THE ENUMERATION OF PERMUTATIONS WHOSE POSETS HAVE A MAXIMAL ELEMENT

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Dedicated to Amitai Regev on the occasion of his 65-th birthday

Abstract

Recently, Tenner [12] studied the set of posets of a permutation of length n with unique maximal element, which arise naturally when studying the set of zonotopal tilings of Elnitsky's polygon. In this paper, we prove that the number of such posets is given by

$$P_{5n} - 4P_{5(n-1)} + 2P_{5(n-2)} - \sum_{j=0}^{n-2} C_j P_{5(n-2-j)},$$

where P_n is the n-th Padovan number and C_n is the n-th Catalan number.

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1. Introduction

Let S_n denote the set of permutations of $\{1, \dots, n\}$, written in one-line notation, and suppose $\alpha \in S_n$ and $\tau \in S_k$. We say α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \dots, \alpha_{i_k})$ is order-isomorphic to τ. In this context τ is usually called a *pattern*; α *avoids* τ, or is τ-*avoiding*, if α does not contain such a subsequence. For any set of patterns $T = \{\tau^1, \dots, \tau^s\}$ we write $S_n(T)$ or $S_n(\tau^1, \dots, \tau^s)$ to denote the set of permutations in S_n which avoid $\tau^1, \dots, \tau^s$ simultaneously. One important and often difficult problem in the study of permutations that avoid a set of patterns is the enumeration problem: given a set T of patterns, enumerate the set $S_n(T)$ consisting of those permutations in S_n which avoid every element of T. It well known that there is no general procedure for finding an explicit formula for $S_n(T)$, in fact it is an art (for example, see [1], [3]-[5], [9]-[10] and references therein).

Recently, a special classe of restricted permutations has arisen in the study of algebraic combinatorics. For instance, Green and Losonczy [6] defined, for any simply laced Coxeter group, a subset of "freely braided elements" (for details, see [6] and [7]), and they suggest as an open problem to enumerate the number of freely-braided permutations in S_n. Mansour [8] showed that the ordinary generating function for the number of freely-braided permutations in S_n is given by

$$\frac{1-3x-2x^2+(1+x)\sqrt{1-4x}}{1-4x-x^2+(1-x^2)\sqrt{1-4x}}.$$

Tenner [12] studied the set of posets $P(\pi)$, $\pi\in S_n$, with an unique maximal element, where $P(\pi)$ arises naturally when studying the set of zonotopal tilings of Elnitsky's polygon (see [12] and the references therein). He proved that the poset $P(\pi)$ has unique maximal element if and only if π avoids each of the three patterns 4231, 4312, and 3421. Thus, it is interesting to present an explicit formula for the number of permutations π in S_n such that the poset $P(\pi)$ has unique maximal element, which is equivalent to find an explicit formula for the number of permutations in S_n that avoid each of the three patterns 4231, 4312, and 3421. Note, however, that a permutation π avoids these three patterns if and only if $r(\pi)$ avoids each of the three patterns 1324, 2134 and 1243, where $r:\pi_1\pi_2\dots\pi_n\rightarrow\pi_n\dots\pi_2\pi_1$. So, for all $n\geq 0$, $\#S_n(1324,2134,1243)=\#S_n(4231,4312,3421)$. This leads us to the following definition.

Definition 1.1. *A permutation π is said to be* maximal-poset *if and only if π avoids each of the three patterns* 1324, 2134, *and* 1243. *We denote the set of all maximal-poset permutations in S_n by $\mathcal{F}_n$, i.e., $\mathcal{F}_n=S_n(1324,2134,1243)$.*

The main reason for the term "maximal-poset permutation" is that for each permutation π with $r(\pi)\in\mathcal{F}_n$ obtains a poset $P(\pi)$ with unique maximal element.

The main result of this paper can be formulated as follows.

Theorem 1.2. *The ordinary generating function for the number of maximal-poset permutations in S_n is given by*

$$\frac{2-C(x)}{2-x-C(x)}.$$

where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ is the ordinary generating function for the Catalan numbers ($C_n=\frac{1}{n+1}\binom{2n}{n}$). Moreover, the number of maximal-poset permutations in S_n is given by

$$P_{5n}-4P_{5(n-1)}+2P_{5(n-2)}-\sum_{j=0}^{n-2}C_jP_{5(n-2-j)},$$

where P_n is the n-th Padovan number (see [11, A012814,A000931]*).*

The proof of the above theorem is presented in Section 2.

2. Proof of main result

Given $b_1,b_2,\dots,b_m\in\mathbb{N}$, we define

$$f_n(b_1,b_2,\dots,b_m)=\#\{\pi_1\pi_2\dots\pi_n\in\mathcal{F}_n\mid \pi_1\pi_2\dots\pi_m=b_1b_2\dots b_m\}.$$

It is natural to extend $f_n(b_1,b_2,\dots,b_m)$ to the case $m=0$ by setting $f_n(\varnothing)=f_n=\#\mathcal{F}_n$. The following properties of the numbers $f_n(b_1,\dots,b_m)$ can be deduced easily from the definitions.

Lemma 2.1. (1) *Let $m \geq 1$ and $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. Then, for all $b_m + 1 \leq j \leq b_1 - 1$,*

$$f_n(b_1, \ldots, b_m, j) = 0.$$

(2) *Let $m \geq 2$ and $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. Then, for all $b_1 + 1 \leq j \leq n-1$,*

$$f_n(b_1, \ldots, b_m, j) = 0.$$

(3) *Let $m \geq 1$ and $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. Then,*

$$f_n(b_1, \ldots, b_m, n) = f_{n-1}(b_1, \ldots, b_m).$$

(4) *Let $m \geq 1$ and $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. Then*

$$f_n(n, b_1, \ldots, b_m) = f_n(n-1, b_1, \ldots, b_m) = f_{n-1}(b_1, \ldots, b_m).$$

Proof. For (1), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m j$, then the entries b_m, j, n, $n-1$ give an occurrence of the pattern 1243 or the entries b_1, j, $n-1$, n give an occurrence of the pattern 2134.

For (2), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m j$, then the entries b_1, b_2, j, n give an occurrence of the pattern 2134.

For (3), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m n$, where $n-2 \geq b_1 > \cdots > b_m \geq 1$, then no occurrence of the patterns 1243, 2134, 1324 in π can involve the entry $\pi_{m+1} = n$. Hence, there is a bijection between the set of permutations $\pi \in \mathcal{F}_n$ with $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m n$ and the set of permutations $\sigma \in \mathcal{F}_{n-1}$ such that $\sigma_1 \ldots \sigma_m = b_1 \ldots b_m$.

Using similar arguments as in the proof of (3) we get that (4) holds. $\square$

Next we introduce objects $A_m(n)$, $C_m(n)$, and $B(n)$ which organize suitably the information about the numbers $f_n(b_1, \ldots, b_m)$ and play an important role in the proof of the main result.

Definition 2.2. *For $1 \leq m \leq n-2$ set*

$$\begin{aligned} A_m(n) &= \sum_{n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1} f_n(b_1, b_2, \ldots, b_m), \\ C_m(n) &= \sum_{n-1 \geq b_1 > b_2 > \cdots > b_m \geq 1} f_n(b_1, b_2, \ldots, b_m), \\ B(n) &= \sum_{i=1}^{n-2} f_n(i, n-1). \end{aligned}$$

We start by deriving recursive expressions for $A_m(n)$ and $B(n)$.

Proposition 2.3. *For all $2 \leq m \leq n-2$,*

$$A_m(n) = A_{m+1}(n) + A_m(n-1) + \cdots + A_2(n+1-m) + f_{n-m} - f_{n-1-m}.$$

Proof. Let $2 \leq m \leq n-2$. Definition 2.2 yields

$$A_m(n) = A_{m+1}(n) + \sum_{n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1} \sum_{j=b_m+1}^{n} f_n(b_1, \ldots, b_m, j).$$

Using Lemma 2.1, parts (1)-(3), we have

$$\begin{aligned} A_m(n) &= A_{m+1}(n) + \sum_{n-2\geq b_1>b_2>\cdots>b_m\geq 1} f_n(b_1,b_2,\ldots,b_m,n) \\ &= A_{m+1}(n) + \sum_{n-2\geq b_1>b_2>\cdots>b_m\geq 1} f_{n-1}(b_1,b_2,\ldots,b_m) \\ &= A_{m+1}(n) + C_m(n-1). \end{aligned} \tag{2.1}$$

Definition 2.2 and Lemma 2.1(4) give

$$\begin{aligned} C_m(n) &= A_m(n) + \sum_{n-2\geq b_2>\cdots>b_m\geq 1} f_n(n-1,b_2,\ldots,b_m) \\ &= A_m(n) + \sum_{n-2\geq b_2>\cdots>b_m\geq 1} f_{n-1}(b_2,\ldots,b_m) \\ &= A_m(n) + C_{m-1}(n-1), \end{aligned} \tag{2.2}$$

and

$$C_1(n) = \sum_{j=1}^{n-2} f_{n-1}(j) = f_{n-1} - f_{n-1}(n-1) = f_{n-1} - f_{n-2}. \tag{2.3}$$

Hence, by induction on m together with (2.1), (2.2) and (2.3) we get the desired result. □

Let $f(x) = \sum_{n\geq 0} f_n x^n$. We next find an explicit expression for $A_2(n)$ in terms of f_n.

Theorem 2.4. *The ordinary generating function for the sequence* $\{A_2(n)\}_{n\geq 0}$ *is given by*

$$\sum_{n\geq 0} A_2(n)x^n = x^2C^2(x)((1-x)f(x)-1).$$

Proof. Define $\mathcal{A}_n(v) = \sum_{j=2}^{n-2} A_j(n)v^{j-2}$ for all $n \geq 0$. Proposition 2.3 gives that

$$\mathcal{A}_n(v) - v\mathcal{A}_{n-1}(v) = \frac{1}{v}(\mathcal{A}_n(v) - \mathcal{A}_n(0)) + \mathcal{A}_{n-1}(0) + f_{n-2} - f_{n-3}$$

for all $n \geq 4$ and $\mathcal{A}_n(v) = 0$ for $n = 0, 1, 2, 3$. Defining $\mathcal{A}(x; v) = \sum_{n\geq 0} \mathcal{A}_n(v)x^n$, multiplying the above recurrence relation by x^n and summing over all $n \geq 4$ we arrive at

$$\left(1 - \frac{1}{v} - xv\right)\mathcal{A}(x; v) = x^2((1-x)f(x)-1) - \frac{1-xv}{v}\mathcal{A}(x; 0).$$

This type of functional equation can be solved systematically using the *kernel method* [2]. In this case, by assuming $v = C(x)$ we get the desired result. □

Next we find an explicit formula for the ordinary generating function $B(x) = \sum_{n\geq 0} B(n)x^n$ of the sequence $\{B(n)\}_{n\geq 0}$ in terms of $f(x)$.

Proposition 2.5. *For all* $n \geq 3$,

$$B(n) = A_2(n-1) + 2f_{n-2} - f_{n-3}.$$

Moreover, the ordinary generating function for the sequence $\{B(n)\}_{n\geq 0}$ *is given by*

$$B(x) = x^3C^2(x)((1-x)f(x)-1) + 2x^2(f(x)-1) - x^3f(x).$$

Proof. By Definition 2.2 we get that

$$B(n) = \sum_{i=1}^{n-3} f_n(i, n-1) + f_n(n-2, n-1).$$

Observe that if a permutation $\pi \in \mathcal{F}_n$ is such that $\pi_1 = n-2$ and $\pi_2 = n-1$, then no occurrence of the patterns 1243, 2134, 1324, can involve either the entry $n-2$ or the entry $n-1$. Thus, $f_n(n-2, n-1) = f_{n-1}(n-2) = f_{n-2}$. On the other hand, by the definitions we get that

$$\sum_{i=1}^{n-3} f_n(i, n-1) = \sum_{i=1}^{n-3}\sum_{j=1}^{i-1} f_n(i, n-1, j) + \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} f_n(i, n-1, j) + \sum_{i=1}^{n-3} f_n(i, n-1, n).$$

Observe that if a permutation $\pi \in \mathcal{F}_n$ is such that $\pi_1 = i$, $\pi_2 = n-1$ and $\pi_3 = j$ with $1 \leq j < i \leq n-2$, then the entries i, $n-1$, j and n give an occurrence of the pattern 1324. Hence the middle sum of the above equation is zero. Using similar arguments as in the proof of Lemma 2.1, part (3), we get that $f_n(i, n-1, n) = f_{n-2}(i)$ for all $1 \leq i \leq n-3$, and $f_n(i, n-1, j) = f_{n-1}(i, j)$ for all $1 \leq j < i \leq n-3$. Thus,

$$\sum_{i=1}^{n-3} f_n(i, n-1) = \sum_{i=1}^{n-3}\sum_{j=1}^{i-1} f_{n-1}(i, j) + \sum_{i=1}^{n-3} f_{n-2}(i).$$

Therefore, by Definition 2.2 we obtain that

$$B(n) = A_2(n-1) + 2f_{n-2} - f_{n-3},$$

for all $n \geq 3$, as claimed. Now multiplying by x^n and summing over all $n \geq 3$ we arrive at

$$\sum_{n\geq 0} B(n)x^n = xA_2(x) + 2x^2(f(x) - 1) - x^3 f(x).$$

Hence, Theorem 2.4 gives that

$$B(x) = x^3C^2(x)((1-x)f(x) - 1) + 2x^2(f(x) - 1) - x^3 f(x),$$

as required. □

We need one more result for the proof of Theorem 1.2.

Proposition 2.6. *We have*

$$(1 - 3x + x^2)f(x) - 1 + 2x = A_2(x) + L(x),$$

where

$$L(x) = \frac{1}{1-x} \sum_{n\geq 0} B(n)x^n.$$

Proof. By Definition 2.2, and defining $L(n) = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} f_n(i, j)$ we get that for all $n \geq 2$,

$$f_n = f_n(n) + f_n(n-1) + \sum_{i=1}^{n-2} f_n(i) = 2f_{n-1} + \sum_{i=1}^{n-2} f_n(i)$$

and

$$\sum_{i=1}^{n-2} f_n(i) = A_2(n) + L(n) + \sum_{i=1}^{n-2} f_n(i, n),$$

which is equivalent to

$$\sum_{i=1}^{n-2} f_n(i) = A_2(n) + L(n) + f_{n-1} - f_{n-2}.$$

In terms of generating functions this becomes

$$f(x) - 1 - x = 3x(f(x) - 1) - x^2 f(x) = L(x) + A_2(x),$$

where $L(x) = \sum_{n\geq 0} L(n)x^n$ is the ordinary generating function for the sequence $\{L(n)\}_{n\geq 0}$.

Also, by Definition 2.2 we obtain that

$$L(n) = B(n) + \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} f_n(i,j) = B(n) + \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} f_n(i,j,j+1).$$

The last equality can be proved as follows: observe that if a permutation $\pi \in \mathcal{F}_n$ is such that $1 \leq \pi_1 = i < \pi_2 = j \leq n-2$, then $\pi_3 = j+1$ otherwise

- if $\pi_3 < \pi_2$ then the entries j, π_3, $n-1$ and n give an occurrence of either 2134 or 1243.
- if $\pi_3 > \pi_2 + 1$ then the entries i, j, π_3 and $j+1$ give an occurrence of 1243.

Now, observe that if a permutation $\pi \in \mathcal{F}_n$ is such that $1 \leq \pi_1 = i < \pi_2 = j \leq n-2$ and $\pi_3 = j+1$, then there no occurrence of the patterns 1243, 2134 and 1324 can involve the entry j and the entry $j+1$. Thus, $f_n(i,j,j+1) = f_{n-1}(i,j)$. Therefore, by the definitions of the sequence $L(n)$ we obtain that for all $n \geq 4$,

$$L(n) = B(n) + \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2} f_{n-1}(i,j) = B(n) + L(n-1).$$

Hence, multiplying by x^n and summing over all $n \geq 4$ we get that $L(x) = \frac{1}{1-x}B(x)$, as required. □

Now we are ready to prove the main result of this paper, namely Theorem 1.2, which is restated here for easy reference.

Theorem 1.2. The ordinary generating function for the number of maximal-poset permutations in S_n is given by

$$\frac{2 - C(x)}{2 - x - C(x)}.$$

Moreover, the number of maximal-poset permutations in S_n is given by

$$P_{5n} - 4P_{5(n-1)} + 2P_{5(n-2)} - \sum_{j=0}^{n-2} C_j P_{5(n-2-j)},$$

where P_n is the n-th Padovan number (see [11, A012814, A000931]) and C_n is the n-th Catalan number.

Proof. Combining Theorem 2.4, Proposition 2.3, and Proposition 2.6 gives

$$\begin{aligned}(1-3x+x^2)f(x) - 1 + 2x &= x^2C^2(x)((1-x)f(x)-1)\\ &\quad + \tfrac{1}{1-x}\Big(x^3C^2(x)((1-x)f(x)-1) + 2x^2(f(x)-1) - x^3 f(x)\Big).\end{aligned}$$

Solving the above equation we get that

$$f(x) = \frac{2 - C(x)}{2 - x - C(x)} = \frac{2 - 9x + 4x^2 + x\sqrt{1-4x}}{2(1 - 5x + 4x^2 - x^3)}.$$

Hence, by using the fact that the ordinary generating function for the fifth term of the Padovan sequence is given by $\frac{1}{1-5x+4x^2-x^3}$ (see [11, A012814]), we obtain that

$$f(x) = \sum_{n\geq 0} x^n \left(P_{5n} - 4P_{5(n-1)} + 2P_{5(n-2)} - \sum_{j=0}^{n-2} C_j P_{5(n-2-j)} \right),$$

as claimed. □

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References

[1] M. D. Atkinson, Permutations which are the union of an increasing and decreasing subsequence, *Electron. J. Combin* **5** (1998) #R6.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, Generating functions for generating trees, Formal power series and algebraic combinatorics (Barcelona, 1999), *Discr. Math.* **246:1-3**:29–55, 2002.

[3] E. Barcucci, A. D. Lungo, E. Pergola, and R. Pinzani, Permutations avoiding and increasing number of length-increasing forbidden subsequences, *Discr. Math. Theor. Comput. Sci.* **4** (2000) 31–44.

[4] M. Bóna, Exact enumeration of 1342-avoiding permutations: A close link with labeled trees and planar maps, *J. Combin. Theory, Series A* **80** (1997) 257–272.

[5] M. Bóna, The permutation classes equinumerous to the smooth class, *Electron. J. Combin.* **5** (1998) #R31.

[6] R.M. Green and J. Losonczy, Freely braided elements in Coxeter groups, *Annals of Combin.* **6** (2002) 337-348.

[7] R.M. Green and J. Losonczy, Freely braided elements in Coxeter groups II, *Adv. Appl. Math.* **33** (2004) 26-39.

[8] T. Mansour, On an open problem of Green and Losonczy: exact enumeration of freely braided permutations, *Discr. Math. Theor. Comput. Sci.* **6:2** (2004) 461–470.

[9] T. Mansour and Z. Stankova, 321-polygon-avoiding permutations and Chebyshev polynomials, *Electron. J. Combin.* **9:2** (2003) #R5 (16pp.).

[10] T. Mansour and A. Vainshtein, Avoiding maximal parabolic subgroups of S_k, *Discr. Math. Theor. Comput. Sci.* **4** (2000) 67–75.

[11] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at *www. research.att.com/ njas/sequences/*.

[12] B.E. Tenner, Reduced decompositions and permutation patterns, preprint arXiv:math.CO/0506242.