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Reducing Hajós' 4-coloring conjecture to 4-connected graphs

Xingxing Yu [a,b,1], Florian Zickfeld [a,2]

[a] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[b] *Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, China*

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Abstract

Hajós conjectured that, for any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. This is true when $k \leqslant 3$, and false when $k \geqslant 6$. Hajós' conjecture remains open for $k = 4, 5$. In this paper, we show that any possible counterexample to this conjecture for $k = 4$ with minimum number of vertices must be 4-connected. This is a step in an attempt to reduce Hajós' conjecture for $k = 4$ to the conjecture of Seymour that any 5-connected non-planar graph contains a K_5-subdivision.

Keywords: Coloring; K_5-subdivision; Connectivity; Cycle

1. Introduction

Graphs considered in this paper are simple and finite. The Four Color Theorem states that every planar graph is 4-colorable. The Kuratowski Theorem states that a graph is planar if, and only if, it contains neither a K_5-subdivision nor a $K_{3,3}$-subdivision. Also, a graph is planar if, and only if, it contains neither a K_5-minor nor a $K_{3,3}$-minor. Based on these characterizations of planar graphs, there are two conjectures that would generalize the Four Color Theorem. One of these was attributed to Hajós (see [1]) which states that, for any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. The other is Hadwiger's conjecture [4]: For any positive integer k, every graph containing no K_{k+1}-minor is k-colorable. Both conjectures are

E-mail address: yu@math.gatech.edu (X. Yu).
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easily seen to be true when $k = 1, 2$. It is also not hard to show that both conjectures are true for $k = 3$.

Hadwiger's conjecture for $k = 4$ is equivalent to the Four Color Theorem [12]. Hadwiger's conjecture for $k = 5$ can also be reduced to the Four Color Theorem [9], and it remains open for $k \geqslant 6$.

On the other hand, Catlin [1] showed that Hajós' conjecture fails when $k \geqslant 6$. In fact, Erdös and Fajtlowicz [3] showed that Hajós' conjecture fails for almost all graphs. Recently, Thomassen [11] discovered more interesting counterexamples to Hajós' conjecture by studying its connections with Ramsey numbers, maximum cuts, and perfect graphs. Thomassen [11] also explored graph classes for which Hajós' conjecture may be true. Kühn and Osthus [6] proved that Hajós' conjecture holds for graphs with sufficiently large girth, and they later [7] improved the bound on girth to 27. However, Hajós' conjecture remains open for $k = 4$ and $k = 5$. It is therefore important to derive structural information about graphs containing no K_5-subdivisions (respectively K_6-subdivisions).

There has been considerable work concerning K_5-subdivisions. Dirac [2] conjectured that every simple graph on n vertices with at least $3n - 5$ edges contains a K_5-subdivision, which was proved by Mader [8]. However, the following conjecture of Seymour [10] remains open: Every 5-connected non-planar graph contains a K_5-subdivision. A result in [5] shows that Seymour's conjecture implies Dirac's conjecture. Our aim is to establish a connection between Hajós' conjecture and Seymour's conjecture by looking at the connectivity of a minimum counterexample to Hajós' conjecture. More specifically, if a counterexample to Hajós' conjecture is 5-connected then, by the Four Color Theorem, Seymour's conjecture implies Hajós' conjecture for $k = 4$.

For convenience, we say that a graph G is a *Hajós graph* if

(i) G is not 4-colorable,
(ii) G contains no K_5-subdivision, and
(iii) subject to (i) and (ii), $|V(G)|$ is minimum.

Note that any non-spanning subgraph of a Hajós graph is 4-colorable. The main result of this paper is the following.

Theorem 1.1. *Every Hajós graph is* 4*-connected.*

Let G be a graph. A *separation* of G is a pair (G_1, G_2) of edge disjoint subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_i) - V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$. (Note that our definition of a separation is different from the usual one in which $V(G_i) - V(G_{3-i}) \neq \emptyset$ is not required.) We call (G_1, G_2) a k*-separation* if $|V(G_1 \cap G_2)| = k$. A set $S \subseteq V(G)$ is a k*-cut* in G, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$.

To prove Theorem 1.1, we need to deal with k-cuts with $k \leqslant 3$. It is very easy to show that no Hajós graph admits k-cuts with $k \leqslant 2$. The main work is to show that no Hajós graph admits a 3-cut, for which we need to combine structural and coloring arguments. Suppose there is a Hajós graph G that admits a 3-cut. Choose a 3-separation (G_1, G_2) of G such that G_2 is minimal with respect to subgraph containment. We shall prove several lemmas showing that G_1 and G_2 admit certain 4-colorings. (This is done in Section 2.) Let G_i' denote the graph obtained from G_i by adding an edge between every pair of distinct vertices from $V(G_1 \cap G_2)$. We shall decide whether G_i' contains a K_5-subdivision. For this reason, we need to know whether G_{3-i}' has a cycle containing $V(G_1 \cap G_2)$. Therefore, we need the following reformulation of a result of Watkins and Mesner [13].

Theorem 1.2. *Let G be a 2-connected graph and let x, y, z be three distinct vertices of G. Then there is no cycle through x, y and z in G if, and only if, at least one of the following statements holds.*

(i) *There exists a 2-cut S in G and there exist three distinct components D_x, D_y, D_z of $G - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.*
(ii) *There exist a vertex v of G, 2-cuts S_x, S_y, S_z in G, and components D_u of $G - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}$, $S_y - \{v\}$, $S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.*
(iii) *There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G and components D_u of $G - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u, for all $u \in \{x, y, z\}$.*

We remark that in order to show that Hajós graphs are 5-connected, one needs to consider the much harder problem of characterizing graphs in which there is no K_4-subdivision at specified locations.

We conclude this section with some notation. Let G be a graph. For $A, B \subseteq V(G)$, an A–B path in G is a path in G which has one end in A and the other in B and is otherwise disjoint from $A \cup B$. If $A = \{x\}$, we speak of x–B path instead of $\{x\}$–B path, and if, in addition, $B = \{y\}$ then we write x–y path instead of $\{x\}$–$\{y\}$ path. Two paths in G are said to be *internally disjoint* if no internal vertex of one is contained in the other. For any two sets $A, B \subseteq V(G)$, we say that a set $S \subseteq V(G)$ *separates A from B* if there is a separation (G_1, G_2) of G with $V(G_1 \cap G_2) = S$, $A \subseteq V(G_1)$, $B \subseteq V(G_2)$, $A - S \neq \emptyset$, and $B - S \neq \emptyset$. If $A = \{v\}$ we simply say S separates v from B, and if $A = \{v\}$ and $B = \{w\}$ then we simply say that S separates v from w.

Let H be a subgraph of a graph G, let $v_1, \ldots, v_k \in V(G)$, and $\{u_i, w_i\} \subseteq V(H) \cup \{v_1, \ldots, v_k\}$, $i = 1, \ldots, m$. Then we let $H + \{v_1, \ldots, v_k, u_1w_1, \ldots, u_mw_m\}$ denote the graph with vertex set $V(H) \cup \{v_1, \ldots, v_k\}$ and edge set $E(H) \cup \{u_1w_1, \ldots, u_kw_k\}$.

Let G be a graph. For any $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. A path P in G is said to be a *branch path* in G if its internal vertices are of degree 2 in G and its ends are of degree at least 3 in G. Vertices of G with degree at least 3 are called *branch* vertices of G.

Given a graph G, we shall view a coloring of G as a mapping c from $V(G)$ to a set of colors such that $c(u) \neq c(v)$ whenever $uv \in E(G)$.

2. 3-Separations

We begin this section by stating an easy fact without proof.

Proposition 2.1. *Every Hajós graph is 3-connected.*

For the remainder of this section, we choose a 3-separation (G_1, G_2) of a Hajós graph such that G_2 is minimal. We shall show two results concerning certain 4-colorings of G_1. First, we need some structural information from G_2.

Lemma 2.2. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then*

(i) $|V(G_2)| \geqslant 5$,
(ii) $G_2 - V(G_1 \cap G_2)$ *is connected, and*
(iii) G_2 *is 2-connected.*

Proof. Suppose $|V(G_2)| \leqslant 4$. Then $|V(G_2)| = 4$. Let $v \in V(G_2) - V(G_1)$. Then by Proposition 2.1, v has degree 3 in G. Since $G - v$ does not contain a K_5-subdivision, $G - v$ is 4-colorable. Because the degree of v in G is 3, any 4-coloring of $G - v$ can easily be extended to v to give a 4-coloring of G, a contradiction. Thus $|V(G_2)| \geqslant 5$ and (i) holds.

Now suppose $G_2 - V(G_1 \cap G_2)$ is not connected. Let D denote a component of $G_2 - V(G_1 \cap G_2)$. Then there is a 3-separation (G_1', G_2') of G with $V(G_1' \cap G_2') = V(G_1 \cap G_2)$ and $G_2' - V(G_1 \cap G_2) = D$. This contradicts the choice of (G_1, G_2), for G_2' is properly contained in G_2. So (ii) holds.

By (ii), $G_2 - V(G_1 \cap G_2)$ is connected, and by Proposition 2.1, every vertex from $V(G_1 \cap G_2)$ has a neighbor in $V(G_2) - V(G_1 \cap G_2)$. So G_2 is connected, and any possible cut vertex of G_2 must be contained in $V(G_2) - V(G_1 \cap G_2)$. Suppose G_2 is not 2-connected, and let v denote a cut vertex of G_2. Then, $V(G_1 \cap G_2)$ cannot be contained in a component of $G_2 - v$, for otherwise v would be a cut vertex in G (contradicting Proposition 2.1). So we may assume that some vertex x from $V(G_1 \cap G_2)$ is contained in the component of $G_2 - v$ which does not contain any other vertex from $V(G_1 \cap G_2)$. Then, since G is 3-connected, v is the only neighbor of x in $V(G_2) - V(G_1 \cap G_2)$. Therefore, since $|V(G_2)| \geqslant 5$, $(V(G_1 \cap G_2) - \{x\}) \cup \{v\}$ is a cut in G, which yields a 3-separation (G_1', G_2') in which $G_2' = G_2 - x$ is a proper subgraph of G_2, a contradiction. Hence (iii) holds. □

Proposition 2.3. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Then there is a 4-coloring c_1 of G_1 such that $c_1(x)$, $c_1(y)$ and $c_1(z)$ are all distinct.*

Proof. Suppose this is not true, that is, $G_1' := G_1 + \{xy, yz, zx\}$ is not 4-colorable. Then since $|V(G_1')| < |V(G)|$, G_1' contains a K_5-subdivision, say Σ.

We claim that x, y, z are branch vertices of Σ. This is easy to see when $\{xy, xz, yz\} \subseteq E(\Sigma)$. So we may assume by symmetry that $yz \notin E(\Sigma)$. By (iii) of Lemma 2.2, there exist internally disjoint paths Y from x to y and Z from x to z in G_2. It is easy to see that $(\Sigma - \{xy, xz\}) \cup Y \cup Z$ (and hence G) contains a K_5-subdivision, a contradiction.

Note that if G_2 contains a cycle C through x, y, z, then $(\Sigma - \{xy, yz, zx\}) \cup C$ (and hence G) contains a K_5-subdivision, a contradiction. So there is no cycle through x, y, z in G_2. Hence by applying Theorem 1.2 to G_2, it suffices to consider the following three cases.

Case 1. There exists a 2-cut S in G_2 and there exist distinct components D_x, D_y, D_z of $G_2 - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.

Let $S = \{a, b\}$. Suppose $|V(D_x)| \geqslant 2$. Then $\{x, a, b\}$ is a 3-cut of G and $(G - V(D_x - x), G[V(D_x) \cup S])$ is a 3-separation of G. But $G[V(D_x) \cup S]$ is properly contained in G_2, contradicting the choice of (G_1, G_2). So $V(D_x) = \{x\}$. Similarly, we have $V(D_y) = \{y\}$, and $V(D_z) = \{z\}$. Hence, a and b are the only vertices of G not contained in G_1.

Since G_1 is a non-spanning subgraph of G, G_1 is 4-colorable. Let c_1 be a 4-coloring of G_1. If $c_1(x), c_1(y), c_1(z)$ are all distinct, then c_1 is a 4-coloring of G_1', a contradiction. So assume $c_1(x), c_1(y), c_1(z)$ are not all distinct. Define $c_1'(u) = c_1(u)$ for all $u \in V(G_1)$, and let $c_1'(a)$ and $c_1'(b)$ be two colors not in $\{c_1(x), c_1(y), c_1(z)\}$. Clearly, c_1' is a 4-coloring of G, a contradiction.

Case 2. There exist a vertex v of G_2, 2-cuts S_x, S_y, S_z in G_2, and components D_u of $G_2 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}$, $S_y - \{v\}$, $S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

As in Case 1, we can show that $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$.

Note that $G_1 + xy$ contains no K_5-subdivision. For otherwise, by replacing xy in such a K_5-subdivision with an x–y path in $G_2 - z$ (which exists by (iii) of Lemma 2.2), we produce a K_5-subdivision in G, a contradiction.

Thus, since $|V(G_1 + xy)| < |V(G)|$, $G_1 + xy$ is 4-colorable. Let c_1 be a 4-coloring of $G_1 + xy$. Then $c_1(x) \neq c_1(y)$. If $c_1(z) \neq c_1(x)$ and $c_1(z) \neq c_1(y)$, then c_1 is a 4-coloring of G_1', a contradiction. Therefore, we may assume (by symmetry between x and y) that $c_1(z) = c_1(y)$.

Since G_2 is a non-spanning subgraph of G, G_2 is 4-colorable. Let c_2 be a 4-coloring of G_2. As y and z together only has three neighbors in G_2, we may choose c_2 so that $c_2(y) = c_2(z)$. Since x has only two neighbors in G_2, we may further choose c_2 so that $c_2(x) \neq c_2(y)$. Now by permuting the colors of vertices of G_2, we may assume that $c_2(u) = c_1(u)$ for all $u \in \{x, y, z\}$. Let c be a coloring of G defined by letting $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then c is a 4-coloring of G, a contradiction.

Case 3. There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G_2 and components D_u of $G_2 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G_2 - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u, for all $u \in \{x, y, z\}$.

As in Case 1, we can show $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$.

Let $S_x := \{a_x, b_x\}$, $S_y := \{a_y, b_y\}$, and $S_z := \{a_z, b_z\}$, and assume that $\{a_x, a_y, a_z\}$ (respectively $\{b_x, b_y, b_z\}$) is contained in the component A (respectively B) of $G_2 - V(D_x \cup D_y \cup D_z)$. Then $|V(A)| = 3 = |V(B)|$; for otherwise, $(G - V(A - \{a_x, a_y, a_z\}), A)$ or $(G - V(B - \{b_x, b_y, b_z\}), B)$ is a 3-separation of G in which A or B is properly contained in G_2, contradicting the choice of (G_1, G_2).

Now $G_1 + \{xy, yz\}$ contains no K_5-subdivision. For otherwise, let Σ be a K_5-subdivision in $G_1 + \{xy, yz\}$. By (iii) of Lemma 2.2, there are internally disjoint paths X, Z from y to x, z, respectively, in G_2. Now $(\Sigma - \{xy, yz\}) \cup X \cup Z$ (and hence G) contains a K_5-subdivision, a contradiction.

Since $|V(G_1 + \{xy, yz\})| < |V(G)|$, $G_1 + \{xy, yz\}$ is 4-colorable. Let c_1 be a 4-coloring of $G_1 + \{xy, yz\}$. Then $c_1(x) \neq c_1(y) \neq c_1(z)$. If $c_1(x) \neq c_1(z)$, then G_1' is 4-colorable, a contradiction. So assume that $c_1(x) = c_1(z)$. For convenience, assume that the colors we use are $\{\alpha, \beta, \gamma, \delta\}$ and $c_1(x) = \alpha$ and $c_1(y) = \beta$. Let c be a coloring of G such that $c(u) = c_1(u)$ for all $u \in V(G_1)$, $c(a_x) = c(b_z) = \gamma$, $c(b_x) = c(a_z) = \beta$, $c(a_y) = \delta$, and $c(b_y) = \alpha$. It is easy to check that c is a 4-coloring of G, a contradiction. □

Proposition 2.4. *Let G be a Hajós graph, let (G_1, G_2) be a* 3*-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Suppose there is a vertex $x' \in V(G_1) - \{x, y, z\}$ separating x from $\{y, z\}$ in G_1. Then there exist* 4*-colorings c_1 and c_2 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$ and $c_2(x) = c_2(z) \neq c_2(y)$.*

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Proof. Note that $xy, xz \notin E(G_1)$ for otherwise x' would not separate x from $\{y, z\}$ in G_1. Also $xx' \in E(G)$, for otherwise $\{x, x'\}$ would be a 2-cut in G, contradicting Proposition 2.1. Let $G_1^* := (G_1 - x) + \{x'y, yz\}$.

We claim that G_1^* contains no K_5-subdivision. For otherwise, let Σ be a K_5-subdivision in G_1^*. Since G contains no K_5-subdivision, $\{x'y, yz\} \cap E(\Sigma) \neq \emptyset$. By (iii) of Lemma 2.2, we see that G_2 contains two internally disjoint paths X, Z from y to x, z, respectively. Now $(\Sigma - \{x'y, yz\}) \cup (X + \{x', xx'\}) \cup Z$, and hence G, contains a K_5-subdivision, a contradiction.

Therefore, since $|V(G_1^*)| < |V(G)|$, G_1^* is 4-colorable. Let c_1^* be a 4-coloring of G_1^*. Then $c_1^*(x') \neq c_1^*(y) \neq c_1^*(z)$. Define a coloring c_1 of G_1 by letting $c_1(x) = c_1^*(y)$ and $c_1(u) = c_1^*(u)$ for all $u \in V(G_1) - \{x\}$. It is easy to see that c_1 gives the desired 4-coloring of G_1.

Similarly, by defining $G_1^* := (G_1 - x) + \{x'z, yz\}$, we can show that G_1 has the desired 4-coloring c_2. □

Next, we show that G_2 admits certain 4-colorings. First, we need the following lemma.

Lemma 2.5. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then G_1 is 2-connected.*

Proof. Suppose G_1 is not 2-connected. Since G is 3-connected (by Proposition 2.1), there must exist vertices $x \in V(G_1 \cap G_2)$ and $x' \in V(G_1) - V(G_1 \cap G_2)$ such that x' separates x from $V(G_1 \cap G_2) - \{x\}$. Let y, z denote the other two vertices in $V(G_1 \cap G_2) - \{x\}$. By Proposition 2.4, there exists a 4-coloring c_1 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$, and there exists a 4-coloring c_1' of G_1 such that $c_1'(x) = c_1'(z) \neq c_1'(y)$.

Note that $G_2 + yz$ contains no K_5-subdivision. For otherwise, let Σ be a K_5-subdivision in $G_2 + yz$. By Proposition 2.1, $G_1 - x$ has a y–z path P. Now $(\Sigma - yz) \cup P$ (and hence G) contains a K_5-subdivision, a contradiction. Since $|V(G_2 + yz)| < |V(G)|$, $G_2 + yz$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + yz$. Then $c_2(y) \neq c_2(z)$.

First, assume that $c_2(y) \neq c_2(x) \neq c_2(z)$. Then c_2 is a 4-coloring of $G_2 + \{xy, xz, yz\}$. By Proposition 2.3, G_1 has a 4-coloring c_1^* such that $c_1^*(x)$, $c_1^*(y)$ and $c_1^*(z)$ are all distinct. We may assume c_1^* and c_2 use the same set of four colors, and by permuting colors of vertices of G_1, we have $c_1^*(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Now define a coloring c of G with $c(u) = c_1^*(u)$ for $u \in V(G_1)$ and $c(u) = c_2(u)$ for $u \in V(G_2)$. This shows that G is 4-colorable, a contradiction.

Now by symmetry between y and z (with respect to c_1 and c_1'), we may assume that $c_2(x) = c_2(y) \neq c_2(z)$. We may also assume that c_1 and c_2 use the same set of four colors, and by permuting colors if necessary, $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Define $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then it is easy to see that c is a 4-coloring of G, a contradiction. □

Proposition 2.6. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Let $F \subseteq \{xy, xz, yz\}$. Then $G_2 + F$ is 4-colorable if, and only if, $|F| \leqslant 2$.*

Proof. First, assume that $|F| = 3$. Then $G_2 + F = G_2 + \{xy, xz, yz\}$. Suppose $G_2 + F$ is 4-colorable, then there is a 4-coloring c_2 of G_2 such that $c_2(x)$, $c_2(y)$ and $c_2(z)$ are all distinct. By Proposition 2.3, let c_1 be a 4-coloring of G_1 such that $c_1(x)$, $c_1(y)$ and $c_1(z)$ are all distinct. Assume that c_1 and c_2 use the same set of four colors. By permuting colors if necessary, we may assume that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$.

Then we see that c is a 4-coloring of G, a contradiction. Hence $G_2 + F$ is not 4-colorable when $|F| = 3$.

Now assume $|F| = 1$. By symmetry, consider $F = \{xy\}$. If $G_2 + xy$ has no K_5-subdivision, then by the choice of G, we see that $G_2 + xy$ is 4-colorable. So assume that $G_2 + xy$ has a K_5-subdivision, say Σ. By Lemma 2.5, we see that $G_1 - z$ has an x–y path P. Now $(\Sigma - xy) \cup P$ (and hence G) contains a K_5-subdivision, a contradiction.

Finally, assume $|F| = 2$. By symmetry, we consider $F = \{xy, xz\}$. If $G_2 + \{xy, xz\}$ contains no K_5-subdivision then, by the choice of G, we see that $G_2 + \{xy, xz\}$ is 4-colorable. So we may assume that $G_2 + \{xy, xz\}$ does contain a K_5-subdivision, and denote it by Σ. By Lemma 2.5, G_1 contains internally disjoint paths Y, Z from x to y, z, respectively. Hence $(\Sigma - \{xy, yz\}) \cup Y \cup Z$ (and hence G) contains a K_5-subdivision, a contradiction. □

We conclude this section with a useful observation.

Lemma 2.7. *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then there is no cycle in G_1 containing $V(G_1 \cap G_2)$, and $V(G_1 \cap G_2)$ is an independent set in G_1.*

Proof. Let $V(G_1 \cap G_2) = \{x, y, z\}$. By Proposition 2.6, $G_2 + \{xy, xz, yz\}$ is not 4-colorable. Hence, $G_2 + \{xy, xz, yz\}$ has a K_5-subdivision Σ. If there is a cycle C in G_1 through x, y, z, then $(\Sigma - \{xy, yz, zx\}) \cup C$ (and hence G) contains a K_5-subdivision, a contradiction. So G_1 contains no cycle through x, y, z. Therefore, by Lemma 2.5 and Theorem 1.2, $\{x, y, z\}$ must be independent in G_1. □

3. 4-Connectivity

In this section, we prove Theorem 1.1. First, we prove a lemma.

Lemma 3.1. *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Let E_x (respectively, E_y) denote the set of edges of G_1 incident with x (respectively, y), and let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^* (and deleting multiple edges). Then $E_x \cap E_y = \emptyset$, G_1^* contains a K_5-subdivision, and for any K_5-subdivision Σ in G_1^*,*

(i) *x^* is a branch vertex of Σ,*
(ii) *$x^*z \notin E(\Sigma)$,*
(iii) *$|E_x \cap E(\Sigma)| = 2 = |E_y \cap E(\Sigma)|$, and*
(iv) *for any two branch vertices u, v of Σ, there are four internally disjoint u–v paths in Σ.*

Proof. For convenience, vertices and edges of G_1 are also viewed as vertices and edges of G_1^*, except for x and y. By Lemma 2.7, $E_x \cap E_y = \emptyset$.

Suppose G_1^* contains no K_5-subdivision. Then since $|V(G_1^*)| < |V(G)|$, G_1^* is 4-colorable. Hence G_1 has a 4-coloring c_1 such that $c_1(x) = c_1(y) \neq c_1(z)$. By Proposition 2.6, $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$. Then $c_2(x) \neq c_2(z) \neq c_2(y)$. If $c_2(x) \neq c_2(y)$ then $G_2 + \{xy, yz, zx\}$ is 4-colorable, contradicting Proposition 2.6. So $c_2(x) = c_2(y)$. We may assume that c_1 and c_2 use the same set of four colors, and we may permute the colors of

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vertices of G_1 so that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then c is a 4-coloring of G, a contradiction.

Now let Σ be a K_5-subdivision in G_1^*. By (iii) of Lemma 2.2, G_2 is 2-connected. So there exists a y–z path (respectively z–x path, x–y path) P_x (respectively P_y, P_z) in $G - x$ (respectively $G - y$, $G - z$). For the same reason, G_2 contains internally disjoint paths X_y, X_z from x to y, z, respectively, and internally disjoint paths Y_x, Y_z from y to x, z, respectively.

Suppose x^* is not a branch vertex of Σ. Then since G_1 has no K_5-subdivision, exactly one branch path of Σ, say R, uses x^*. Let q, r be the neighbors of x^* in R. First assume that $z \in \{q, r\}$, say $z = r$. If $qy \in E(G_1)$ then $((\Sigma - x^*) + \{y, qy\}) \cup P_x$ is a K_5-subdivision in G, a contradiction. So assume $qx \in E(G_1)$ then $((\Sigma - x^*) + \{x, qx\}) \cup P_y$ is a K_5-subdivision in G, a contradiction. Now assume that $z \notin \{q, r\}$. If $qx, rx \in E(G_1)$ then $(\Sigma - x^*) + \{x, qx, rx\}$ is a K_5-subdivision in G_1, a contradiction. If $qy, ry \in E(G_1)$ then $(\Sigma - x^*) + \{y, qy, ry\}$ is a K_5-subdivision in G_1, a contradiction. So assume by symmetry that $qx, ry \in E(G_1)$. Then $((\Sigma - x^*) + \{x, y, qx, ry\}) \cup P_z$ is a K_5-subdivision in G, a contradiction. Thus x^* is a branch vertex of Σ, and (i) holds.

Suppose $x^*z \in E(\Sigma)$. Then either $|E_x \cap E(\Sigma)| \leqslant 1$ or $|E_y \cap E(\Sigma)| \leqslant 1$. By symmetry, assume that $|E_x \cap E(\Sigma)| \leqslant 1$. If $|E_x \cap E(\Sigma)| = 0$ then let $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$, and we see that $((\Sigma - x^*) + \{y, yy_1, yy_2, yy_3\}) \cup P_x$ is a K_5-subdivision in G, a contradiction. So assume $|E_x \cap E(\Sigma)| = 1$. Let $yy_1, yy_2 \in E_y \cap E(\Sigma)$ and $xx' \in E_x \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, yy_1, yy_2, xx'\}) \cup Y_x \cup Y_z$ is a K_5-subdivision in G, a contradiction. So $x^*z \notin E(\Sigma)$, and (ii) holds.

If $|E_x \cap E(\Sigma)| = 0$ or $|E_y \cap E(\Sigma)| = 0$, then by (ii), Σ gives a K_5-subdivision in G (by simply renaming x^* as x or y), a contradiction. Suppose (iii) fails, and assume by symmetry that $|E_x \cap E(\Sigma)| = 1$ and $|E_y \cap E(\Sigma)| = 3$. Let $xx' \in E_x \cap E(\Sigma)$, $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, xx', yy_1, yy_2, yy_3\}) \cup P_z$ is a K_5-subdivision in G, a contradiction. So (iii) must hold.

Clearly, (iv) holds. □

Proof of Theorem 1.1. Suppose the assertion of Theorem 1.1 is not true. Let G be a Hajós graph, and assume that G is not 4-connected. By Proposition 2.1, G is 3-connected. Let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$.

By Lemma 2.7, $\{x, y, z\}$ is an independent set in G_1. Let E_x (respectively E_y) denote the set of edges of G_1 incident with x (respectively y). Let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^* (and deleting multiple edges). Then by Lemma 3.1, $E_x \cap E_y = \emptyset$, and G_1^* contains a K_5-subdivision, say Σ. Note that Σ satisfies (i)–(iv) of Lemma 3.1.

Note that G_1 is 2-connected (by Lemma 2.5) and G_1 has no cycle containing $\{x, y, z\}$ (by Lemma 2.7). Therefore, by applying Theorem 1.2 to G_1, it suffices to consider the following three cases.

Case 1. There exists a 2-cut S in G_1 and there exist three distinct components D_x, D_y, D_z of $G_1 - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.

Let $S := \{a, b\}$. By (i) of Lemma 3.1, x^* is a branch vertex of Σ. Therefore, D_z contains no branch vertex of Σ because S and the edge zx^* show that G_1^* contains at most three internally disjoint paths between x^* and D_z, contradicting (iv) of Lemma 3.1. Similarly, either $D_x - x$ or $D_y - y$ has no branch vertex of Σ since $S \cup \{x^*\}$ is a 3-cut in G_1^* separating $D_x - x$ from $D_y - y$.

Therefore, we may assume that all branch vertices of Σ are in $V(D_x) \cup S \cup \{x^*\}$. By (iii) of Lemma 3.1, $|E_y \cap E(\Sigma)| = 2$. Because there are no branch vertices of Σ in D_y, Σ contains two paths P_a, P_b from x^* to a, b, respectively, each using one edge of E_y, that are contained in $G_1^*[V(D_y) \cup S \cup \{x^*\}]$. If a and b are branch vertices of Σ then the branch path P of Σ between a and b may use edges outside of $G_1^*[V(D_x) \cup S \cup \{x^*\}]$. Except for edges in the paths P_a, P_b and possibly P, all edges of Σ appear in $G_1^*[V(D_x) \cup S \cup \{x^*\}]$.

If $G[V(D_y) \cup S]$ (respectively $G[V(D_z) \cup S]$) contains internally disjoint paths Y from a to y (respectively z) and B from a to b, then we can produce a K_5-subdivision in G as follows: replace P_a, P by Y, B, respectively, replace P_b by a path in $G[V(D_z) \cup \{b\}]$ (respectively $G[V(D_y) \cup \{b\}]$) from z (respectively y) to b, and add two internally disjoint paths from x to $\{y, z\}$ in G_2 (which exist by (iii) of Lemma 2.2). This gives a contradiction. So we may assume that such paths Y, B do not exist in $G[V(D_y) \cup S]$ (respectively $G[V(D_z) \cup S]$). Then there is a cut vertex a_y (respectively a_z) of $G[V(D_y) \cup S]$ (respectively $G[V(D_z) \cup S]$) separating a from $\{y, b\}$ (respectively $\{z, b\}$). Since $\{a, a_y\}$ is not a 2-cut in G, we see that a_y (respectively a_z) is the only neighbor of a in $G[V(D_y) \cup S]$ (respectively $G[V(D_z) \cup S]$).

Similarly, we conclude that b has only one neighbor b_y in $G[V(D_y) \cup S]$, and b has only one neighbor b_z in $G[V(D_z) \cup S]$.

Next we use the above structural information to color vertices of G. By Proposition 2.3, G_1 has a 4-coloring c_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. We shall obtain a new 4-coloring c_1' of G_1 such that x, y, z use exactly two colors. For convenience, let $\{\alpha, \beta, \gamma, \delta\}$ denote the four colors used by c_1, and let H_{ij} denote the subgraph of G_1 induced by vertices of color i or j, for all $\{i, j\} \subseteq \{\alpha, \beta, \gamma, \delta\}$. Let $c_1(x) = \alpha$, $c_1(y) = \beta$, and $c_1(z) = \gamma$. Note that $\{y, z\}$ must be contained in a component of $H_{\beta\gamma}$, as otherwise we could switch colors in the component of $H_{\beta\gamma}$ containing y, yielding the desired 4-coloring c_1' of G_1. Therefore by symmetry between a and b, we may assume that $c_1(a_y) = \beta = c_1(a_z)$ and $c_1(a) = \gamma$, or $c_1(a_y) = \gamma = c_1(a_z)$ and $c_1(a) = \beta$. By the same argument, $\{x, z\}$ must be contained in a component of $H_{\alpha\gamma}$, and $\{x, y\}$ must be contained in a component of $H_{\alpha\beta}$. Therefore, $c_1(b_y) = \beta$, $c_1(b) = \alpha$, and $c_1(b_z) = \gamma$. But then, neither x nor z can be in the component of $H_{\beta\delta}$ containing y, and neither y nor z is in the component of $H_{\alpha\delta}$ containing x. Thus we can switch the colors in the component of $H_{\beta\delta}$ containing y and in the component of $H_{\alpha\delta}$ containing x. This yields the desired 4-coloring c_1' of G_1, with $c_1'(x) = c_1'(y) = \delta$ and $c_1'(z) = \gamma$.

Now by symmetry, assume that $c_1'(x) = c_1'(y) \neq c_1'(z)$. By Proposition 2.6, $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$ using the colors from $\{\alpha, \beta, \gamma, \delta\}$. If $c_2(x) \neq c_2(y)$ then c_2 is a 4-coloring of $G_2 + \{xy, yz, zx\}$, contradicting Proposition 2.6. So $c_2(x) = c_2(y)$. By permuting colors if necessary, we may assume that $c_2(u) = c_1'(u)$ for all $u \in \{x, y, z\}$. Now let $c(u) = c_1'(u)$ for all $u \in V(G_1)$ and $c(u) = c_2(u)$ for all $u \in V(G_2)$. Then c is a 4-coloring of G, a contradiction.

Case 2. There exist a vertex v of G_1, 2-cuts S_x, S_y, S_z in G_1, and components D_u of $G_1 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

By (i) of Lemma 3.1, x^* is a branch vertex of Σ. Therefore, D_z contains no branch vertex of Σ because S_z and the edge zx^* shows that G_1^* contains at most three internally disjoint paths between x^* and D_z, contradicting (iv) of Lemma 3.1. In fact, all branch vertices of Σ must be contained in $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. For otherwise, Σ has a branch vertex $v \notin R$, and Σ must have four disjoint path leaving R. But this forces $x^*z \in E(\Sigma)$, contradicting (ii) of Lemma 3.1.

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We claim that, for each $u \in \{x, y\}$, not all branch vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, suppose by symmetry that all branch vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma 3.1, let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let B_q, B_r be the branch paths in Σ containing x^*q, x^*r, respectively. Since $x^*z \notin E(\Sigma)$ (by (ii) of Lemma 3.1), both B_q and B_r have an x^*–S_y subpath whose internal vertices are all contained in D_y. Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z, respectively, which exist by (iii) of Lemma 2.2. Note that there exists an $(S_z - \{v\})$–$(S_x - \{v\})$ path Q_{xz} in $(G_1 - v) - V(D_x \cup D_y \cup D_z)$; for otherwise, one of $\{v, x\}, \{v, z\}$ is a 2-cut in G, contradicting Proposition 2.1. Let Y be a y–v path in $G[V(D_y) \cup \{v\}]$ and let Z be a z–$(S_z - \{v\})$ path in $G[V(D_z) \cup (S_z - \{v\})]$. Then

$$\Big(\big((\Sigma - x^*) + \{x, xs, xt\}\big) - \Big(V(B_q \cup B_r) - \big(V(D_x) \cup S_x\big)\Big)\Big) \\ \cup (P_{xy} \cup Y) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5-subdivision in G, a contradiction.

Since $|\{x^*\} \cup S_x \cup S_y| = 4$, there must exist a branch vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may assume that $x' \in V(D_x - x)$. Hence by the above claim, there is also a branch vertex y' of Σ such that $y' \in V(D_y - y) \cup (S_y - \{v\})$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y', contradicting (iv) of Lemma 3.1.

Case 3. There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G_1 and components D_u of $G_1 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G_1 - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u, for all $u \in \{x, y, z\}$.

Let $S_x = \{a_x, b_x\}$, $S_y = \{a_y, b_y\}$, and $S_z = \{a_z, b_z\}$ such that $\{a_x, a_y, a_z\}$ is contained in a component A of $G_1 - V(D_x \cup D_y \cup D_z)$, and $\{b_x, b_y, b_z\}$ is contained in another component B of $G_1 - V(D_x \cup D_y \cup D_z)$.

As in Cases 1 and 2, we can show that all branch vertices of Σ are in $R \cup S_z$, where $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. In fact, all branch vertices of Σ must be in R. For otherwise, assume by symmetry that a_z is a branch vertex of Σ. Then, since $x^*z \notin E(\Sigma)$ (by (ii) of Lemma 3.1), $\{b_z, a_x, a_y\}$ shows that Σ cannot contain four internally disjoint paths between a_z and x^*, contradicting (iv) of Lemma 3.1.

We claim that, for each $u \in \{x, y\}$, not all branch vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, we may assume that all branch vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma 3.1, let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let A_q, B_r be the branch paths in Σ containing x^*q, x^*r, respectively. Since $x^*z \notin E(\Sigma)$, both A_q and B_r have an x^*–S_y subpath whose internal vertices are all contained in D_y. Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z, respectively, which exist by (iii) of Lemma 2.2. Note that there exists an a_y–a_x path Q_{xy} in A (since A is connected) and there exists a b_z–b_x path Q_{xz} in B (since B is connected). Let Y be a y–a_y path in $G[V(D_y) \cup \{a_y\}]$ and let Z be an z–b_z path in $G[V(D_z) \cup \{b_z\}]$. Then,

$$\Big(\big((\Sigma - x^*) + \{x, xs, xt\}\big) - \Big(V(A_q \cup B_r) - \big(V(D_x) \cup S_x\big)\Big)\Big) \\ \cup (P_{xy} \cup Y \cup Q_{xy}) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5-subdivision in G, a contradiction.

We further claim that the set of branch vertices of Σ is $S_x \cup S_y \cup \{x^*\}$. For otherwise, there must be a branch vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may

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assume that $x' \in V(D_x - x)$. Then by the above claim, there is a branch vertex y' of Σ such that $y' \in V(D_y - y) \cup S_y$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y', contradicting (iv) of Lemma 3.1.

Since $x^*z \notin E(\Sigma)$, we see that Σ must contain two branch paths from $\{a_x, a_y\}$ to $\{b_x, b_y\}$ which must be contained in $G_1 - V(D_x \cup D_y)$. But this is impossible, because a_z separates $\{a_x, a_y\}$ from $\{b_x, b_y\}$ in $G_1 - V(D_x \cup D_y)$, a contradiction. □

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