Quasilocally Connected, Almost Locally Connected or Triangularly Connected Claw-free Graphs

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Abstract. The definitions of quasilocally connected graphs, almost locally connected graphs and triangularly connected graphs are introduced by Zhang, Teng and Lai et al. They are all different extensions of locally connected graphs. Many known results on the condition of local connectivity have been extended to these weaker conditions mentioned above. In this paper, we study the relations among these different conditions. In particular, we prove that every triangularly connected claw-free graph without isolated vertices is also quasilocally connected claw-free.

1 Introduction and Notations

In this paper, we consider only finite undirected graphs without loops and multiple edges. For terminologies and notations not defined here we refer to [1]. Let $\langle S\rangle$ denote the subgraph of G induced by $S \subseteq V(G)$. A vertex $x \in V(G)$ is locally connected if $\langle N(x)\rangle$ is connected, where $N(x)$ stands for the open neighborhood of x. A graph G is said to be locally connected if each vertex of G is locally connected. A graph G is said to be claw-free if it has no $K_{1,3}$ as its induced subgraph. For any two distinct vertices x and y in $V(G)$, (x, y)-path denotes a path with x, y as its two end vertices. For $3 \leq l \leq |V(G)|$, l-cycle denotes a cycle of length l. A cycle of length l containing the vertex set S in $V(G)$ is called a (S, l)-cycle. For convenience, we write $(\{v\}, l)$-cycle as (v, l)-cycle. A graph G is vertex pancyclic if every vertex of G is contained in cycles of all lengths l for $3 \leq l \leq |V(G)|$. A graph G is called quasilocally connected if each vertex cut of G contains a locally connected vertex. A graph G is called almost locally connected if $B(G) = \{x \in V(G) \colon \langle N(x)\rangle \text{ is not connected}\}$ is independent and for any $x \in B(G)$, there is a vertex y in $V(G)\backslash\{x\}$ such that $\langle N(x) \cup \{y\}\rangle$ is connected. A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \ldots, C_m$ such that $e_1 \in E(C_1)$, $e_2 \in E(C_m)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq m-1)$.

Obviously, the condition of triangularly connected, quasilocally connected or almost locally connected is different extension of locally connected, respectively.

Clark [2] proved the following result.

Theorem 1. (Clark, 1981) *Every connected locally connected claw-free graph with $|V(G)| \geq 3$ is vertex pancyclic.*

Zhang introduced the definition of quasilocally connected graphs and obtained the following result in [3].

Theorem 2. (Zhang, 1989) *If the vertex v of a quasilocally connected claw-free graph G with $|V(G)| \geq 3$ is contained in a cycle of length r, then v is contained in a cycle of length h for any $h = r, r+1, \ldots, n$.*

Some extensions of Theorem 2 are known.

Theorem 3. (Veldman et al., 1990) *Every quasilocally connected claw-free graph with $|V(G)| \geq 3$ is vertex pancyclic.*

Theorem 4. (Teng et al., 2002) *Every connected almost locally connected claw-free graph with $|V(G)| \geq 3$ is vertex pancyclic.*

Theorem 5. (Lai et al., preprint) *Every triangularly connected claw-free graph G with $|E(G)| \geq 3$ is vertex pancyclic.*

In this paper, we study the relations among these different conditions and prove the following theorem.

Theorem 6. *Every triangularly connected claw-free graph without isolated vertices is also quasilocally connected claw-free.*

Note that we can easily conclude Theorem 5 by Theorem 3 and Theorem 6.

2 Claw-Free Graphs in Different Conditions

Note that G_1, G_2 and G_3 are showed in Fig. 1, Fig. 2 and Fig. 3, respectively. Obviously, G_1 is an almost locally connected claw-free graph. We can easily obtain that both of v_1 and v_2 are not locally connected vertices in G_1. However, $\{v_1, v_2\}$ is a vertex cut of G_1. Hence, G_1 is not a quasilocally connected claw-free graph. By the definition of triangularly connected, we know that G_1 is not a triangularly connected claw-free graph. Obviously, G_2 is a quasilocally connected claw-free graph but it is not a triangularly connected claw-free graph. Obviously, both of v_1 and v_2 are not locally connected vertex in G_3. Furthermore, we can not find a vertex u in $V(G_3)\backslash\{v_1\}$ or $V(G_3)\backslash\{v_2\}$ such that $\langle N(v_1) \cup \{u\}\rangle$ or $\langle N(v_2) \cup \{u\}\rangle$ is connected. Hence, G_3 is not an almost locally connected claw-free graph. We can easily obtain that G_3 is a quasilocally connected claw-free graph and it is also a triangularly connected claw-free graph by the definition of quasilocally connected and triangularly connected.

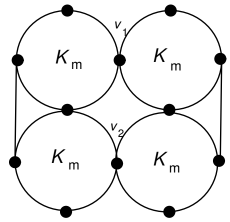

Fig. 1 G_1 (m≥4)

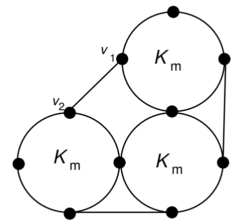

Fig. 2 G_2 (m≥4)

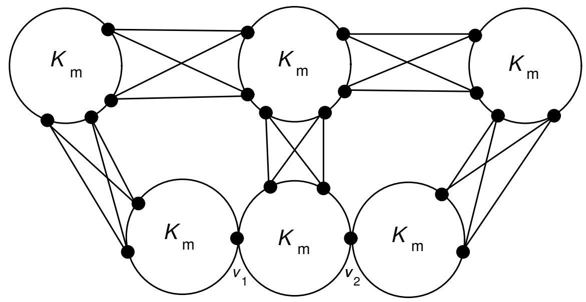

Fig. 3 G_3 (m≥6)

3 Proof of Theorem 6

Let G be a triangularly connected claw-free graph without isolated vertices. Let S be a vertex cut of G. It suffices to show that there is a locally connected vertex in S. Since G is claw-free, if a vertex u is not locally connected then $\langle N(u)\rangle$ is a disjoint union of two cliques. Hence, if $N(u)$ has a pair of nonadjacent vertices x_1, x_2 such that there is an (x_1, x_2)-path in $N(u)$, then u should be a locally connected vertex. Let A be a component of $G-S$ and write $\overline{A} = G-S\cup A$. Then there is an edge e_1 such that $V(e_1)\cap V(A) \neq \phi$, where $V(e_1)$ stands for the set of two end vertices of e_1. Also there is an edge e_2 such that $V(e_2)\cap V(\overline{A}) \neq \phi$. Since G is triangularly connected, G has a sequence of 3-cycles $C_1,\ C_2, \dots, C_m$ such that $e_1 \in E(C_1)$, $e_2 \in E(C_m)$ and $E(C_i)\cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq m-1)$. Choose e_1, e_2 and C_1, $C_2, \dots, C_m$ so that m is as small as possible. Since $V(e_1)\cap V(A) \neq \phi$ and $V(e_2) \cap V(\overline{A}) \neq \phi$, we observe that $V(C_1) \neq V(C_m)$, which implies that $m \geq 2$. Let $V(C_1) = \{v_1, v_2, v_3\}$. Without loss of generality, we may assume $e_1 = v_1v_2$, $v_1 \in V(A)$ and $E(C_1)\cap E(C_2) = \{v_2v_3\}$. If $\{v_2, v_3\}\cap V(A) \neq \phi$, then we can take v_2v_3 as e_1 instead of v_1v_2 and we get a shorter sequence C_2, $C_3, \dots, C_m$, which contradicts the choice of e_1, e_2 and C_1, $C_2, \dots, C_m$. Hence $\{v_2, v_3\} \cap V(A) = \phi$, which means that $v_2, v_3 \in S$. Let $V(C_2) = \{v_2, v_3, v_4\}$. We show $v_1v_4 \notin E(G)$. Assume $v_1v_4 \in E(G)$, which implies $v_4 \notin \overline{A}$ and there is the next 3-cycle C_3. If $E(C_2) \cap E(C_3) = v_2v_3$, then we can skip C_2 and we get a shorter sequence C_1, $C_3, \dots, C_m$, which contradicts the choice of e_1, e_2 and C_1, $C_2, \dots, C_m$. Hence, we know that either $E(C_2)\cap E(C_3) = \{v_2v_4\}$ or $E(C_2)\cap$

$E(C_3) = \{v_3v_4\}$. By symmetry, we may assume that $E(C_2) \cap E(C_3) = \{v_3v_4\}$. Then taking $e_1^{'} = v_1v_4$ instead of $e_1 = v_1v_2$ and $V(C_1^{'}) = \{v_1, v_3, v_4\}$ instead of C_1, we again get a shorter sequence $C_1^{'}, C_3, \ldots, C_m$, which contradicts the choice of e_1, e_2 and $C_1, C_2, \ldots, C_m$. Now it is shown that $v_1v_4 \notin E(G)$. Then, since $v_1v_3v_4$ is a (v_1, v_4)-path and $\{v_1, v_3, v_4\} \subseteq N(v_2)$, we observe that there is a pair of nonadjacent vertices v_1, v_4 in $N(v_2)$ such that $\langle N(v_2)\rangle$ has a (v_1, v_4)-path. This assures us that v_2 is a desired locally connected vertex and the proof of Theorem 6 is completed.

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