we have $\operatorname{ord}(\sigma(S)) < n$. But this implies that for all minimal zero-sum sequences $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{D}(G)$ we have $\max\{\operatorname{ord}(g) \mid g \in \operatorname{supp}(S)\} < n$, a contradiction. □

Note that Conjecture 4.1 implies Condition (c) of Lemma 5.5. Using this condition we immediately obtain the following corollary.

Corollary 5.6. *If* $\mathsf{d}^*(G) = \mathsf{d}(G)$ *and* $g \in G$ *with* $\operatorname{ord}(g) = \exp(G)$, *then* $\mathsf{D}_g(G) = \mathsf{d}(G)$.

6. On the invariants $\eta(G)$, $\mathsf{s}(G)$ and their analogues

We start with a key result first obtained by W. Gao (see [60]). Its proof is based on the Addition Theorem of Kemperman-Scherk (for the version below we refer to [96, Theorem 5.7.3]).

Theorem 6.1. *Let* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| \ge |G|$. *Then* S *has a non-empty zero-sum subsequence* T *of length* $|T| \le \min\big\{\mathsf{h}(S), \ \max\{\operatorname{ord}(g) \mid g \in \operatorname{supp}(S)\}\big\}$.

Now we discuss the invariants $\eta(G)$, $\mathsf{s}(G)$ and their relationship. Both invariants have received a lot of attention in the literature. The various contributions and the present state of knowledge are well-described in [40], where also the connection with finite geometry is discussed (see also [82]). Therefore we only mention some of the most recent results, and then we discuss the relationship of $\eta(G)$ and $\mathsf{s}(G)$ in greater detail. A simple observation shows that

$$\mathsf{D}(G) \le \eta(G) \le \mathsf{s}(G) - \exp(G) + 1\,.$$

Using Theorem 6.1 we obtain the following upper bounds on $\eta(G)$ and $\mathsf{s}(G)$ (see [96, Theorem 5.7.4]) which are sharp for cyclic groups.

Theorem 6.2. $\eta(G) \le |G|$ *and* $\mathsf{s}(G) \le |G| + \exp(G) - 1$.

Both invariants, $\eta(G)$ and $\mathsf{s}(G)$ are completely determined for groups of rank at most two (see [96, Theorem 5.8.3]). Theorem 6.3 is based on the result by C. Reiher which states that $\mathsf{s}(C_p \oplus C_p) = 4p - 3$ for all $p \in \mathbb{P}$ (see [154], and also [155]), and it contains the Theorem of Erdős-Ginzburg-Ziv (set $n_1 = 1$). Theorem 6.4 may be found in [158].

Theorem 6.3. *Let* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$. *Then*

$$\eta(G) = 2n_1 + n_2 - 2 \quad \text{and} \quad \mathsf{s}(G) = 2n_1 + 2n_2 - 3\,.$$

Theorem 6.4. *Let* G *be a* p*-group for some odd prime* p *with* $\exp(G) = n$ *and* $\mathsf{D}(G) \le 2n - 1$. *Then*

$$2\mathsf{D}(G) - 1 \le \eta(G) + n - 1 \le \mathsf{s}(G) \le \mathsf{D}(G) + 2n - 2\,.$$

In particular, if $\mathsf{D}(G) = 2n - 1$, *then* $\mathsf{s}(G) = \eta(G) + n - 1 = 4n - 3$.

We continue with the following conjecture

Conjecture 6.5. $\eta(G) = \mathsf{s}(G) - \exp(G) + 1$.

Theorem 6.6. *Conjecture 6.5 holds true in each of the following cases*:
1. $\exp(G) \in \{2, 3, 4\}$.
2. $\mathsf{r}(G) \le 2$.
3. *G is a p-group for some odd prime p and* $\mathsf{D}(G) = 2\exp(G) - 1$.
4. $G = C_5^3$.

Proof. 1. is proved in [73], 2. follows from Theorem 6.3, and 3. follows from Theorem 6.4. In order to give an idea of the arguments we are going to prove 4. We need the following two results:

F1 If $n \in \mathbb{N}_{\ge 3}$ is odd, then $\eta(C_n^3) \ge 8n - 7$ (this is due to C. Elsholtz [43], see also [40, Lemma 3.4]).

F2 If $\exp(G) = n$ and $S \in \mathcal{F}(G)$ such that

$$|S| \ge \eta(G) + n - 1 \quad \text{and} \quad \mathsf{h}(S) \ge n - \lfloor n/2 \rfloor - 1\,,$$

then S has a zero-sum subsequence of length n (see [73, Proposition 2.7]).

Let $G = C_5^3$. It suffices to show that $\mathsf{s}(G) \le \eta(G) + 4$. Let $S \in \mathcal{F}(G)$ be a sequence of length $\eta(G) + 4$. We have to verify that S has a zero-sum subsequence of length 5. By **F1** we have, $|S| \ge 37$. If we can prove that $\mathsf{h}(S) \ge 2$, then the assertion follows from **F2**.

Assume to the contrary that S is squarefree. Let $G = H \oplus \langle g \rangle$ where $H \subset G$ is a subgroup with $|H| = 25$ and $g \in G$ with $\mathrm{ord}(g) = 5$. Then

$$S = \prod_{i=1}^{l} (g_i + h_i), \quad \text{where} \quad g_i \in \langle g \rangle,\ h_i \in H, \quad \text{and we set} \quad T = \prod_{i=1}^{l} g_i\,.$$

If $\mathsf{h}(T) \ge 9$, say $g_1 = \ldots = g_9$, then $h_1, \ldots, h_9$ are pairwise distinct. Since $\mathsf{g}(C_5^2) = 9$ (see [124] and Conjecture 10.2), the sequence $h_1 \cdot \ldots \cdot h_9$ has a zero-sum subsequence of length 5, and therefore S has a zero-sum subsequence of length 5.

Suppose that $\mathsf{h}(T) \le 8$. Then $T = 0^{l_0} g^{l_1} (2g)^{l_2} (3g)^{l_3} (4g)^{l_4}$ with $l_0, l_1, l_2, l_3, l_4 \in [5, 8]$, and we write S in the form

$$S = \prod_{i=0}^{4} \prod_{j=1}^{l_i} (ig + h_{i,j}) \quad \text{with all} \quad h_{i,j} \in H\,.$$

Since S is squarefree, for every $i \in [0, 4]$ the elements $h_{i,1}, \ldots, h_{i,l_i}$ are pairwise distinct, and we set $A_i = \{h_{i,1}, \ldots, h_{i,l_i}\}$. Note that $0 + g + 2g + 3g + 4g = 0 \in G$. So if

$$0 \in A = A_0 + A_1 + A_2 + A_3 + A_4\,,$$

then S has a zero-sum subsequence of length 5. Let K be the maximal subgroup of H such that $A+K=A$. By Kneser's Addition Theorem (see [96, Theorem 5.2.6.2]) we obtain that

$$|A| \ge \sum_{i=0}^{4} |A_i + K| - 4|K| \,.$$

If $|K|=1$, then $|A| \ge |A_0|+|A_1|+|A_2|+|A_3|+|A_4|-4 = |S|-4 \ge 33$, a contradiction.

Assume to the contrary that $|K|=5$. Since $|A_0|+|A_1|+|A_2|+|A_3|+|A_4| = |S| \ge 37$ and $|A_i| = l_i \in [5,8]$, it follows that $|A_i| \ge 6$ for at least four indices $i \in [0,4]$. Therefore we obtain that

$$|A| \ge \sum_{i=0}^{4} |A_i + K| - 4|K| \ge 4 \cdot 2|K| + |K| - 4|K| = 5|K| = 25 \,,$$

a contradiction. Thus it follows that $K=H$ whence $A=H$ and we are done.

□

For recent progress on Conjecture 6.5 we refer to [82]. Next we consider the invariant $\mathsf{s}_{n\mathbb{N}}(G)$. Theorem 6.3 allows to determine $\mathsf{s}_{n\mathbb{N}}(G)$ for groups G of rank $\mathsf{r}(G) \le 2$.

Theorem 6.7. *Let* $\exp(G) = n \ge 2$.

1. $\mathsf{d}(G) + n \le \mathsf{s}_{n\mathbb{N}}(G) \le \min\{\mathsf{s}(G), \mathsf{D}(G \oplus C_n)\}$.
2. *We have* $\mathsf{s}_{n\mathbb{N}}(G) = \mathsf{d}(G) + n$ *in each of the following cases*:
   (a) G *is a* p*-group.*
   (b) $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$.

Proof. 1. is simple (see [79, Lemma 3.5]) and 2.(a) is a consequence of 1. To verify 2.(b), let $G = C_{n_1} \oplus C_{n_2}$ with $1 \le n_1 \mid n_2$. Then 1. implies that $\mathsf{d}(G) + n_2 \le \mathsf{s}_{n\mathbb{N}}(G)$ whence it remains to prove that $\mathsf{s}_{n\mathbb{N}}(G) \le \mathsf{d}(G) + n_2 = n_1 + 2n_2 - 2$. If $n_1 = 1$, this follows from 1. Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = n_1 + 2n_2 - 2$. We have to show that S has a zero-sum subsequence of length n_2 or $2n_2$.

Let $H = G \oplus C_{n_2} = G \oplus \langle e \rangle$ with $\operatorname{ord}(e) = n_2$, so that every $h \in G \oplus C_{n_2}$ has a unique representation $h = g + je$, where $g \in G$ and $j \in [0, n_2 - 1]$. We define $\psi \colon G \to H$ by $\psi(g) = g + e$ for every $g \in G$. Thus it suffices to show that $\psi(S)$ has a non-empty zero-sum subsequence. We distinguish two cases.

CASE 1: $n_1 = n_2$.

We set $n = n_1$ and proceed by induction on n. If n is prime, the assertion follows from 2.(a). Suppose that n is composite, p a prime divisor of n and $\varphi \colon H \to H$ the multiplication by p. Then $pG \cong C_{n/p} \oplus C_{n/p}$ and $\operatorname{Ker}(\varphi) \cong C_p^3$. Since $\mathsf{s}(pG) = 4(n/p) - 3$ and $|S| = 3n - 2 \ge (3p-4)(n/p) + 4n/p - 3$, S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_{3p-3} S'$ such that, for all $i \in [1, 3p-3]$, $\varphi(S_i)$ has sum zero and length $|S_i| = n/p$ (for details see [96, Lemma 5.7.10]). Then $|S'| = 3n/p - 2 = \mathsf{s}_{n\mathbb{N}}(C_{n/p} \oplus C_{n/p})$, and thus S' has a subsequence S_{3p-2} such that $\varphi(S_{3p-2})$ has sum zero and length $|S_{3p-2}| \in \{n/p, 2n/p\}$. This implies that

$$\prod_{i=1}^{3p-2} \sigma\big(\psi(S_i)\big) \in \mathcal{F}\big(\operatorname{Ker}(\varphi)\big) \,.$$

Since $\mathsf{D}(\operatorname{Ker}(\varphi)) = 3p - 2$, there exists a non-empty subset $I \subset [1, 3p-2]$ such that

$$\sum_{i \in I} \sigma\big(\psi(S_i)\big) = 0 \quad \text{whence} \quad \prod_{i \in I} \psi(S_i)$$

is a non-empty zero-sum subsequence of $\psi(S)$.

CASE 2: $n_2 > n_1$.

Let $m = n_1^{-1} n_2$ and let $\varphi \colon H = C_{n_1} \oplus C_{n_2}^2 \to C_{n_1} \oplus mC_{n_2}^2$ be a map which is the identity on the first component and the multiplication by m on the second and on the third component whence $\operatorname{Ker}(\varphi) \cong C_m \oplus C_m$ and $\varphi(G) \cong C_{n_1} \oplus C_{n_1}$. Since $\mathsf{s}(C_{n_1} \oplus C_{n_1}) = 4n_1 - 3$ and $|S| = n_1 + 2n_2 - 2 \ge (2m-3)n_1 + (4n_1 - 3)$,

S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_{2m-2} S'$, where for all $i \in [1, 2m-2]$, $\varphi(S_i)$ has sum zero and length $|S_i| = n_1$. Then $|S'| = 3n_1 - 2$, and since by CASE 1, $\mathsf s_{n\mathbb N}(C_{n_1} \oplus C_{n_1}) = 3n_1 - 2$, the sequence S' has a subsequence S_{2m-1} such that $\varphi(S_{2m-1})$ has sum zero and length $|S_{2m-1}| \in \{n_1, 2n_1\}$. This implies that

$$\prod_{i=1}^{2m-1} \sigma\big(\psi(S_i)\big) \in \mathcal F\big(\mathrm{Ker}(\varphi)\big) .$$

Since $\mathsf D(\mathrm{Ker}(\varphi)) = 2m-1$, there exists a non-empty subset $I \subset [1, 2m-1]$ such that

$$\sum_{i \in I} \sigma\big(\psi(S_i)\big) = 0 \quad \text{whence} \quad \prod_{i \in I} \psi(S_i)$$

is a non-empty zero-sum subsequence of $\psi(S)$. □

Next we deal with zero-sum subsequences of length $|G|$. The following result is due to W. Gao and Y. Caro (see [21], [22], [62] and also [96, Proposition 5.7.9]). In Section 9 we discuss generalizations due to Y. ould Hamidoune. The structure of sequences S of length $|S| = |G| + \mathsf d(G) - 1$ which have no zero-sum subsequence of length $|G|$ is studied in [94].

Theorem 6.8. $\mathsf s_{|G|}(G) = |G| + \mathsf d(G)$.

Note that Theorem 6.8 yields immediately a generalization of a Theorem of Hall (see [134, Section 3]).

Conjecture 6.9. *Let G be cyclic of order $n \ge 2$, q the smallest prime divisor of n and $S \in \mathcal F(G^\bullet)$ be a sequence of length $|S| = n$. If $h = \mathsf h(S) \ge n/q - 1$, then $\sum_{\le h}(S) = \sum(S)$.*

Conjecture 6.9 has been verified for cyclic groups of prime power order in [93]. The following example shows that the conclusion of Conjecture 6.9 does not hold whenever $nq/(2n-q) \le h \le n/q - 2$.

Let all notations be as in Conjecture 6.9, $N = \{0, a_1, \cdots, a_{n/q-1}\}$ a subgroup of G with $|N| = n/q$, $g \in G$ with $\mathrm{ord}(g) = n$ and

$$W = a_1^h \cdot \ldots \cdot a_{n/q-1}^h g^h (g + a_1)^h \cdot \ldots \cdot (g + a_{n/q-1})^h \ \in \mathcal F(G) .$$

Since $h \ge nq/(2n-q)$, we have $|W| = (\frac{n}{q} - 1)h + \frac{n}{q}h \ge n$. Now let S be a subsequence of W of length $|S| = n$ such that $g^h(g + a_i)$ is a subsequence of S for some $i \in [1, (n/q) - 1]$. Then $\mathsf h(S) = h$,

$$((h+1)g + N) \cap \Sigma(S) \ne \emptyset \quad \text{but} \quad ((h+1)g + N) \cap \Sigma_{\le h}(S) = \emptyset$$

whence $\Sigma_{\le h}(S) \ne \Sigma(S)$.

Next we discuss the invariants $\mathsf s_{kn}(G)$ where $\exp(G) = n$ and $k \in \mathbb N$. If $S \in \mathcal F(G)$ is a zero-sumfree sequence of length $|S| = \mathsf d(G)$ elements, then the sequence

$$T = 0^{kn-1} S$$

has no zero-sum subsequence of length kn whence $\mathsf s_{kn}(G) \ge |T| + 1 = kn + \mathsf d(G)$. The following result may be found in [73].

Theorem 6.10. *Let $\exp(G) = n \ge 2$ and $k \in \mathbb N$.*

1. *If $k < \mathsf D(G)/n$, then $\mathsf s_{kn}(G) > kn + \mathsf d(G)$.*
2. *If $k \ge |G|/n$, then $\mathsf s_{kn}(G) = kn + \mathsf d(G)$.*
3. *If G a finite abelian p-group and $p^l \ge \mathsf D(G)$, then $\mathsf s_{p^l k}(G) = p^l k + \mathsf d(G)$.*

Theorem 6.10 motivates the following definition.

Definition 6.11. We denote by $\mathsf{l}(G)$ the smallest integer $l \in \mathbb{N}$ such that

$$\mathsf{s}_{k \exp(G)}(G) = k \exp(G) + \mathsf{d}(G) \quad \text{for every} \quad k \ge l\,.$$

Theorem 6.10 shows that

$$\frac{\mathsf{D}(G)}{n} \le \mathsf{l}(G) \le \frac{|G|}{n} \quad \text{whence} \quad \mathsf{l}(C_n) = 1\,.$$

Theorem 6.12. *Let* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 < n_1 \mid n_2$. *Then* $\mathsf{l}(G) = 2$.

Proof. Since $\mathsf{s}(G) = 2n_1 + 2n_2 - 3 > n_2 + \mathsf{d}(G)$, it follows that $\mathsf{l}(G) \ge 2$. Let $k \ge 2$ and $S \in \mathcal{F}(G)$ a sequence of length $|S| = kn_2 + \mathsf{d}(G) = (k-2)n_2 + 3n_2 + n_1 - 2$. We prove that S has a zero-sum subsequence of length kn which implies that $\mathsf{l}(G) \le 2$. Since $\mathsf{s}(G) = 2n_1 + 2n_2 - 3$, S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_{k-1} S'$ where for all $i \in [1, k-1]$, S_i has sum zero and length $|S_i| = n_2$ (for details see [96, Lemma 5.7.10]). Since $|S'| = |S| - (k+1)n_2 = 2n_2 + n_1 - 1$, Theorem 6.7.2.(b) implies that S' has a zero-sum subsequence S_k of length $|S_k| \in \{n_2, 2n_2\}$ whence either $S_1 \cdot \ldots \cdot S_{k-1} S_k$ or $S_1 \cdot \ldots \cdot S_{k-2} S_k$ is a zero-sum subsequence of length kn_2. □

The invariant $\mathsf{E}_k(G)$ was introduced in [72] (in connection with investigations on $\mathsf{s}(G)$, see also [90]). Clearly, we have $\mathsf{D}(G) \le \mathsf{E}_k(G) \le \mathsf{s}(G)$, and if $\mathsf{D}(G) < k$, then $\mathsf{D}(G) = \mathsf{E}_k(G)$ (see [156, Lemma 2.1]).

Theorem 6.13.

1. [72, Section 3] *If* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$ *and* n_2 *odd, then* $\mathsf{E}_2(G) = 2n_1 + 2n_2 - 3$.
2. *If* $G = C_n \oplus C_n$ *with* $n \ge 2$ *and* $3 \nmid n$, *then* $\mathsf{E}_3(G) = 3n - 2$.
3. [156] *If* G *is a* p*-group and* $k \in \mathbb{N}_{\ge 2}$ *with* $\gcd(p, k) = 1$, *then*

$$\mathsf{E}_k(G) = \left\lfloor \frac{k}{k-1} \mathsf{d}^*(G) \right\rfloor + 1.$$

Proof. 2. By [156, Lemma 2.4], we have $3n-2 \le \mathsf{E}_3(G)$. Since $\mathsf{s}_{n\mathbb{N}}(G) = 3n-2$, every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge 3n-2$ has a zero-sum subsequence T of length $|T| \in \{n, 2n\}$ whence $\mathsf{E}_3(G) \le 3n-2$. □

7. Inverse problems associated with $\eta(G)$ and $\mathsf{s}(G)$

In this section we investigate the structure of sequences $S \in \mathcal{F}(G)$ of length

$$\begin{aligned} \eta(G) - 1 &\quad \text{without a zero-sum subsequence } T \text{ of length} \quad |T| \in [1, \exp(G)]\,, \\ \mathsf{s}(G) - 1 &\quad \text{without a zero-sum subsequence } T \text{ of length} \quad |T| = \exp(G)\,. \end{aligned}$$

We formulate two properties and two conjectures.

Conjecture 7.1. *Let* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| = \mathsf{s}(G) - 1$. *If* S *has no zero-sum subsequence of length* $\exp(G)$, *then* $\mathsf{h}(S) = \exp(G) - 1$.

Note that Conjecture 7.1 and Fact **F2** (formulated in the proof of Theorem 6.6) imply Conjecture 6.5.

Property C. Every sequence $S \in \mathcal{F}(G)$ of length $|S| = \eta(G) - 1$ which has no short zero-sum subsequence has the form $S = T^{n-1}$ for some sequence $T \in \mathcal{F}(G)$.

Property D. Every sequence $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{s}(G) - 1$ which has no zero-sum subsequence of length n has the form $S = T^{n-1}$ for some sequence $T \in \mathcal{F}(G)$.

Suppose that G has Property **D**. We show that G satisfies Property **C** as well. Let $S \in \mathcal{F}(G)$ be a sequence of length $\eta(G)-1$ which has no short zero-sum subsequence. We consider the sequence

$$T = 0^{n-1} S\,.$$

If T has a zero-sum subsequence T' of length $|T'| = n$, then $T' = 0^k S'$ with $k' \in [0, n-1]$ whence S' is a short zero-sum subsequence of S, a contradiction. Thus T has no zero-sum subsequence of length n. Since Property **D** holds, Conjecture 7.1 and Conjecture 6.5 hold in G whence $|T| = \eta(G)-1+(n-1) = \mathsf{s}(G)-1$. Therefore Property **D** implies that S has the required form.

Conjecture 7.2. *Every group $G = C_n^r$, where $r \in \mathbb{N}$ and $n \in \mathbb{N}_{\ge 2}$, has Property* **D**.

An easy observation shows that

$$\mathsf{s}(G) \le \big(\mathsf{g}(G)-1\big)(n-1)+1\,.$$

Moreover, if $G = C_n^r$ and equality holds, then C_n^r has Property **D** (see [40, Lemma 2.3]). Thus [118, Hilfssatz 3] implies that C_3^r has Property **D** for every $r \in \mathbb{N}$. However, only little is known for groups $G = C_n^r$ in case $r \ge 3$ (see [91] and [82]).

We continue with some results on $\Sigma_{|G|}(S)$ for general groups which arose from generalizations of the Erdős-Ginzburg-Ziv Theorem (see also [85], [57], [167] [113] and note that Theorem 7.3 implies Theorem 6.8). Then we discuss cyclic groups and groups of the form $G = C_n \oplus C_n$.

Theorem 7.3. [60, 61] *Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| \ge |G|$ and let $g \in G$ with $\mathsf{v}_g(S) = \mathsf{h}(S)$.*

1. $$\Sigma_{|G|}(S) \;=\; \Sigma_{\ge (|G|-\mathsf{h}(S))}(-g + g^{-\mathsf{h}(S)} S)\,.$$
2. *Suppose that for every $a \in G$ and every subsequence T of S of length $|T| = |S|-|G|+1$ we have $0 \in \Sigma(a+T)$. Then*
$$\Sigma_{|G|}(S) \;=\; \bigcap_{y \in G} \Sigma(y+S) \;=\; \Sigma(-g+S)\,.$$

Next we present a result by D.J. Grynkiewicz ([105, Theorem 1]) which confirms a conjecture of Y. ould Hamidoune (see [115, Theorem 3.6] and [59] for special cases).

Theorem 7.4. *Let $S \in \mathcal{F}(G)$ be a sequence of length $|S| \ge |G|+1$, $k \in \mathbb{N}$ with $|\mathrm{supp}(S)| \ge k$ and $\mathsf{h}(S) \le |G|-k+2$. Then one of the following two statements holds*:

(a) $|\Sigma_{|G|}(S)| \ge \min\{|G|, |S|-|G|+k-1\}$.

(b) *There exists a non-trivial subgroup $H \subset G$, some $g \in G$ and a subsequence T of S such that the following conditions hold*:
- *$H \subset \sum_{|G|}(S)$, $\Sigma_{|G|}(S)$ is H-periodic and $|\sum_{|G|}(S)| \ge (|T|+1)|H|$.*
- *$\mathrm{supp}(T^{-1}S) \subset g+H$ and $|T| \le \min\big\{\frac{|S-|G|+k-2}{|H|}, (G\colon H)-2\big\}$.*

Now we consider cyclic groups. Several authors ([170], [13], [20], [50], [23]) showed independently that a sequence $S \in \mathcal{F}(C_n)$ of length $|S| = 2n-2$, which has no zero-sum subsequence of length n, has the form $S = a^{n-1} b^{n-1}$ where $a, b \in C_n$ and $\mathrm{ord}(a-b) = n$. Based on Theorem 7.3 the following stronger result was obtained in [65, Theorem 1] (see also [88]).

Theorem 7.5. *Let G be cyclic of order $n \ge 2$, $k \in [2, \lfloor n/4 \rfloor + 2]$ and $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 2n-k$. If S has no zero-sum subsequence of length n, then*

$$S = a^u b^v c_1 \cdot \ldots \cdot c_l, \quad \textit{where} \quad \mathrm{ord}(a-b) = n,\ u \ge v \ge n-2k+3 \quad \textit{and}$$

$u+v \ge 2n-2k+1$ (equivalently, $l \le k-1$). In particular, we have

- *If* $k = 2$, *then* $S = a^{n-1}b^{n-1}$.
- *If* $k = 3$ *and* $n \ge 4$, *then* $S = a^{n-1}b^{n-2}$ *or* $S = a^{n-1}b^{n-3}(2b-a)$

Closely related to the inverse problem is the investigation of the Brakemeier function (see [18, 15], [58, 56, 57], [120, 121]).

Conjecture 7.6. *Let* G *be cyclic of order* $n \ge 2$, q *the smallest prime divisor of* n *and* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| \ge n + n/q - 1$. *If* $0 \notin \Sigma_n(S)$ *then* $\mathsf{h}(S) \ge |S| - n + 1$.

Conjecture 7.6 has been verified for cyclic groups of prime power order (see [92], [93]). The following example shows that the conclusion of Conjecture 7.6 does not hold whenever $q \le |S| - n \le n/q - 2$.

Let all notations be as in Conjecture 7.6, $N = \{0, a_1, a_2, \cdots, a_{n/q-1}\}$ be the subgroup of G with $|N| = n/q$, $k \in [q, n/q-2]$, $g \in G$ with $\operatorname{ord}(g) = n$ and

$$W = a_1^k \cdot \ldots \cdot a_{n/q-1}^k g^k (g+a_1)^k \cdot \ldots \cdot (g + a_{n/q-1})^k \in \mathcal{F}(G)$$

a sequence of length $|W| = k(2n/q - 1)$. Since $k \in [q, n/q-2]$, one can choose a subsequence S of W such that $|S| = n + k$ such that g^k is a subsequence of S and $\sigma(S) \in (k+1)g + N$. Therefore $\mathsf{h}(S) = k$ and $((k+1)g + N) \cap \Sigma_k(S) = \emptyset$ which implies that $\sigma(S) \notin \Sigma_k(S)$ and $0 \notin \sum_n(S)$.

Now suppose that $G = C_n \oplus C_n$. It was P. van Emde Boas who studied Property **C** for such groups in connection with his investigations on the Davenport constant for groups of rank three (see [44] and [69, Lemma 4.7]). Property **D** was introduced in [70], where it is shown that both Property **C** and Property **D** are multiplicative in the following sense.

Theorem 7.7. *Let* $n_1, n_2 \in \mathbb{N}_{\ge 2}$. *If the groups* $C_{n_1} \oplus C_{n_1}$ *and* $C_{n_2} \oplus C_{n_2}$ *both have Property* **C** (*or Property* **D** *respectively*), *then the group* $C_{n_1 n_2} \oplus C_{n_1 n_2}$ *has Property* **C** (*or Property* **D** *respectively*).

The next result follows from Theorem 6.7.2.(b), from Theorem 7.7 and from [79, Theorem 6.2].

Theorem 7.8. *Let* $n \ge 2$ *and suppose that* $n = m_1 \cdot \ldots \cdot m_s$ *where* $s \in \mathbb{N}$ *and* $m_1, \ldots, m_s \in \mathbb{N}_{\ge 2}$. *If for all* $i \in [1, s]$ *the groups* $C_{m_i} \oplus C_{m_i}$ *satisfy the equivalent conditions of Theorem 4.4, then* $C_n \oplus C_n$ *has Property* **C**.

In [124] it is shown that $C_p \oplus C_p$ has Property **D** for $p \in \{2, 3, 5\}$ and in [164] the same is shown for $p = 7$. We end with a result which could be a first step on the way showing that $C_n \oplus C_n$ has Property **C**.

Theorem 7.9. *Let* $G = C_n \oplus C_n$ *with* $n \ge 3$ *and* $S = f_1^{n-1} f_2^{n-1} g_1 \cdot \ldots \cdot g_{n-1} \in \mathcal{F}(G)$ *be a sequence of length* $|S| = 3n - 3$ *which has no short zero-sum subsequence. Then there exists a basis* (e_1, e_2) *of* G *such that*

$$S = (e_1 + e_2)^{n-1} e_2^{n-1} \prod_{i=1}^{n-1} (a_i e_1 + b e_2)$$

where $a_i \in [0, n-1]$ *for all* $i \in [1, n-1]$ *and* $b \in [0, n-1] \setminus \{1\}$.

Proof. By [96, Lemma 5.8.6] it follows that (f_1, f_2) is a basis of G whence $g_i = y_i f_1 + x_i f_2$ with $x_i, y_i \in [0, n-1]$ for all $i \in [1, n-1]$. We assert that

$$x_1 + y_1 = \ldots = x_{n-1} + y_{n-1} \,.$$

Assume to the contrary that this does not hold. Then Theorem 4.2.2 implies that the sequence

$$\prod_{i=1}^{n-1} \big((x_i + y_i - 1) e_1 \big) \quad \text{is not zero-sumfree} \,.$$

Hence after some renumeration we may suppose that

$$\sum_{i=1}^{t}(x_i+y_i-1)\equiv 0 \mod n \quad \text{for some} \quad t\in[1,n-1]\,.$$

Then the sequence

$$W=f_2^{n-x}f_1^{n-y}\prod_{i=1}^{t}(y_if_1+x_if_2),$$

where $x,y\in[1,n]$ such that $x\equiv x_1+\ldots+x_t \mod n$ and $y\equiv y_1+\ldots+y_t \mod n$, is a zero-sum subsequence of S of length $|W|=(n-x)+(n-y)+t\equiv 0 \mod n$. Since S has no short zero-sum subsequence, it follows that $|W|=2n$. But then $|W|>\mathsf d(C_n\oplus C_n)$ whence W (and thus S) has a short zero-sum subsequence, a contradiction.

Now we obtain that $(e_1,e_2)=(f_2-f_1,f_1)$ is a basis of G and

$$g_i=y_if_1+x_if_2=x_ie_1+(x_i+y_i)e_2 \quad \text{for all} \quad i\in[1,n-1]\,.$$

Thus it remains to show that $x_1+y_1\not\equiv 1 \mod n$. Assume to the contrary that $(x_1+y_1)e_2=e_2$. Since $\mathsf s(C_n)=2n-1$, the sequence $e_1^{n-1}0^{n-1}(x_1e_1)$ has a zero-sum subsequence of length n whence $(e_1+e_2)^{n-1}e_2^{n-1}(x_1e_1+e_2)$ has a zero-sum subsequence of length n, a contradiction. □

8. On the number of zero-sum subsequences

The enumeration of zero-sum subsequences of a given (long) sequence over G, which have some prescribed properties, is a classical topic in combinatorial number theory going back to P. Erdős, J.E. Olson and others. Many zero-sum results (such as the proof of $\mathsf d^*(G)=\mathsf d(G)$ for p-groups or the proof that $\mathsf s(C_p\oplus C_p)=4p-3$) are based on enumeration results.

Definition 8.1. Let $S=g_1\cdot\ldots\cdot g_l\in\mathcal F(G)$ be a sequence of length $|S|=l\in\mathbb N_0$ and let $g\in G$.

1. For every $k\in\mathbb N_0$ let

$$\mathsf N_g^k(S)=\Big|\Big\{I\subset[1,l]\;\Big|\;\sum_{i\in I}g_i=g \text{ and } |I|=k\Big\}\Big|$$

denote the number of subsequences T of S having sum $\sigma(T)=g$ and length $|T|=k$ (counted with the multiplicity of their appearance in S). In particular, $\mathsf N_0^0(S)=1$ and $\mathsf N_g^0(S)=0$ if $g\in G^\bullet$.

2. We define

$$\mathsf N_g(S)=\sum_{k\ge 0}\mathsf N_g^k(S)\,,\quad \mathsf N_g^+(S)=\sum_{k\ge 0}\mathsf N_g^{2k}(S) \quad\text{and}\quad \mathsf N_g^-(S)=\sum_{k\ge 0}\mathsf N_g^{2k+1}(S)\,.$$

Thus $\mathsf N_g(S)$ denotes the number of subsequences T of S having sum $\sigma(T)=g$, $\mathsf N_g^+(S)$ denotes the number of all such subsequences of even length, and $\mathsf N_g^-(S)$ denotes the number of all such subsequences of odd length (each counted with the multiplicity of its appearance in S).

We start with two results on p-groups. The first one (see [75]) sharpens results of J.E. Olson and I.Koutis (see [142, Theorem 1] and [127, Theorems 7, 8, 9 and 10]). It is proved via group algebras.

Theorem 8.2. *Let G be a p-group, $g\in G$, $k\in\mathbb N_0$ and $S\in\mathcal F(G)$ be a sequence of length $|S|>k\exp(G)+\mathsf d^*(G)$.*

1. $\mathsf N_g^+(S)\equiv\mathsf N_g^-(S) \mod p^{k+1}$.
2. *If $p=2$, then $\mathsf N_g(S)\equiv 0 \mod 2^{k+1}$.*

The next result (proved in [73]) is based on Theorem 8.2.

Theorem 8.3. *Let G be a p-group and $S \in \mathcal{F}(G)$ be a sequence of length $|S| \in [|G| + \mathsf{d}(G), 2|G| - 1]$. Then*

$$\mathsf{N}_g^{|G|}(S) \equiv \begin{cases} 0 \mod p & \text{if } g \in G^\bullet, \\ 1 \mod p & \text{if } g = 0. \end{cases}$$

An easy argument shows that in an elementary 2-group we have $\mathsf{N}_0(S) = \mathsf{N}_g(S)$ for every $S \in \mathcal{F}(G)$ and every $g \in \Sigma(S)$ (see [75, Proposition 3.3]). For more enumeration results in $G = C_p$ see [64], and in $G = C_p \oplus C_p$ see [96, Theorems 5.8.1 and 5.8.2]).

We continue with some results of the following type: A sequence $S \in \mathcal{F}(G)$, for which $|S|$ is long and $|\Sigma(S)|$ is small, has a very special form. The first result is due to J.E. Olson ([148, Theorems 1 and 2]).

Theorem 8.4. *Let $S \in \mathcal{F}(G^\bullet)$ be a sequence of length $|S| = |G|$. If $\mathsf{N}_0(S) < |G|$, then G is cyclic and $S = g^{|G|}$ for some $g \in G^\bullet$.*

For cyclic groups there are the following two sharper results: For Theorem 8.5 see [67, Theorem 1] (note that there is a misprint in the formulation of Theorem 1), and for Theorem 8.6 see [67, Theorems 2, 3 and 4].

Theorem 8.5. *Let G be cyclic of order $n \ge 2$, $k \in [1, \lfloor n/4 \rfloor + 1]$ and $S \in \mathcal{F}(G)$. If $\mathsf{N}_0(S) < 2^{|S|-n+k+1}$, then there exists some $g \in G$ with $\operatorname{ord}(g) = n$ such that*

$$S \;=\; g^u(-g)^v(x_1g) \cdot \ldots \cdot (x_{k-1}g)(y_1g) \cdot \ldots \cdot (y_lg)$$

where $u \ge v \ge 0$, $u + v = n - 2k + 1$, $y_i \in [0, n-1]$ for all $i \in [1, l]$, $x_i \in [1, n-1]$ for all $i \in [1, k-1]$ and $\sum_{x_i \le n/2} x_i + \sum_{x_i > n/2}(n - x_i) \le 2k - 2$.

Theorem 8.6. *Let G be cyclic of order $n \ge 22$ and $S \in \mathcal{F}(G^\bullet)$ be a sequence of length $|S| = n - 1$. If $\mathsf{N}_0(S) \le n$, then there exists some $g \in G$ with $\operatorname{ord}(g) = n$ such that S has one of the following forms:*

$$(-g)g^{n-2},\ (2g)(-g)g^{n-3},\ (3g)(-g)g^{n-3},\ (2g)^2(-g)g^{n-4},\ g^{n-1},\ (2g)g^{n-2},\ (3g)g^{n-2},\ (2g)^2g^{n-3}.$$

The next result deals with the number of zero-sum subsequences of length $\exp(G)$ in cyclic groups (see [68]).

Theorem 8.7. *Let G be cyclic of order $n \ge 2$ and $S \in \mathcal{F}(G)$ be a sequence of length $|S| = 2n - 1$.*

1. *For every $g \in G^\bullet$ we have $\mathsf{N}_g^n(S) = 0$ or $\mathsf{N}_g^n(S) \ge n$.*
2. *$\mathsf{N}_0^n(S) \ge n + 1$ or $S = a^n b^{n-1}$ for some $a, b \in G$ with $\operatorname{ord}(a - b) = n$.*

The following examples show that the inequalities in Theorem 8.7 cannot be improved. Let $g \in G$ with $\operatorname{ord}(g) = n$. If

$$S = 0^{n-1}g^{n-1}(-g), \quad \text{then} \quad \mathsf{N}_{-g}^n(S) = n,$$

and if

$$S = 0^{n+1}g^{n-2}, \quad \text{then} \quad \mathsf{N}_0^n(S) = n + 1.$$

A problem related to Theorem 8.7 on $\mathsf{N}_0^n(S)$ is the following conjecture formulated by A. Bialostocki and M. Lotspeich ([125], [55]):

Conjecture 8.8. *Let G be cyclic of order $n \ge 2$ and $S \in \mathcal{F}(G)$. Then*

$$\mathsf{N}_0^n(S) \ge \binom{\lfloor |S|/2 \rfloor}{n} + \binom{\lceil |S|/2 \rceil}{n}.$$

Z. Füredi and D. Kleitman, M. Kisin, W. Gao and D.J. Grynkiewicz gave partial positive answers to the above conjecture.

Theorem 8.9. *Conjecture 8.8 holds true in each of the following cases:*

1. [55] $n = p^a q^b$ *where* p, q *are distinct primes,* $a \in \mathbb{N}$ *and* $b \in \{0, 1\}$.
2. [125] $|S| \ge n^{6n}$.
3. [68] $|S| < 5n/2$.
4. [102] $|S| \le 19n/3$.

The next result (see [84]) settles a conjecture of B. Bollobás and I. Leader (see [16]).

Theorem 8.10. *Let* $S \in \mathcal{F}(G)$ *be a sequence. If* $0 \notin \Sigma_{|G|}(S)$*, then there is a zero-sumfree sequence* $T \in \mathcal{F}(G)$ *of length* $|T| = |S| - |G| + 1$ *such that* $|\Sigma_{|G|}(S)| \ge |\Sigma(T)|$.

We conclude with an explicit formula for the number of all zero-sum sequences of given length, which was recently derived by V. Ponomarenko (see [153]).

Theorem 8.11. *Let* G *be cyclic of order* $n \ge 10$ *and* $k > 2n/3$*. Then*

$$\bigl|\bigl\{S \in \mathcal{A}(G) \bigm| |S| = k\bigr\}\bigr| \;=\; \varphi(n)\mathsf{p}_k(n)\,,$$

where φ *is Euler's Phi Function and* $\mathsf{p}_k(n)$ *denotes the number of partitions of* n *into* k *parts.*

9. Weighted sequences and the cross number

We start with a recent result due to D.J. Grynkiewicz (see [103, Theorem 1.1]) which may be considered as a weighted version of the Theorem of Erdős-Ginzburg-Ziv (the case where G is cyclic, $k = |G|$ and $w_1 = \ldots = w_k = 1$ gives the classical result). It completely affirms a conjecture of Y. Caro formulated in 1996 (see [23, Conjecture 2.4]. Special cases were settled by N. Alon, A. Bialostocki and Y. Caro ([23]), by W. Gao and X. Jin ([83]) and by Y. ould Hamidoune ([112, Theorem 2.1]).

Theorem 9.1. *Let* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| = |G| + k - 1$*, for some* $k \ge 2$*, and* $(w_1, \ldots, w_k) \in \mathbb{Z}^k$ *a* k*-tuple of integers such that* $w_1 + \ldots + w_k \equiv 0 \mod \exp(G)$*. Then* S *has a subsequence* $T = g_1 \cdot \ldots \cdot g_k$ *such that* $w_1 g_1 + \ldots + w_k g_k = 0$.

We continue with a result by Y. ould Hamidoune ([112, Theorem 3.2] which implies Theorem 6.8 (for results of a similar flavor see [110], [34], [116]).

Theorem 9.2. *Let* $S \in \mathcal{F}(G)$ *be a sequence of length* $|S| = \mathsf{D}(G) + k$ *with* $k \ge |G| - 1$ *and let* $g \in G$ *with* $\mathsf{v}_g(S) = \mathsf{h}(S)$*. Then* S *has a subsequence* T *of length* $|T| = k$ *such that* $\sigma(T) = kg$.

Next we discuss the cross number of a finite abelian group. It was introduced by U. Krause (see [128], [129]), and its relevance stems from the theory of non-unique factorizations (see [157] and [96, Chapter 6]).

Definition 9.3. The invariant

$$\mathsf{K}(G) = \max\bigl\{\mathsf{k}(S) \bigm| S \in \mathcal{A}(G)\bigr\}$$

is called the *cross number* of G and

$$\mathsf{k}(G) = \max\bigl\{\mathsf{k}(S) \bigm| S \in \mathcal{F}(G) \text{ is zero-sumfree}\bigr\}$$

is called the *little cross number* of G.

If $\exp(G) = n$ and q is the smallest prime divisor of n, then a straightforward argument (see [96, Proposition 5.1.8]) shows that

$$\frac{1}{n} + \mathsf{k}^*(G) \le \frac{1}{n} + \mathsf{k}(G) \le \mathsf{K}(G) \le \frac{1}{q} + \mathsf{k}(G)\,.$$

Conjecture 9.4. $\frac{1}{n} + \mathsf{k}^*(G) = \mathsf{K}(G)$.

Conjecture 9.4 has been verified for p-groups and various other classes of groups (see [96, Theorem 5.5.9 and Section 5.7]).

Theorem 9.5.

1. $1 + n\mathsf{k}(G)$ *is the smallest integer* $l \in \mathbb{N}$ *such that every sequence* $S \in \mathcal{F}(G)$ *with* $n\mathsf{k}(S) \ge l$ *has a non-empty zero-sum subsequence.*
2. *Every sequence* $S \in \mathcal{F}(G)$ *of length* $|S| \ge |G|$ *has a non-empty zero-sum subsequence* T *with* $\mathsf{k}(T) \le 1$.

Whereas Theorem 9.5.1 is straightforward, Theorem 9.5.2 settles a conjecture of D. Kleitman and P. Lemke (see [126], [95], and [42] for a recent graph theoretical approach). For more information on the cross number we refer to [99], [28], [100], [101], [7].

10. On the Olson constant, the critical number and some analogues

We summarize some basic relationships of the invariants introduced in Definition 2.2. Note that $\max\big\{|U| \bigm| U \in \mathcal{A}(G) \text{ squarefree}\big\}$ is called the *strong Davenport constant* of G (see [51, 26, 27, 150]).

Lemma 10.1.

1. $1 + \mathsf{ol}(G) = \mathsf{Ol}(G) \le \mathsf{g}(G) \le |G| + 1$.
2. $\mathsf{g}(G) = |G| + 1$ *if and only if* G *is either cyclic of even order or an elementary* 2*-group.*
3. $1 + \max\big\{|S| \bigm| S \in \mathcal{F}(G) \text{ squarefree}, \Sigma(S) = G^\bullet\big\} \le \mathsf{Ol}(G) \le \min\{\mathsf{D}(G), \mathsf{cr}(G)\}$.
4. $\max\big\{|\mathrm{supp}(U)| \bigm| U \in \mathcal{A}(G)\big\} = \max\big\{|U| \bigm| U \in \mathcal{A}(G) \text{ squarefree}\big\} \le \mathsf{Ol}(G)$.
5. *If* $\mathsf{f}(G, l) \ge 1 + c^{-2} l^2$ *for some* $l \in \mathbb{N}$ *and* $c \in \mathbb{R}_{>0}$, *then* $\mathsf{ol}(G) < c\sqrt{|G| - 1}\,$.

Proof. We show the upper bound on $\mathsf{g}(G)$ and 2. A proof of 4. may be found in [26, Theorem 7]), and the remaining assertions follow either by the very definitions or by [96, Lemma 5.1.17]. Since there are no squarefree sequences $S \in \mathcal{F}(G)$ of length $|S| \ge |G| + 1$, every such sequence has a zero-sum subsequence of length $|T| = \exp(G)$ whence $\mathsf{g}(G) \le |G| + 1$. If G is cyclic of even order or an elementary 2-group, then the squarefree sequence $S \in \mathcal{F}(G)$ consisting of all group elements has no zero-sum subsequence T of length $|T| = \exp(G)$ whence $\mathsf{g}(G) > |G|$. Suppose that $G = H \oplus \langle g \rangle$ with some (possibly trivial) subgroup $H \subset G$ and some $g \in G$ with $\mathrm{ord}(g) = \exp(G) = n \ge 3$. We have to show that the squarefree sequence $S \in \mathcal{F}(G)$ consisting of all group elements has a zero-sum subsequence T of length $|T| = n$. If n is odd, then $T = g(2g) \cdot \ldots \cdot (ng)$ has the required property. If n is even and $h \in H \setminus \{0\}$, then $T = g(2g) \cdot \ldots \cdot \big((n-2)g\big)\big(h + (n-1)g\big)\big(-h + (n/2)g\big)$ has the required property. $\square$

We start with the g-invariant which was first studied by H. Harborth and A. Kemnitz (see [118, 124]). Let $G = C_n \oplus C_n$ with $n \ge 3$ and let (e_1, e_2) be a basis of G. If n is odd, then

$$S = \prod_{i=0}^{n-2} (ie_2) \prod_{i=1}^{n-1} (e_1 + ie_2) \in \mathcal{F}(G)$$

is a squarefree sequence of length $|S| = 2n - 2$ which has no zero-sum subsequence of length n whence $\mathsf{g}(G) \ge 2n - 1$. If n is even, then

$$S = \prod_{i=0}^{n-1} (ie_2) \prod_{i=0}^{n-1} (e_1 + ie_2) \in \mathcal{F}(G)$$

is a squarefree sequence of length $|S| = 2n$ which has no zero-sum subsequence of length n whence $\mathsf{g}(G) \ge 2n + 1$.

Conjecture 10.2. *Let* $G = C_n \oplus C_n$ *with* $n \ge 3$. *Then*

$$\mathsf{g}(G) = \begin{cases} 2n - 1 & \text{if} \quad n \quad \text{is odd,} \\ 2n + 1 & \text{if} \quad n \quad \text{is even.} \end{cases}$$

Conjecture 10.2 holds true for some small integers and for all primes $p \ge 67$ (see [92]).

We continue with the Olson constant. For some basic bounds for the f-invariant (and hence for the Olson constant) we refer to [96, Section 5.3]. Proving a conjecture of P. Erdős and H. Heilbronn, E. Szemerédi [165] showed that there is some $c \in \mathbb{R}_{>0}$ (not depending on the group) such that $\mathsf{Ol}(G) \le c\sqrt{|G|}$. J.E. Olson [144] proved the result for $c = 3$. The following result is due to Y. ould Hamidoune and G. Zémor [117, Theorems 3.3 and 4.5].

Theorem 10.3.

1. *If* G *is prime cyclic, then* $\mathsf{Ol}(G) \le \sqrt{2|G|} + 5\log(|G|)$.
2. $\mathsf{Ol}(G) \le \sqrt{2|G|} + \varepsilon(|G|)$ *for some real-valued function* ε *with* $\varepsilon(x) = O(x^{1/3}\log x)$.

The result for prime cyclic groups is essentially the best possible. However, the situation is completely different for non-cyclic groups. We have $\mathsf{ol}(G) \le \mathsf{d}(G)$, and obviously equality holds for elementary 2-groups, and by [162] also for elementary 3-groups. In the following theorem we summarize two results. The first one (see [77, Theorem 7.3]) shows in particular that in p-groups of large rank we have $\mathsf{ol}(G) = \mathsf{d}(G)$ (which is in contrast to the situation in $C_p \oplus C_p$, see Theorem 4.4). The second result was recently achieved in [89].

Theorem 10.4.

1. *Let* $G = H \oplus C_n^{s+1}$ *where* $\exp(G) = n \ge 2$, $s \in \mathbb{N}_0$, $H \subset G$ *a (possibly trivial) subgroup and* $\exp(H)$ *a proper divisor of* n. *If* $\mathsf{r}(H) + s/2 \ge n$, *then* $1 + \mathsf{d}^*(G) \le \max\{|U| \mid U \in \mathcal{A}(G) \ \text{squarefree}\}$.
2. $\mathsf{Ol}(C_p \oplus C_p) = \mathsf{Ol}(C_p) + p - 1$ *for all primes* $p > 4 \cdot 67 \times 10^{34}$.

Let $G = H \oplus C_n = H \oplus \langle e \rangle$ where $H \subset G$ is a subgroup with $|H| \ge n-1$ and $e \in G$ with $\mathrm{ord}(e) = n$. If $T \in \mathcal{F}(H)$ is a squarefree zero-sumfree sequence of length $|T| = \mathsf{ol}(G)$ and $h_1, \ldots, h_{n-1} \in H$ are pairwise distinct, then

$$S \;=\; T \prod_{i=1}^{n-1} (e + h_i) \in \mathcal{F}(G)$$

is a squarefree zero-sumfree sequence of length $|S| = |T| + n - 1$ whence $\mathsf{ol}(G) \ge \mathsf{ol}(H) + n - 1$. Let n be a prime power. Assume to the contrary that $\mathsf{Ol}(C_n^r) = \mathsf{Ol}(C_n^{r-1}) + n - 1$ for all $r \ge 2$. Then Theorem 10.4.1 implies that $\mathsf{Ol}(C_n) = \mathsf{D}(C_n)$, a contradiction. Thus there exists some $r \ge 2$ such that $\mathsf{Ol}(C_n^r) > \mathsf{Ol}(C_n^{r-1}) + n - 1$.

Finally we discuss the critical number $\mathsf{cr}(G)$ of G. It was first studied by P. Erdős and H. Heilbronn (see [48, Theorem I]) for cyclic groups of prime order, and in the sequel this problem found a lot of attention (see [137], [38], [37], [138], [152], [30], [33], [133], [86], [114]). Following [87] (where the inverse problem associated to the critical number is studied) we summarize what is known on $\mathsf{cr}(G)$.

Theorem 10.5. *Let q denote the smallest prime divisor of $\exp(G)$.*

1. *Suppose that $|G| = q$. Then $\mathsf{cr}(G) \le \lfloor \sqrt{4q-7} \rfloor$, and equality holds if the upper bound is odd (see [33, Example 4.2]).*
2. *Suppose that $|G|/q$ is prime.*
   (a) $\mathsf{cr}(C_2 \oplus C_2) = 3$*, and if q is odd, then* $\mathsf{cr}(C_q \oplus C_q) = 2q - 2$.
   (b) $|G|/q + q - 2 \le \mathsf{cr}(G) \le |G|/q + q - 1$.
3. *Suppose that $|G|/q$ is composite. We have $\mathsf{cr}(C_8) = \mathsf{cr}(C_2 \oplus C_4) = 5$, and otherwise*

$$\mathsf{cr}(G) = \frac{|G|}{q} + q - 2\,.$$

C. Peng (see [151], [152], [66]) investigated the following variant of the critical number. He studied the smallest integer $l \in \mathbb{N}_0$ with the following property: Every sequence $S \in \mathcal{F}(G^\bullet)$ of length $|S| \ge l$ and with $|\mathrm{supp}(S) \cap H| \le |H| - 1$ for all proper subgroups $H \subset G$, satisfies $\Sigma(S) = G$.

Van H. Vu (see [168]) showed the existence of a constant C with the following property: If G is a sufficiently large cyclic group and $S \in \mathcal{F}(G)$ a squarefree sequence with $\mathrm{supp}(S) \subset \{g \in G \mid \mathrm{ord}(g) = |G|\}$ and $|S| \ge C\sqrt{|G|}$, then $\Sigma(S) = G^\bullet$.

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Note added in proof: When this article went to press in June 2006, we were informed on the following progress:

- S. Savchev and F. Chen announced an improvement of Theorem 4.2
- D.J. Grynkiewiecz, O. Ordaz, M.T. Varela and F. Villarroel announced progress on Conjectures 6.9 and 7.6.

References

[1] W.R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmicheal numbers*, Ann. Math. **140** (1994), 703 – 722.
[2] N. Alon, *Tools from higher algebra*, Handbook of Combinatorics, vol. 2, North Holland, 1995, pp. 1749 – 1783.
[3] ———, *Combinatorial Nullstellensatz*, Comb. Probab. and Comput. **8** (1999), 7 – 29.
[4] N. Alon, S. Friedland, and G. Kalai, *Regular subgraphs of almost regular graphs*, J. Comb. Theory, Ser. B **37** (1984), 79 – 91.
[5] P.C. Baayen, *Een combinatorisch problem voor eindige abelse groepen*, MC Syllabus 5, Colloquium Discrete Wiskunde, Math. Centre, Amsterdam, 1968.
[6] ———, $C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}$!, Reports ZW-1969-006, Math. Centre, Amsterdam, 1969.
[7] P. Baginski, S.T. Chapman, K. McDonald, and L. Pudwell, *On cross numbers of minimal zero sequences in certain cyclic groups*, Ars Comb. **70** (2004), 47 – 60.
[8] R.C. Baker and W. Schmidt, *Diophantine problems in variables restricted to the values of* 0 *and* 1, J. Number Theory **12** (1980), 460 – 486.
[9] E. Balandraud, *Un nouveau point de vue isopérimétrique appliqué au théorème de Kneser*, manuscript.
[10] P. Balister, Y. Caro, C. Rousseau, and R. Yuster, *Zero-sum square matrices*, Eur. J. Comb. **23** (2002), 489 – 497.
[11] A. Bialostocki, G. Bialostocki, Y. Caro, and R. Yuster, *Zero-sum ascending waves*, J. Comb. Math. Comb. Comput. **32** (2000), 103 – 114.
[12] A. Bialostocki and P. Dierker, *Zero sum Ramsey theorems*, Congr. Numerantium **70** (1990), 119 – 130.
[13] ———, *On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math. **110** (1992), 1 – 8.
[14] A. Bialostocki, P. Dierker, D. Grynkiewicz, and M. Lotspeich, *On some developments of the Erdős-Ginzburg-Ziv Theorem II*, Acta Arith. **110** (2003), 173 – 184.
[15] A. Bialostocki and M. Lotspeich, *Some developments of the Erdős-Ginzburg-Ziv Theorem*, Sets, Graphs and Numbers, vol. 60, Coll. Math. Soc. J. Bolyai, 1992, pp. 97 – 117.
[16] B. Bollobás and I. Leader, *The number of k-sums modulo k*, J. Number Theory **78** (1999), 27 – 35.
[17] J.D. Bovey, P. Erdős, and I. Niven, *Conditions for zero sum modulo n*, Can. Math. Bull. **18** (1975), 27 – 29.

[18] W. Brakemeier, *Eine Anzahlformel von Zahlen modulo n*, Monatsh. Math. **85** (1978), 277 – 282.

[19] J. Brüdern and H. Godinho, *On Artin's conjecture. II. Pairs of additive forms*, Proc. Lond. Math. Soc. **84** (2002), 513 – 538.

[20] Y. Caro, *Zero-sum Ramsey numbers-stars*, Discrete Math. **104** (1992), 1 – 6.

[21] ———, *Zero-sum subsequences in abelian non-cyclic groups*, Isr. J. Math. **92** (1995), 221 – 233.

[22] ———, *Remarks on a zero-sum theorem*, J. Comb. Theory, Ser. A **76** (1996), 315 – 322.

[23] ———, *Zero-sum problems - a survey*, Discrete Math. **152** (1996), 93 – 113.

[24] ———, *Problems in zero-sum combinatorics*, J. London Math. Soc. **55** (1997), 427 – 434.

[25] S.T. Chapman, M. Freeze, W. Gao, and W.W. Smith, *On Davenport's constant of finite abelian groups*, Far East J. Math. Sci. **2** (2002), 47 – 54.

[26] S.T. Chapman, M. Freeze, and W.W. Smith, *Minimal zero sequences and the strong Davenport constant*, Discrete Math. **203** (1999), 271 – 277.

[27] ———, *Equivalence classes of minimal zero-sequences modulo a prime*, Ideal Theoretic Methods in Commutative Algebra, Lect. Notes Pure Appl. Math., vol. 220, Marcel Dekker, 2001, pp. 133 – 145.

[28] S.T. Chapman and A. Geroldinger, *On cross numbers of minimal zero sequences*, Australas. J. Comb. **14** (1996), 85 – 92.

[29] S.T. Chapman and W.W. Smith, *A characterization of minimal zero-sequences of index one in finite cyclic groups*, Integers **5(1)** (2005), Paper A27, 5p.

[30] G. Chiaselotti, *Sums of distinct elements in finite abelian groups*, Boll. Unione Mat. Ital. **7** (1993), 243 – 251.

[31] J.A. Dias da Silva, *Linear algebra and additive theory*, Unusual Applications of Number Theory, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 64, Am. Math. Soc., 2004, pp. 61 – 69.

[32] J.A. Dias da Silva and H. Godinho, *Generalized derivatives and additive theory*, Linear Algebra Appl. **342** (2002), 1 – 15.

[33] J.A. Dias da Silva and Y. ould Hamidoune, *Cyclic spaces for Grassmann derivatives and additive theory*, Bull. Lond. Math. Soc. **26** (1994), 140 – 146.

[34] C. Delorme, I. Marquez, O. Ordaz, and A. Ortuño, *Existence conditions for barycentric sequences*, Discrete Math. **281** (2004), 163 – 172.

[35] C. Delorme, O. Ordaz, and D. Quiroz, *Some remarks on Davenport constant*, Discrete Math. **237** (2001), 119 – 128.

[36] G.T. Diderrich, *On Kneser's addition theorem in groups*, Proc. Am. Math. Soc. **38** (1973), 443 – 451.

[37] ———, *An addition theorem for abelian groups of order pq*, J. Number Theory **7** (1975), 33 – 48.

[38] G.T. Diderrich and H.B. Mann, *Combinatorial problems in finite abelian groups*, A Survey of Combinatorial Theory, North-Holland, 1973, pp. 95 – 100.

[39] V. Dimitrov, *On the strong Davenport constant of nonabelian finite p-groups*, Math. Balk. **18** (2004), 131 – 140.

[40] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, and L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, manuscript.

[41] R.B. Eggleton and P. Erdős, *Two combinatorial problems in group theory*, Acta Arith. **21** (1972), 111 – 116.

[42] S. Elledge and G.H. Hurlbert, *An application of graph pebbling to zero-sum sequences in abelian groups*, Integers **5(1)** (2005), Paper A17, 10p.

[43] C. Elsholtz, *Lower bounds for multidimensional zero sums*, Combinatorica **24** (2004), 351 – 358.

[44] P. van Emde Boas, *A combinatorial problem on finite abelian groups II*, Reports ZW-1969-007, Math. Centre, Amsterdam, 1969.

[45] P. van Emde Boas and D. Kruyswijk, *A combinatorial problem on finite abelian groups*, Reports ZW-1967-009, Math. Centre, Amsterdam, 1967.

[46] ———, *A combinatorial problem on finite abelian groups III*, Reports ZW-1969-008, Math. Centre, Amsterdam, 1969.

[47] P. Erdős, A. Ginzburg, and A. Ziv, *Theorem in the additive number theory*, Bull. Research Council Israel **10** (1961), 41 – 43.

[48] P. Erdős and H. Heilbronn, *On the addition of residue classes modulo p*, Acta Arith. **9** (1964), 149 – 159.

[49] B.W. Finklea, T. Moore, V. Ponomarenko, and Z.J. Turner, *On block monoid atomic structure*, manuscript.

[50] C. Flores and O. Ordaz, *On the Erdős-Ginzburg-Ziv theorem*, Discrete Math. **152** (1996), 321 – 324.

[51] M. Freeze, *Lengths of factorizations in Dedekind domains*, Ph.D. thesis, University of North Carolina at Chapel Hill, 1999.

[52] M. Freeze and W.W. Smith, *Sumsets of zerofree sequences*, Arab. J. Sci. Eng. Sect. C Theme Issues **26** (2001), 97 – 105.

[53] G.A. Freiman, *Foundations of a Structural Theory of Set Addition*, Translations of Mathematical Monographs, vol. 37, American Mathematical Society, 1973.

[54] ———, *Structure theory of set addition*, Structure Theory of Set Addition, vol. 258, Astérisque, 1999, pp. 1 – 21.

[55] Z. Füredi and D.J. Kleitman, *The minimal number of zero sums*, Combinatorics, Paul Erdős is Eighty, vol. 1, J. Bolyai Math. Soc., 1993, pp. 159 – 172.

[56] L. Gallardo and G. Grekos, *On Brakemeier's variant of the Erdős-Ginzburg-Ziv problem*, Tatra Mt. Math. Publ. **20** (2000), 91 – 98.

[57] L. Gallardo, G. Grekos, L. Habsieger, F. Hennecart, B. Landreau, and A. Plagne, *Restricted addition in $\mathbb{Z}/n\mathbb{Z}$ and an application to the Erdős-Ginzburg-Ziv problem*, J. Lond. Math. Soc. **65** (2002), 513 – 523.

[58] L. Gallardo, G. Grekos, and J. Pihko, *On a variant of the Erdős-Ginzburg-Ziv problem*, Acta Arith. **89** (1999), 331 – 336.

[59] W. Gao, *Subsequence sums in finite cyclic groups*, manuscript.

[60] ______, *Some problems in additive group theory and number theory*, Ph.D. thesis, Sichuan University, Sichuan, P.R. China, 1994.

[61] ______, *Addition theorems for finite abelian groups*, J. Number Theory **53** (1995), 241 – 246.

[62] ______, *A combinatorial problem on finite abelian groups*, J. Number Theory **58** (1995), 100 – 103.

[63] ______, *An improvement of Erdős-Ginzburg-Ziv theorem*, Acta Math. Sin. **39** (1996), 514 – 523.

[64] ______, *Two addition theorems on groups of prime order*, J. Number Theory **56** (1996), 211 – 213.

[65] ______, *An addition theorem for finite cyclic groups*, Discrete Math. **163** (1997), 257 – 265.

[66] ______, *Addition theorems and group rings*, J. Comb. Theory, Ser. A **77** (1997), 98 – 109.

[67] ______, *On the number of zero sum subsequences*, Discrete Math. **163** (1997), 267 – 273.

[68] ______, *On the number of subsequences with given sum*, Discrete Math. **195** (1999), 127 – 138.

[69] ______, *On Davenport's constant of finite abelian groups with rank three*, Discrete Math. **222** (2000), 111 – 124.

[70] ______, *Two zero sum problems and multiple properties*, J. Number Theory **81** (2000), 254 – 265.

[71] ______, *Zero sums in finite cyclic groups*, Integers **0** (2000), Paper A14, 9p.

[72] ______, *On zero sum subsequences of restricted size III*, Ars Comb. **61** (2001), 65 – 72.

[73] ______, *On zero sum subsequences of restricted size II*, Discrete Math. **271** (2003), 51 – 59.

[74] W. Gao and A. Geroldinger, *Group algebras of finite abelian groups and their applications to combinatorial problems*, manuscript.

[75] ______, *On the number of subsequences with given sum of sequences over finite abelian p-groups*, Rocky Mt. J. Math., to appear.

[76] ______, *On the structure of zerofree sequences*, Combinatorica **18** (1998), 519 – 527.

[77] ______, *On long minimal zero sequences in finite abelian groups*, Period. Math. Hung. **38** (1999), 179 – 211.

[78] ______, *On the order of elements in long minimal zero-sum sequences*, Period. Math. Hung. **44** (2002), 63 – 73.

[79] ______, *On zero-sum sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$*, Integers **3** (2003), Paper A08, 45p.

[80] ______, *Zero-sum problems and coverings by proper cosets*, Eur. J. Comb. **24** (2003), 531 – 549.

[81] ______, *On a property of minimal zero-sum sequences and restricted sumsets*, Bull. Lond. Math. Soc. **37** (2005), 321 – 334.

[82] W. Gao, Q.H. Hou, W.A. Schmid, and R. Thangadurai, *On short zero-sum subsequences II*, manuscript.

[83] W. Gao and X. Jin, *Weighted sums in finite cyclic groups*, Discrete Math. **283** (2004), 243 – 247.

[84] W. Gao and I. Leader, *Sums and k-sums in abelian groups of order k*, J. Number Theory, to appear.

[85] W. Gao and Y. ould Hamidoune, *Zero sums in abelian groups*, Combin. Probab. Comput. **7** (1998), 261 – 263.

[86] ______, *On additive bases*, Acta Arith. **88** (1999), 233 – 237.

[87] W. Gao, Y. ould Hamidoune, A. Llado, and O. Serra, *Covering a finite abelian group by subset sums*, Combinatorica **23** (2003), 599 – 611.

[88] W. Gao, A. Panigrahi, and R. Thangadurai, *On the structure of p-zero-sum free sequences and its application to a variant of Erdős-Ginzburg-Ziv theorem*, Proc. Indian Acad. Sci., Math. Sci. **115** (2005), 67 – 77.

[89] W. Gao, I. Ruzsa, and R. Thangadurai, *Olson's constant for the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$*, J. Comb. Theory, Ser. A **107** (2004), 49 – 67.

[90] W. Gao and R. Thangadurai, *On zero-sum sequences of prescribed length*, Aequationes Math., to appear.

[91] ______, *On the structure of sequences with forbidden zero-sum subsequences*, Colloq. Math. **98** (2003), 213 – 222.

[92] ______, *A variant of Kemnitz conjecture*, J. Comb. Theory, Ser. A **107** (2004), 69 – 86.

[93] W. Gao, R. Thangadurai, and J. Zhuang, *Addition theorems on the cyclic groups $\mathbb{Z}_{p^n}$*, manuscript.

[94] W. Gao and J. Zhuang, *Sequences not containing long zero-sum subsequences*, Eur. J. Comb. **27** (2006), 777 – 787.

[95] A. Geroldinger, *On a conjecture of Kleitman and Lemke*, J. Number Theory **44** (1993), 60 – 65.

[96] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[97] A. Geroldinger and Y. ould Hamidoune, *Zero-sumfree sequences in cyclic groups and some arithmetical application*, J. Théor. Nombres Bordx. **14** (2002), 221 – 239.

[98] A. Geroldinger and R. Schneider, *On Davenport's constant*, J. Comb. Theory, Ser. A **61** (1992), 147 – 152.

[99] ______, *The cross number of finite abelian groups II*, Eur. J. Comb. **15** (1994), 399 – 405.

[100] ______, *The cross number of finite abelian groups III*, Discrete Math. **150** (1996), 123 – 130.

[101] ______, *On minimal zero sequences with large cross number*, Ars Comb. **46** (1997), 297 – 303.

[102] D.J. Grynkiewicz, *On the number of m-term zero-sum subsequences*, Acta Arith., to appear.

[103] ———, *A weighted version of the Erdős-Ginzburg-Ziv Theorem*, Combinatorica, to appear.
[104] ———, *On four colored sets with non-decreasing diameter and the Erdős-Ginzburg-Ziv Theorem*, J. Comb. Theory, Ser. A **100** (2002), 44 – 60.
[105] ———, *On a conjecture of Hamidoune for subsequence sums*, Integers **5(2)** (2005), Paper A07, 11p.
[106] ———, *On a partition analog of the Cauchy-Davenport Theorem*, Acta Math. Hung. **107** (2005), 161 – 174.
[107] ———, *On an extension of the Erdős-Ginzburg-Ziv Theorem to hypergraphs*, Eur. J. Comb. **26** (2005), 1154 – 1176.
[108] ———, *Quasi-periodic decompositions and the Kemperman structure theorem*, Eur. J. Comb. **26** (2005), 559 – 575.
[109] F. Halter-Koch, *A generalization of Davenport's constant and its arithmetical applications*, Colloq. Math. **63** (1992), 203 – 210.
[110] Y. ould Hamidoune, *On weighted sequence sums*, Comb. Probab. Comput. **4** (1995), 363 – 367.
[111] ———, *An isoperimetric method in additive theory*, J. Algebra **179** (1996), 622 – 630.
[112] ———, *On weighted sums in abelian groups*, Discrete Math. **162** (1996), 127 – 132.
[113] ———, *Subsequence sums*, Comb. Probab. Comput. **12** (2003), 413 – 425.
[114] Y. ould Hamidoune, A.S. Lladó, and O. Serra, *On sets with a small subset sum*, Comb. Probab. Comput. **8** (1999), 461 – 466.
[115] Y. ould Hamidoune, O. Ordaz, and A. Ortuño, *On a combinatorial theorem of Erdős, Ginzburg and Ziv*, Comb. Probab. Comput. **7** (1998), 403 – 412.
[116] Y. ould Hamidoune and D. Quiroz, *On subsequence weighted products*, Comb. Probab. Comput. **14** (2005), 485 – 489.
[117] Y. ould Hamidoune and G. Zémor, *On zero-free subset sums*, Acta Arith. **78** (1996), 143 – 152.
[118] H. Harborth, *Ein Extremalproblem für Gitterpunkte*, J. Reine Angew. Math. **262** (1973), 356 – 360.
[119] G. Harcos and I.Z. Ruzsa, *A problem on zero subsums in abelian groups*, Period. Math. Hung. **35** (1997), 31 – 34.
[120] F. Hennecart, *La fonction de Brakemeier dans le problème d'Erdős-Ginzburg-Ziv*, Acta Arith. **117** (2005), 35 – 50.
[121] ———, *Restricted addition and some developments of the Erdős-Ginzburg-Ziv theorem*, Bull. Lond. Math. Soc. **37** (2005), 481 – 490.
[122] G.H. Hurlbert, *Recent progress in graph pebbling*, Graph Theory Notes of New York, to appear.
[123] F. Kainrath, *On local half-factorial orders*, Arithmetical Properties of Commutative Rings and Monoids, Lect. Notes Pure Appl. Math., vol. 241, Chapman & Hall/CRC, 2005, pp. 316 – 324.
[124] A. Kemnitz, *On a lattice point problem*, Ars Comb. **16-B** (1983), 151 – 160.
[125] M. Kisin, *The number of zero sums modulo m in a sequence of length n*, Mathematika **41** (1994), 149 – 163.
[126] D. Kleitman and P. Lemke, *An addition theorem on the integers modulo n*, J. Number Theory **31** (1989), 335 – 345.
[127] I. Koutis, *Dimensionality restrictions on sums over $\mathbb{Z}_p^d$*, manuscript.
[128] U. Krause, *A characterization of algebraic number fields with cyclic class group of prime power order*, Math. Z. **186** (1984), 143 – 148.
[129] U. Krause and C. Zahlten, *Arithmetic in Krull monoids and the cross number of divisor class groups*, Mitt. Math. Ges. Hamb. **12** (1991), 681 – 696.
[130] G. Lettl and W.A. Schmid, *Minimal zero-sum sequences in $C_n \oplus C_n$*, Eur. J. Comb., to appear.
[131] G. Lettl and Zhi-Wei Sun, *On covers of abelian groups by cosets*, manuscript.
[132] V.F. Lev, *Restricted set addition in abelian groups: results and conjectures*, J. Théor. Nombres Bordx. **17** (2005), 181 – 193.
[133] E. Lipkin, *Subset sums of sets of residues*, Structure Theory of Set Addition, vol. 258, Astérisque, 1999, pp. 187 – 192.
[134] A.D. Lungo, *Reconstructing permutation matrices from diagonal sums*, Theor. Comput. Sci. **281** (2002), 235 – 249.
[135] H.B. Mann, *Additive group theory - a progress report*, Bull. Am. Math. Soc. **79** (1973), 1069 – 1075.
[136] ———, *Addition Theorems: The Addition Theorems of Group Theory and Number Theory*, R.E. Krieger, 1976.
[137] H.B. Mann and J.E. Olson, *Sums of sets in the elementary abelian group of type (p, p)*, J. Comb. Theory, Ser. A **2** (1967), 275 – 284.
[138] H.B. Mann and Ying Fou Wou, *An addition theorem for the elementary abelian group of type (p, p)*, Monatsh. Math. **102** (1986), 273 – 308.
[139] M. Mazur, *A note on the growth of Davenport's constant*, Manuscr. Math. **74** (1992), 229 – 235.
[140] R. Meshulam, *An uncertainty inequality and zero subsums*, Discr. Math. **84** (1990), 197 – 200.
[141] M.B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.
[142] J.E. Olson, *A combinatorial problem on finite abelian groups I*, J. Number Theory **1** (1969), 8 – 10.
[143] ———, *A combinatorial problem on finite abelian groups II*, J. Number Theory **1** (1969), 195 – 199.
[144] ———, *Sums of sets of group elements*, Acta Arith. **28** (1975), 147 – 156.
[145] ———, *On a combinatorial problem of Erdős, Ginzburg and Ziv*, J. Number Theory **8** (1976), 52 – 57.
[146] ———, *On the sum of two sets in a group*, J. Number Theory **18** (1984), 110 – 120.
[147] ———, *On the symmetric difference of two sets in a group*, Eur. J. Comb. **7** (1986), 43 – 54.
[148] ———, *A problem of Erdős on abelian groups*, Combinatorica **7** (1987), 285 – 289.

[149] J.E. Olson and E.T. White, *Sums from a sequence of group elements*, Number Theory and Algebra (H. Zassenhaus, ed.), Academic Press, 1977, pp. 215 – 222.
[150] O. Ordaz and D. Quiroz, *On zero-free sets*, Divulg. Mat. **14** (2006), 1 – 10.
[151] C. Peng, *Addition theorems in elementary abelian groups I*, J. Number Theory **27** (1987), 46 – 57.
[152] ______, *Addition theorems in elementary abelian groups II*, J. Number Theory **27** (1987), 58 – 62.
[153] V. Ponomarenko, *Minimal zero sequences of finite cyclic groups*, Integers **4** (2004), Paper A24, 6p.
[154] C. Reiher, *On Kemnitz' conjecture concerning lattice points in the plane*, Ramanujan J., to appear.
[155] S. Savchev and F. Chen, *Kemnitz' conjecture revisited*, Discrete Math. **297** (2005), 196 – 201.
[156] W.A. Schmid, *On zero-sum subsequences in finite abelian groups*, Integers **1** (2001), Paper A01, 8p.
[157] ______, *Half-factorial sets in finite abelian groups: a survey*, Grazer Math. Ber. **348** (2005), 41 – 64.
[158] W.A. Schmid and J.J. Zhuang, *On short zero-sum subsequences over p-groups*, Ars Comb., to appear.
[159] M. Skałba, *The relative Davenport's constant of the group* $\mathbb{Z}_n \times \mathbb{Z}_n$, Grazer Math. Ber. **318** (1992), 167 – 168.
[160] ______, *On numbers with a unique representation by a binary quadratic form*, Acta Arith. **64** (1993), 59 – 68.
[161] ______, *On the relative Davenport constant*, Eur. J. Comb. **19** (1998), 221 – 225.
[162] J. Subocz, *Some values of Olson's constant*, Divulg. Mat. **8** (2000), 121 – 128.
[163] Zhi-Wei Sun, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announc. Am. Math. Soc. **9** (2003), 51 – 60.
[164] B. Sury and R. Thangadurai, *Gao's conjecture on zero-sum sequences*, Proc. Indian Acad. Sci., Math. Sci. **112** (2002), 399 – 414.
[165] E. Szemerédi, *On a conjecture of Erdős and Heilbronn*, Acta Arith. **17** (1970), 227 – 229.
[166] R. Thangadurai, *Interplay between four conjectures on certain zero-sum problems*, Expo. Math. **20** (2002), 215 – 228.
[167] ______, *Non-canonical extensions of Erdős-Ginzburg-Ziv Theorem*, Integers **2** (2002), Paper A08, 14p.
[168] V.H. Vu, *Olson's Theorem for cyclic groups*, manuscript.
[169] T. Yuster, *Bounds for counter-examples to addition theorems in solvable groups*, Arch. Math. **51** (1988), 223 – 231.
[170] T. Yuster and B. Peterson, *A generalization of an addition theorem for solvable groups*, Can. J. Math. **36** (1984), 529 – 536.
[171] J. Zhuang and W. Gao, *Erdős-Ginzburg-Ziv theorem for dihedral groups of large prime index*, Eur. J. Comb. **26** (2005), 1053 – 1059.

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