Finite symmetric graphs with two-arc transitive quotients (II)

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Abstract

Let Γ be a finite G-symmetric graph whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$. It was observed that the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ can be $(G, 2)$-arc transitive even if Γ itself is not necessarily $(G, 2)$-arc transitive. In a previous paper of Iranmanesh, Praeger and Zhou, this observation motivated a study of G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and, for blocks $B, C \in \mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, there are exactly $|B| - 2$ (≥ 1) vertices in B which have neighbours in C. In the present paper we investigate the general case where $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and is not multi-covered by Γ, that is, at least one vertex in B has no neighbour in C for adjacent $B, C \in \mathcal{B}$. In this case it is natural to analyse the dual $\mathcal{D}^*(B)$ of the 1-design $\mathcal{D}(B) := (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ and the

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dual $\overline{\mathcal{D}}^*(B)$ of the complement $\overline{\mathcal{D}}(B)$ of $\mathcal{D}(B)$, where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$ and $\alpha \mathrm{I} C$ ($\alpha \in B$, $C \in \Gamma_{\mathcal{B}}(B)$) in $\mathcal{D}(B)$ if and only if α has at least one neighbour in C. A crucial feature is that $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$ admit G as a group of automorphisms acting 2-transitively on points and transitively on blocks. The case when no point of $\mathcal{D}(B)$ is incident with two blocks can be reduced to multicovers, and the case when no point of $\overline{\mathcal{D}}(B)$ is incident with two blocks can be partially reduced to the 3-arc graph construction. In the general situation $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$ are $(G,2)$-point-transitive and G-block-transitive 2-designs, and exploring relationships between them and Γ is an attractive research direction. In this paper we investigate the degenerate case where $\mathcal{D}^*(B)$ or $\overline{\mathcal{D}}^*(B)$ is a trivial Steiner system with block size 2, that is, a complete graph. In each of these cases we give a construction which produces symmetric graphs with the corresponding properties, and we prove further that every such graph Γ can be constructed from $\Gamma_{\mathcal{B}}$ by using the construction.

Keywords: Symmetric graph; two-arc transitive graph; quotient graph; three-arc graph; 2-design

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1 Introduction

This paper is a continuation of [24, 20], where two classes of symmetric graphs with 2-arc transitive quotients were investigated. Although most quotient graphs of 2-arc transitive graphs are not themselves 2-arc transitive (see e.g. [29]), the main results of [24] suggest that under certain circumstances a quotient of a symmetric graph can be 2-arc transitive even when the original graph is not 2-arc transitive. This observation motivated the following general questions [20, Question 1.1]: When does a quotient of a symmetric graph admit a natural 2-arc transitive group action? If there is such a quotient, what information does this give us about the original graph? This paper is an attempt on the second question under the assumption that the original is not a multicover of the quotient. Thus, the objects of investigation in the paper are symmetric graphs with 2-arc transitive quotients; and the main results of the paper, which will be summarised in Section 1.2, are concerned with the structure and construction of such graphs.

There has been a lot of interest in 2-arc transitive graphs since the classification of finite simple groups. See for example [7, 14, 15, 21, 22, 23, 28], [1, 25, 27] for 2-arc transitive Cayley graphs, and [17, 20, 24, 33, 35] for 2-arc transitive quotients of symmetric graphs. The present paper is a contribution towards symmetric graphs with 2-arc transitive quotients, and it forms part of our study of imprimitive symmetric graphs.

The reader is referred to [4, 19] for basic results about symmetric graphs, and to [29, 30] for more recent development in the area.

1.1 Notation and terminology

Let us first introduce the notation and terminology that will be used throughout the paper.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite graph and $s \geq 1$ an integer. An *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of $s+1$ vertices of Γ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. Denote by $\mathtt{Arc}_s(\Gamma)$ the set of s-arcs of Γ. Let G be a finite group acting on $V(\Gamma)$. The graph Γ is said to *admit* G as a group of automorphisms if G preserves the adjacency of Γ, that is, for any $\alpha, \beta \in V(\Gamma)$ and $g \in G$, α and β are adjacent in Γ if and only if α^g and β^g are adjacent in Γ. In the case where G is transitive on $V(\Gamma)$ and, under the induced action, transitive on $\mathtt{Arc}_s(\Gamma)$, Γ is said to be *(G, s)-arc transitive*. From this definition it is clear that a (G, s)-arc transitive graph must be $(G, s-1)$-arc transitive, where $(G, 0)$-arc transitivity is interpreted as G-vertex transitivity. A 1-arc is usually called an *arc*, and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. In this paper we will use $\mathtt{Arc}(\Gamma)$ in place of $\mathtt{Arc}_1(\Gamma)$. Evidently, a G-vertex transitive graph Γ is G-symmetric ($(G, 2)$-arc transitive, respectively) if and only if G_α is transitive (2-transitive, respectively) on $\Gamma(\alpha)$.

Roughly speaking, in most cases a G-symmetric graph Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$, where $B^g := \{\alpha^g : \alpha \in B\}$. In such a case Γ is called an *imprimitive G-symmetric graph*. From permutation group theory [12, Corollary 1.5A], this happens precisely when the stabilizer G_α in G of a vertex $\alpha \in V(\Gamma)$ is not a maximal subgroup of G. In this case the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exist $\alpha \in B$, $\beta \in C$ such that α, β are adjacent in Γ. In this paper we will always assume that $\Gamma_{\mathcal{B}}$ contains at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ (see e.g. [4, Proposition 22.1]). It is not difficult to see that $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced action (possibly unfaithful) of G on $\mathcal{B}$. Although $\Gamma_{\mathcal{B}}$ stores a lot of information about the original graph Γ, a genuine picture of Γ would need the bipartite subgraph induced on two adjacent blocks and a 1-design with point set B. Let $\Gamma(\alpha)$ denote the *neighbourhood* of α in Γ. For $B \in \mathcal{B}$, define $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$, and denote by $\Gamma_{\mathcal{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathcal{B}}$. For adjacent blocks B, C of $\mathcal{B}$, let $\Gamma[B, C]$ be the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Define $\mathcal{D}(B) := (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ to be the incidence structure in which $\alpha \mathrm{I} C$ for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$ if and only if $\alpha \in \Gamma(C)$; such a pair (α, C) is called a *flag* of $\mathcal{D}(B)$. The triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$ mirrors the structure of Γ, and this approach to imprimitive symmetric graphs was suggested in [16] and further developed in [17, 24, 20, 32, 33, 35, 36]. Let

$$v := |B|, \ \ k := |\Gamma(C) \cap B|, \ \ b := |\Gamma_{\mathcal{B}}(B)|, \ \ r := |\Gamma_{\mathcal{B}}(\alpha)|$$

where $\Gamma_{\mathcal{B}}(\alpha) := \{C \in \Gamma_{\mathcal{B}}(B) : \alpha \mathrm{I} C\}$. In the case where $k = v$, Γ is called a *multicover* [29] of $\Gamma_{\mathcal{B}}$. Since Γ is G-symmetric, it can be checked easily that $\mathcal{D}(B)$ is a 1-(v, k, r) design with b blocks, and is independent of the choice of B up to isomorphism. The number of blocks of $\mathcal{D}(B)$ with the same trace is a constant, m, called the *multiplicity* of $\mathcal{D}(B)$. Also, up to isomorphism, $\Gamma[B, C]$ is independent of the choice of adjacent blocks B, C of $\mathcal{B}$. As we will see later sometimes it is convenient to analyse the *complementary structure* $\overline{\mathcal{D}}(B) := (B, \Gamma_{\mathcal{B}}(B), \bar{\mathrm{I}})$ of $\mathcal{D}(B)$, for which $\alpha \bar{\mathrm{I}} C$ if and only if $\alpha \notin \Gamma(C)$, that is, (α, C) is a flag of $\overline{\mathcal{D}}(B)$ if and only if it is an *antiflag* of $\mathcal{D}(B)$ and vice versa. *In the following we will take* $\mathcal{D}(B)$ *and* $\overline{\mathcal{D}}(B)$ *respectively as having blocks* $\Gamma(C) \cap B$ *and* $B \setminus \Gamma(C)$ *(where* $C \in \Gamma_{\mathcal{B}}(B)$*), each repeated* m *times, and interpret their incidence relations as set-theoretic inclusion.*

The notation and terminology for graphs, groups and designs used in the paper are standard; see e.g. [4], [12] and [3], respectively. For a group G acting on a set Ω and for $X \subseteq \Omega$, G_X and $G_{(X)}$ are the setwise and pointwise stabilisers of X in G, respectively. For $\alpha \in \Omega$, $\alpha^G := \{\alpha^g : g \in G\}$ is the *G-orbit* on Ω containing α, and $G_\alpha := \{g \in G : \alpha^g = \alpha\}$ is the *stabiliser* of α in G. The action of G on Ω is said to be *faithful* if $G_{(\Omega)} = 1$, and *regular* if it is transitive and $G_\alpha = 1$ for $\alpha \in \Omega$. Suppose that a group G acts on two sets Ω_1 and Ω_2. If there exists a bijection $\psi : \Omega_1 \to \Omega_2$ such that $\psi(\alpha^g) = (\psi(\alpha))^g$ for all $\alpha \in \Omega_1$ and $g \in G$, then the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent*.

A graph Γ is called *regular* if all vertices of it have the same valency, denoted by $\mathtt{val}(\Gamma)$. The union of n vertex-disjoint copies of Γ is denoted by $n \cdot \Gamma$. For two graphs Γ and Σ, the *lexicographic product* of Γ by Σ, $\Gamma[\Sigma]$, is the graph with vertex set $V(\Gamma) \times V(\Sigma)$ such that $(\alpha, \beta), (\gamma, \delta)$ are adjacent if and only if either α, γ are adjacent in Γ, or $\alpha = \gamma$ and β, δ are adjacent in Σ.

1.2 A summary of the main results

Let Γ be a G-symmetric graph with $V(\Gamma)$ admitting a nontrivial G-invariant partition $\mathcal{B}$. In this summary we assume that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, and that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Then both $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$ admit G as a group of automorphisms acting transitively on points and 2-transitively on blocks. Thus, G is 2-transitive on the point sets of $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$, the *dual designs* of $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$, respectively.

Of course, $\overline{\mathcal{D}}(B)$, $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$ are all induced from $\mathcal{D}(B)$. However, sometimes it seems handy to think of these derived designs instead of $\mathcal{D}(B)$.

It may happen that no two points of $\mathcal{D}^*(B)$ lie in a block of $\mathcal{D}^*(B)$ simultaneously. In this case Γ admits a second G-invariant partition $\mathcal{Q}$ such that Γ is a multicover of $\Gamma_{\mathcal{Q}}$ (Theorem 3.5). It may also happen that no two points of $\overline{\mathcal{D}}^*(B)$ lie in a block of $\overline{\mathcal{D}}^*(B)$ simultaneously. This second possibility can be partially reduced to the 3-arc graph construction, which was introduced in [24] and developed in [32]. (See the beginning of Section 3.2 for the 3-arc graph construction.) More precisely, in this case Γ admits a second G-invariant partition $\mathcal{P}$ such that $\Gamma_{\mathcal{P}}$ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ or otherwise Γ, $\Gamma_{\mathcal{P}}$ and $\Gamma_{\mathcal{B}}$ are all known (Theorem 3.7). Combining the 3-arc graph construction and the well-known covering graph construction [4], we then give (Construction 3.8) a method of constructing symmetric graphs for which the second possibility occurs.

In the general case, $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$ are both 2-designs with parameters $\lambda \geq 1$ and $\overline{\lambda} \geq 1$, and with G acting 2-transitively on points. These 2-designs may contain key information about the structure of Γ and $\Gamma_{\mathcal{B}}$, yet to be dug out, and many natural and interesting problems arise from these designs. For example, we may ask when a 2-point-transitive, flag-transitive (antiflag-transitive) and block-transitive 2-design can occur as $\mathcal{D}^*(B)$ ($\overline{\mathcal{D}}^*(B)$), and how much structural information of Γ and/or $\Gamma_{\mathcal{B}}$ we can get if we know $\mathcal{D}^*(B)$ or $\overline{\mathcal{D}}^*(B)$. No one may expect simple answers to these questions due to the complex of both imprimitive symmetric graphs and 2-designs. Perhaps it is not realistic to try all possible cases in one time. So in this paper we will focus on the following "trivial" cases: (i) $\mathcal{D}^*(B)$ is the trivial design with $\lambda = 1$ and block size $r = 2$; (ii) $\overline{\mathcal{D}}^*(B)$ is the trivial design with $\overline{\lambda} = 1$ and block size $b - r = 2$. Interesting combinatorics appears even in such degenerate cases. In fact, we find a simple construction which can produce imprimitive symmetric graphs in case (i), and moreover every imprimitive symmetric graph in this case can be constructed from its quotient by using this method. See Construction 4.2, Theorem 4.4 and Theorem 4.10. This is another kind of "3-arc construction" since it involves a self-paired orbit of the 3-arcs of a regular graph. However, unlike the 3-arc construction in [24, 32]. This time the vertices of the constructed graph are paths of length two of the given graph. The main results for case (i) imply the following somewhat strange consequence (Corollary 4.8): every connected $(G, 3)$-arc transitive graph of valency at least 3 is a quotient graph of at least one G-symmetric but not $(G, 2)$-arc transitive graph. Thus, a connected graph can be just symmetric but not 2-arc transitive, but a quotient of it can be 7-arc transitive. (There are infinitely many such graphs since there are infinitely many 7-arc transitive graphs [11].) For case (ii) we find another construction and prove that every imprimitive symmetric graph in this case can be constructed from its quotient by using this construction. See Construction 4.11, Theorem 4.12 and Theorem 4.14.

As preliminaries we will derive general information about $\mathcal{D}(B)$ and $\overline{\mathcal{D}}(B)$ in the next section. The most striking result in this section says that either $\lambda = 0$ or $\mathcal{D}(B)$ is connected as a hypergraph. See Theorem 2.2 for details.

2 Induced designs

In this section we do not assume 2-arc transitivity of the quotient graph. The following simple facts were observed in [16, Section 3] under the assumption that G_α is primitive on $\Gamma(\alpha)$, and stated explicitly in [32, Lemma 2.1] for any imprimitive symmetric graph.

Lemma 2.1 *Let Γ be a finite G-symmetric graph whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$, and let B be a block of $\mathcal{B}$. Then G_B induces a group of automorphisms of $\mathcal{D}(B)$ which is transitive on the points, the blocks and the flags of $\mathcal{D}(B)$.*

The following theorem is obtained by viewing $\overline{\mathcal{D}}(B)$ as a uniform hypergraph [2] with vertex set B and hyperedges $B \setminus \Gamma(C)$, $C \in \Gamma_{\mathcal{B}}(B)$. It says that in general $\overline{\mathcal{D}}(B)$ is connected as a hypergraph. This was known in [20, Theorem 2.1] in the special case when $k = v - 2 \geq 1$, for which $\overline{\mathcal{D}}(B)$ is the graph Γ^B as defined in [20]. (A hypergraph is *connected* [2] if, for any two vertices α, β there exists a sequence $E_0, E_1, \ldots, E_n$ of hyperedges such that $\alpha \in E_0, \beta \in E_n$ and $E_{i-1} \cap E_i \neq \emptyset$ for $i = 1, \ldots, n$.) Let $d := \mathtt{val}(\Gamma[B, C])$ for adjacent blocks B, C of $\mathcal{B}$.

Theorem 2.2 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Then $\overline{\mathcal{D}}(B)$ is a* 1-$(v, v-k, b-r)$ *design, and it admits G_B as a group of automorphisms acting transitively on the points, the blocks and the antiflags. Moreover, one of the following (a)-(b) occurs.*

(a) *$\overline{\mathcal{D}}(B)$ is connected as a hypergraph.*

(b) *The blocks of $\overline{\mathcal{D}}(B)$ form a G_B-invariant partition of B, and the blocks of $\overline{\mathcal{D}}(B)$ for B running over $\mathcal{B}$ form a G-invariant partition of $V(\Gamma)$, namely*

$$\mathcal{P} := \bigcup_{B \in \mathcal{B}} \{B \setminus \Gamma(C) : C \in \Gamma_{\mathcal{B}}(B)\}, \tag{1}$$

which is a nontrivial refinement of $\mathcal{B}$. In this case $v-k$ divides v and k, $b = m(t+1)$, $r = mt$, where $t = k/(v-k)$, either $k = v/2$ or $k \geq 2v/3$, $G_{(\mathcal{P})} = G_{(\mathcal{B})}$, and the *parameters with respect to* $(\Gamma, \mathcal{P})$ *satisfy* $v_{\mathcal{P}} = v - k$ *and* $r_{\mathcal{P}} d_{\mathcal{P}} = rd$. *Moreover,* $\mathcal{P}$ *admits a* G*-invariant partition* $\hat{\mathcal{B}}$ *induced by* $\mathcal{B}$ *such that* $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$*, and the parameters with respect to* $(\Gamma_{\mathcal{P}}, \hat{\mathcal{B}})$ *satisfy* $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = t \geq 1$, $b_{\hat{\mathcal{B}}} = b$, $r_{\hat{\mathcal{B}}} = r$ *and* $m_{\hat{\mathcal{B}}} = m$.

Proof From Lemma 2.1 it follows immediately that G_B induces a group of automorphisms of $\overline{\mathcal{D}}(B)$ which is transitive on the points, the blocks and the antiflags of $\overline{\mathcal{D}}(B)$. In particular, $\overline{\mathcal{D}}(B)$ must be a 1-design with v points and b blocks of size $v-k$. Hence each point is incident with $b(v-k)/v = b - r$ blocks in $\overline{\mathcal{D}}(B)$. In other words, $\overline{\mathcal{D}}(B)$ is a 1-$(v, v-k, b-r)$ design.

Since G_B is transitive on the points and blocks of $\overline{\mathcal{D}}(B)$, the connected components $B^{(1)}, \ldots, B^{(\omega)}$ of $\overline{\mathcal{D}}(B)$ (as a hypergraph) form a G_B-invariant partition $\mathcal{Q}$ of B. It is straightforward to show that G_B is transitive on such components. Note that the vertices in the same block of $\overline{\mathcal{D}}(B)$ must be in the same connected component of $\overline{\mathcal{D}}(B)$. Thus, each $B^{(i)}$ is the union of some blocks of $\overline{\mathcal{D}}(B)$. In the following we will show that either $\overline{\mathcal{D}}(B)$ is connected (that is, $\omega = 1$) and (a) holds, or each part of $\mathcal{Q}$ is a block of $\overline{\mathcal{D}}(B)$ (that is, $\omega = b/m$).

Let us first assume that each block $B^{(i)}$ of $\mathcal{Q}$ contains more than $v-k$ vertices, that is, each $B^{(i)}$ is the union of at least two non-repetitive blocks of $\overline{\mathcal{D}}(B)$. In this case we must have $\omega = 1$. Suppose to the contrary that $\omega \geq 2$, and let $\gamma \in B^{(2)}$. Let $X = B \setminus \Gamma(C)$ be a block of $\overline{\mathcal{D}}(B)$ contained in $B^{(1)}$, where $C \in \Gamma_{\mathcal{B}}(B)$. Then $X \neq B^{(1)}$ and hence we may take a vertex $\alpha \in B^{(1)} \setminus X$. Thus $\alpha, \gamma \in \Gamma(C) \cap B$ and consequently there exist $\beta, \delta \in \Gamma(B) \cap C$ such that $(\alpha, \beta), (\gamma, \delta) \in \mathtt{Arc}(\Gamma)$. Since Γ is G-symmetric, there exists $g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$. Since $\mathcal{B}$ is a G-invariant partition of $V(\Gamma)$, this implies that g fixes each of B, C setwise. Thus, $X^g = B^g \setminus \Gamma(C^g) = B \setminus \Gamma(C) = X$. However, $\mathcal{Q}$ is a G_B-invariant partition of B and $X \subset B^{(1)}$. Hence g fixes $B^{(1)}$ setwise. This together with $\alpha \in B^{(1)}$ implies $\gamma = \alpha^g \in B^{(1)}$, which contradicts the assumption that $\gamma \in B^{(2)}$. This contradiction shows that $\omega = 1$, that is, $\overline{\mathcal{D}}(B)$ is connected and hence (a) holds.

In the remaining case each $B^{(i)}$ consists of only one block of $\overline{\mathcal{D}}(B)$. In this case the blocks (ignoring the multiplicity) of $\overline{\mathcal{D}}(B)$ form a G_B-invariant partition of B, and this partition is exactly $\mathcal{Q}$. Based on this fact it is straightforward to show that the blocks of $\overline{\mathcal{D}}(B)$ for B running over $\mathcal{B}$ form a G-invariant partition $\mathcal{P}$ of $V(\Gamma)$ with block size $v_{\mathcal{P}} = v - k$. Clearly, $\mathcal{P} = \bigcup_{B \in \mathcal{B}} \{B \setminus \Gamma(C) : C \in \Gamma_{\mathcal{B}}(B)\}$, and $\mathcal{P}$ is a refinement of $\mathcal{B}$. Thus, $v - k$ is a divisor of v and hence a divisor of k. Also, setting $t = k/(v-k)$, we have $b = mv/(v-k) = m(t+1)$ and $r = b - m = mt$. Note that $\mathcal{P} \neq \mathcal{B}$ and so we have $v/(v-k) \geq 2$, that is, $k \geq v/2$. Moreover, if $t = 1$ then $k = v/2$; if $t > 1$ then

$k/(v-k) \geq 2$, which implies $k \geq 2v/3$. Evidently, we have $r_{\mathcal{P}} d_{\mathcal{P}} = rd = \mathtt{val}(\Gamma)$. Since $\mathcal{B}$ is G-invariant and $\mathcal{P}$ is a refinement of $\mathcal{B}$, we have $G_{(\mathcal{P})} \subseteq G_{(\mathcal{B})}$. On the other hand, if $g \in G_{(\mathcal{B})}$, then g fixes setwise each block of $\mathcal{B}$ and hence fixes setwise each block of $\mathcal{P}$, that is, $g \in G_{(\mathcal{P})}$. Therefore, $G_{(\mathcal{P})} = G_{(\mathcal{B})}$.

Let $\hat{B} := \{B \setminus \Gamma(C) \in \mathcal{P} : C \in \Gamma_{\mathcal{B}}(B)\}$ (ignoring the multiplicity of each $B \setminus \Gamma(C)$). Then $\hat{B}$ is the set of blocks of $\mathcal{P}$ contained in B. Let

$$\hat{\mathcal{B}} := \{\hat{B} : B \in \mathcal{B}\}. \tag{2}$$

It is straightforward to show that $\hat{\mathcal{B}}$ is a G-invariant partition of $\mathcal{P}$, and that $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$ via the bijection $B \leftrightarrow \hat{B}$ between $\mathcal{B}$ and $\hat{\mathcal{B}}$. Let $v_{\hat{\mathcal{B}}}, k_{\hat{\mathcal{B}}}, b_{\hat{\mathcal{B}}}, r_{\hat{\mathcal{B}}}, m_{\hat{\mathcal{B}}}$ be the parameters with respect to $(\Gamma_{\mathcal{P}}, \hat{\mathcal{B}})$. For blocks B, C of $\mathcal{B}$ adjacent in $\Gamma_{\mathcal{B}}$, each "vertex" $B \setminus \Gamma(D)$ of $\hat{B}$ other than $B \setminus \Gamma(C)$ (where $D \in \Gamma_{\mathcal{B}}(B) \setminus \{C\}$) is adjacent to at least one "vertex" of $\hat{C}$ other than $C \setminus \Gamma(B)$. Thus we have $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = (v/v_{\mathcal{P}}) - 1 = t \geq 1$. Also, the isomorphism above between $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}}$ and $\Gamma_{\mathcal{B}}$ implies that $b_{\hat{\mathcal{B}}} = b$. Since $v_{\hat{\mathcal{B}}} r_{\hat{\mathcal{B}}} = b_{\hat{\mathcal{B}}} k_{\hat{\mathcal{B}}}$, it follows that $(v/(v-k)) r_{\hat{\mathcal{B}}} = b(k/(v-k))$ and hence $r_{\hat{\mathcal{B}}} = r$. Also, it is clear that $m_{\hat{\mathcal{B}}} = m$. $\square$

Remark 2.3 (a) In contrast with $\mathcal{D}(B)$ (Lemma 2.1), the group G_B is not necessarily transitive on the flags of $\overline{\mathcal{D}}(B)$.

(b) In possibility (b) of Theorem 2.2, the case $k = v/2$ occurs if and only if $t = 1$, and in this case we have $b = 2m$, $r = m$, and $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = 1$. In the other case where $k \geq 2v/3$ (that is, $t \geq 2$), we have $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = t \geq 2$. In both cases the graph $\Gamma_{\mathcal{P}}$ together with its vertex-partition $\hat{\mathcal{B}}$ satisfies the assumptions in [24]. This connection will be explored in the next section under the condition that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In particular, we will give a construction of graphs with 2-arc transitive quotients satisfying the conditions of Theorem 2.2(b).

(c) In possibility (b) of Theorem 2.2, since the blocks of $\overline{\mathcal{D}}(B)$ form a G_B-invariant partition of B with block size $v - k < v$, G_B is not 2-transitive on B unless $k = v - 1$.

(d) Let $(\Gamma, G, \mathcal{B})$ be as in Theorem 2.2. In the case where in addition $v - k$ is a prime, either (b) in Theorem 2.2 occurs, or G is faithful on $\mathcal{B}$. In fact, since $G_{(\mathcal{B})}$ is a normal subgroup of G, the set $\mathcal{O}$ of $G_{(\mathcal{B})}$-orbits on the vertices of Γ is a G-invariant partition of $V(\Gamma)$, which is a refinement of $\mathcal{B}$. Since $G_{(\mathcal{B})}$ fixes each block of $\mathcal{B}$ setwise, each block of $\overline{\mathcal{D}}(B)$ must be invariant under the action of $G_{(\mathcal{B})}$ and hence is a union of some blocks of $\mathcal{O}$. Under the condition that G is unfaithful on $\mathcal{B}$, $\mathcal{O}$ is a nontrivial partition of $V(\Gamma)$. Thus, if in addition $v - k$ is a prime, then each block of $\overline{\mathcal{D}}(B)$ is a block of $\mathcal{O}$; in other words, case (b) in Theorem 2.2 occurs.

Similarly, if k is a prime, then either the blocks of $\mathcal{D}(B)$ for B running over $\mathcal{B}$ form a G-invariant partition of $V(\Gamma)$, or G is faithful on $\mathcal{B}$. Also, if v and k are coprime, then $\mathcal{O}$ is forced to be the trivial partition $\{\{\alpha\} : \alpha \in V(\Gamma)\}$, and consequently G must be faithful on $\mathcal{B}$.

(e) For $\mathcal{D}(B)$ the counterpart of Theorem 2.2 is not true, that is, under the same conditions $\mathcal{D}(B)$ is not necessarily connected as a hypergraph if its blocks do not form a partition of B.

For each block X of $\mathcal{D}(B)$, define

$$\langle X \rangle := \{C \in \Gamma_{\mathcal{B}}(B) : \Gamma(C) \cap B = X\}.$$

Thus, $\langle X \rangle$ is an m-element subset of $\Gamma_{\mathcal{B}}(B)$. Let

$$\mathcal{L}(B) := \{\langle X \rangle : X \text{ a block of } \mathcal{D}(B)\}.$$

The following lemma is straightforward, and it is a generalisation of [20, Lemma 3.1]. Note that to establish (b) we require that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. A G-symmetric graph is called G-locally primitive or G-locally imprimitive according to whether the stabiliser of a vertex in G is primitive or imprimitive on its neighbourhood.

Lemma 2.4 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Then $\mathcal{L}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$ with block size m. The induced action of G_B on $\mathcal{L}(B)$, the action of G_B on the blocks of $\mathcal{D}(B)$ (ignoring the multiplicity of each block) and the action of G_B on the blocks of $\overline{\mathcal{D}}(B)$ (ignoring the multiplicity of each block) are permutationally equivalent with respect to the bijections $X \mapsto \langle X \rangle$, $\overline{X} \mapsto \langle X \rangle$, for blocks X of $\mathcal{D}(B)$. Moreover, if $m = 1$ then the actions of G_B on $\Gamma_{\mathcal{B}}(B)$, on the blocks of $\mathcal{D}(B)$ and on the blocks of $\overline{\mathcal{D}}(B)$ are permutationally equivalent; and if $m \geq 2$ then $\Gamma_{\mathcal{B}}$ is G-locally imprimitive and hence $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive.*

3 Two-arc transitive quotients

From now on we will deal with G-symmetric graphs $(\Gamma, \mathcal{B})$ with $(G, 2)$-arc transitive quotients $\Gamma_{\mathcal{B}}$. The following consequence of Lemma 2.4 will be the starting point for our investigation.

Lemma 3.1 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive.*

Then $\mathcal{D}(B)$ contains no repeated blocks, and moreover G_B is 2-transitive on the blocks of $\mathcal{D}(B)$ and 2-transitive on the blocks of $\overline{\mathcal{D}}(B)$.

Thus, when $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, any two distinct blocks of $\mathcal{D}(B)$ ($\overline{\mathcal{D}}(B)$, respectively) intersect in the same number of points. That is,

$$\lambda := |X \cap Y|, \ \ \overline{\lambda} := |\overline{X} \cap \overline{Y}|$$

are independent of the choice of distinct blocks X, Y of $\mathcal{D}(B)$, where $\overline{X} = B\backslash X, \overline{Y} = B\backslash Y$ are blocks of $\overline{\mathcal{D}}(B)$. Evidently, we have

$$\overline{\lambda} = v - 2k + \lambda. \tag{3}$$

Recall that $\mathcal{D}^*(B)$, $\overline{\mathcal{D}}^*(B)$ are the dual designs of $\mathcal{D}(B)$, $\overline{\mathcal{D}}(B)$, respectively. From Lemmas 2.1 and 3.1 we obtain the following results immediately.

Theorem 3.2 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. If $\lambda \geq 1$, then $\mathcal{D}^*(B)$ is a 2-(b,r,λ) design with parameters $(v^*, b^*, r^*, k^*, \lambda^*) = (b, v, k, r, \lambda)$, and it admits G_B as a group of automorphisms acting 2-transitively on its points and transitively on its blocks and flags. If $\overline{\lambda} \geq 1$, then $\overline{\mathcal{D}}^*(B)$ is a 2-$(b, b-r, \overline{\lambda})$ design with parameters $(\overline{v}^*, \overline{b}^*, \overline{r}^*, \overline{k}^*, \overline{\lambda}^*) = (b, v, v-k, b-r, \overline{\lambda})$, and it admits G_B as a group of automorphisms acting 2-transitively on its points and transitively on its blocks and antiflags.*

Corollary 3.3 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive. Then*

$$\lambda(b-1) = k(r-1). \tag{4}$$

Proof In the case where $\lambda \geq 1$, $\mathcal{D}^*(B)$ is a 2-design by Theorem 3.2 and hence (4) follows from $\lambda^*(v^*-1) = r^*(k^*-1)$. Similarly, if $\overline{\lambda} \geq 1$, then $\overline{\mathcal{D}}^*(B)$ is a 2-design and from $\overline{\lambda}^*(\overline{v}^*-1) = \overline{r}^*(\overline{k}^*-1)$ we have $\overline{\lambda}(b-1) = (v-k)(b-r-1)$, which gives (4) after simplification.

In the remaining case $\lambda = \overline{\lambda} = 0$ we have $k = v/2$ by (3), and hence $r = 1$ and (4) is valid as well. □

The discussion above suggests that we may distinguish the following (not exclusive) three cases:

Case 1: $\lambda = 0$;

Case 2: $\overline{\lambda} = 0$;

Case 3: $\lambda \geq 1, \overline{\lambda} \geq 1$.

The remainder of this section is devoted to the first two cases.

3.1 Case 1: $\lambda = 0$

This case can be partially characterised by the following simple construction, which appeared in [20, Example 2.4] when Σ is trivalent.

Construction 3.4 Let Σ be a G-symmetric graph. Define $\Gamma_{\mathtt{arc}}(\Sigma)$ to be the graph with vertex set $\mathtt{Arc}(\Sigma)$ and edge set $\{\{(\sigma,\tau),(\tau,\sigma)\} : (\sigma,\tau) \in \mathtt{Arc}(\Sigma)\}$. Then $\Gamma_{\mathtt{arc}}(\Sigma)$ is a G-symmetric graph admitting $\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$ as a G-invariant partition such that $k_{\mathcal{B}(\Sigma)} = 1$, $\Gamma_{\mathtt{arc}}(\Sigma) \cong \ell \cdot K_2$ and $(\Gamma_{\mathtt{arc}}(\Sigma))_{\mathcal{B}(\Sigma)} \cong \Sigma$, where $\ell = |E(\Sigma)|$ and $B(\sigma) = \{(\sigma,\tau) : \tau \in \Sigma(\sigma)\}$ for $\sigma \in V(\Sigma)$.

Note that, when $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, the case $\lambda = 0$ occurs if and only if $\mathcal{D}(B)$ contains two disjoint blocks.

Theorem 3.5 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive and $\lambda = 0$ (that is, $\mathcal{D}(B)$ contains two disjoint blocks). Then Γ admits a second G-invariant partition, namely*

$$\mathcal{Q} := \bigcup_{B \in \mathcal{B}} \{\Gamma(C) \cap B : C \in \Gamma_{\mathcal{B}}(B)\}, \tag{5}$$

which is a refinement of $\mathcal{B}$, such that $\Gamma_{\mathcal{Q}}$ is $(G,2)$-arc transitive, Γ is a multicover of $\Gamma_{\mathcal{Q}}$, $\Gamma_{\mathcal{Q}} \cong \Gamma_{\mathtt{arc}}(\Gamma_{\mathcal{B}}) \cong \ell \cdot K_2$ where $\ell = |E(\Gamma_{\mathcal{B}})|$, and the parameters with respect to $(\Gamma, \mathcal{Q})$ satisfy $v_{\mathcal{Q}} = k_{\mathcal{Q}} = k = v/b$ and $b_{\mathcal{Q}} = r_{\mathcal{Q}} = r = 1$. Moreover, $\mathcal{Q}$ admits a G-invariant partition $\hat{\mathcal{B}}$ induced by $\mathcal{B}$ such that $(\Gamma_{\mathcal{Q}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$, $\mathcal{D}(\hat{B})$ contains no repeated blocks (where $\hat{B}$ is a block of $\hat{\mathcal{B}}$), and the parameters with respect to $(\Gamma_{\mathcal{Q}}, \hat{\mathcal{B}})$ are given by $v_{\hat{\mathcal{B}}} = b_{\hat{\mathcal{B}}} = b$ and $r_{\hat{\mathcal{B}}} = k_{\hat{\mathcal{B}}} = 1$.

Proof Since $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, $\mathcal{D}(B)$ contains no repeated blocks by Lemma 2.4. Also, from Theorem 3.2 the assumption that $\mathcal{D}(B)$ contains two disjoint blocks implies that any two blocks of $\mathcal{D}(B)$ are disjoint, that is, $\lambda = 0$. Hence the blocks of $\mathcal{D}(B)$ form a G_B-invariant partition of B. From this it is straightforward to show that the blocks of $\mathcal{D}(B)$, for B running over $\mathcal{B}$, form a G-invariant partition of $V(\Gamma)$, which is $\mathcal{Q}$ given in (5). Clearly, $\mathcal{Q}$ is a refinement of $\mathcal{B}$, $v_{\mathcal{Q}} = k_{\mathcal{Q}} = k = v/b$ and $b_{\mathcal{Q}} = r_{\mathcal{Q}} = r = 1$ (noting that $vr = bk$ and $v_{\mathcal{Q}} r_{\mathcal{Q}} = b_{\mathcal{Q}} k_{\mathcal{Q}}$). Thus, $\Gamma_{\mathcal{Q}}$ is a matching (and hence $(G,2)$-arc transitive automatically) and Γ is a multicover of $\Gamma_{\mathcal{Q}}$. Moreover, one can check that $\Gamma_{\mathcal{Q}} \cong \Gamma_{\mathtt{arc}}(\Gamma_{\mathcal{B}})$ with respect to the bijection which maps the vertex "$\Gamma(C) \cap B$" of $\Gamma_{\mathcal{Q}}$ to the vertex "(B,C)" of $\Gamma_{\mathtt{arc}}(\Gamma_{\mathcal{B}})$. Note that $\Gamma_{\mathtt{arc}}(\Gamma_{\mathcal{B}}) \cong \ell \cdot K_2$, where $\ell = |E(\Gamma_{\mathcal{B}})|$.

Let $\hat{B} := \{\Gamma(C) \cap B : C \in \Gamma_{\mathcal{B}}(B)\}$ and $\hat{\mathcal{B}} := \{\hat{B} : B \in \mathcal{B}\}$. It is straightforward to show that $\hat{\mathcal{B}}$ is a G-invariant partition of $\mathcal{Q}$ and the parameters with respect to $(\Gamma_{\mathcal{Q}}, \hat{\mathcal{B}})$ are given by $v_{\hat{\mathcal{B}}} = b_{\hat{\mathcal{B}}} = b$ and $r_{\hat{\mathcal{B}}} = k_{\hat{\mathcal{B}}} = 1$. Furthermore, $(\Gamma_{\mathcal{Q}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$ and $\mathcal{D}(\hat{B})$ contains no repeated blocks. $\square$

Remark 3.6 Unless $\Gamma_{\mathcal{B}} \cong K_2$ the graph Γ in Theorem 3.5 is disconnected since it is a multicover of the disconnected graph $\Gamma_{\mathcal{Q}} \cong \ell \cdot K_2$, whilst $\Gamma_{\mathcal{B}}$ is not necessarily disconnected. Thus, for connected imprimitive G-symmetric graph $(\Gamma, G, \mathcal{B})$ such that $\Gamma_{\mathcal{B}} \ncong K_2$ is $(G,2)$-arc transitive, Case 1 does not appear at all.

3.2 Case 2: $\overline{\lambda} = 0$

In this subsection we will prove that Case 2 can be partially reduced to the 3-arc graph construction. Let Σ be a regular graph, and let Δ be a *self-paired* subset of $\mathtt{Arc}_3(\Sigma)$, that is, $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$. The 3-*arc graph* $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ [24, 32] is the graph with vertex set $\mathtt{Arc}(\Sigma)$ in which $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. In the case where Σ is G-symmetric and G is transitive on Δ under the induced action of G on $\mathtt{Arc}_3(\Sigma)$, $\Gamma := \Xi(\Sigma, \Delta)$ is a G-symmetric graph [24, Section 6] which admits

$$\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$$

as a G-invariant partition such that $\Sigma \cong \Gamma_{\mathcal{B}(\Sigma)}$ under the natural bijection $\sigma \mapsto B(\sigma)$, where $B(\sigma) := \{(\sigma, \tau) : \tau \in \Sigma(\sigma)\}$.

In Theorem 2.2 we discussed Case 2 without assuming the $(G,2)$-arc transitivity of $\Gamma_{\mathcal{B}}$. By Lemma 3.1, under the assumption that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, Case 2 occurs if and only if the blocks of $\overline{\mathcal{D}}(B)$ form a partition of B, which in turn is true if and only if $\overline{\mathcal{D}}(B)$ contains two disjoint blocks. In this case the following theorem says that either both $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{P}}$ can be determined, or $\Gamma_{\mathcal{P}}$ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ with respect to a self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, where $\mathcal{P}$ is as defined in (1). Note that Cases 1 and 2 have overlap, and this happens only when $k = v/2$.

Theorem 3.7 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive and $\overline{\lambda} = 0$ (that is, $\overline{\mathcal{D}}(B)$ contains two disjoint blocks). Then*

(a) $(2k - v)(b - 1) = k(r - 1)$;

(b) *$v - k$ divides v and k, $r = k/(v - k)$, $b = r + 1 = v/(v - k)$, and either $k = v/2$ or $k \geq 2v/3$.*

Moreover, Γ admits a second G-invariant partition of block size $v - k$, namely $\mathcal{P}$ defined in (1), which is a refinement of $\mathcal{B}$, and $\mathcal{P}$ admits a natural G-invariant partition $\hat{\mathcal{B}}$ as defined in (2), such that $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$, $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = r \geq 1$, $b_{\hat{\mathcal{B}}} = b$, $r_{\hat{\mathcal{B}}} = r$ and $\mathcal{D}(\hat{B})$ contains no repeated blocks (where $\hat{B}$ is a block of $\hat{\mathcal{B}}$). In the case where $k = v/2$, we have $\Gamma_{\mathcal{B}} \cong \ell \cdot C_n$, $\Gamma_{\mathcal{P}} \cong (\ell n) \cdot K_2$ and $\Gamma \cong (\ell n) \cdot \Gamma[B, C]$ for some integers $\ell \geq 1, n \geq 3$. In the general case where $k \geq 2v/3$, there exists a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$ such that $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$; moreover, $\Gamma_{\mathcal{B}} \cong K_{b+1}$ if and only if Δ contains a 3-cycle, and in this case $\Gamma_{\mathcal{P}} \cong (b+1) \cdot K_b$, Δ is the set of all 3-cycles of $\Gamma_{\mathcal{B}}$, and G is 3-transitive on $\mathcal{B}$.

Proof Since $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, by Lemma 3.1 the assumption that $\overline{\mathcal{D}}(B)$ contains two disjoint blocks is equivalent to saying that $\overline{\lambda} = 0$, or equivalently the blocks of $\overline{\mathcal{D}}(B)$ form a G_B-invariant partition of B. In this case, Theorem 2.2(b) applies. Thus, $V(\Gamma)$ admits a second G-invariant partition $\mathcal{P}$ (defined in (1)) which is a refinement of $\mathcal{B}$, and $\mathcal{P}$ admits a natural G-invariant partition $\hat{\mathcal{B}}$ (defined in (2)) such that $(\Gamma_{\mathcal{P}})_{\hat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$, $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 = k/(v-k) \geq 1$, $b_{\hat{\mathcal{B}}} = b$, $r_{\hat{\mathcal{B}}} = r$ and $\mathcal{D}(\hat{B})$ contains no repeated blocks. Also, since $m = 1$ by Lemma 3.1, from Theorem 2.2(b) it follows that $v - k$ is a divisor of v and k, $r = k/(v - k)$, $b = r + 1 = v/(v - k)$, and either $k = v/2$ or $k \geq 2v/3$. Moreover, since $\overline{\lambda} = 0$, we have $\lambda = 2k - v$ by (3) and hence $(2k - v)(b - 1) = k(r - 1)$ by (4).

In the case where $k = v/2$, we have $k_{\hat{\mathcal{B}}} = 1$, $\mathcal{D}(B)$ has precisely two blocks which form a partition of B (hence $\lambda = 0$), and $\mathcal{P}$ is also the partition of $V(\Gamma)$ consisting of the blocks of $\mathcal{D}(B)$ for B running over $\mathcal{B}$. In this case it is clear that $b = 2$ and hence $\Gamma_{\mathcal{B}} \cong \ell \cdot C_n$, $\Gamma_{\mathcal{P}} \cong (\ell n) \cdot K_2$, $\Gamma \cong (\ell n) \cdot \Gamma[B, C]$ for some integers $\ell \geq 1, n \geq 3$, where B, C are adjacent blocks of $\mathcal{B}$. In the general case where $k \geq 2v/3$, we have $k_{\hat{\mathcal{B}}} = v_{\hat{\mathcal{B}}} - 1 \geq 2$. Since $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive it follows from [24, Theorem 1] that $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. The statements for the case $\Gamma_{\mathcal{B}} \cong K_{b+1}$ follow from [24, Theorems 8(b) and 10(c)] immediately. $\square$

In the case where $k \geq 2v/3$ and $\Gamma_{\mathcal{B}} \cong K_{b+1}$, from the classification of finite 3-transitive groups (see e.g. [6, pp.8]) it follows that $G/G_{(\mathcal{B})}$ is one of S_{b+1} ($b \geq 3$), A_{b+1} ($b \geq 4$),

M_{b+1} ($b = 10, 11, 21, 22, 23$), M_{11} ($b = 11$), $\mathrm{PSL}(2, b) \le G \le \mathrm{P\Gamma L}(2, b)$ ($b \ge 3$ a prime power), $G = \mathrm{AGL}(d, 2)$ ($b = 2^d - 1 \ge 3$), or $\mathbb{Z}_2^4.A_7$ ($b = 15$). Each of these possibilities can happen, as we will see in Example 3.9.

Parts (a) and (b) of Theorem 3.7 together imply the following inequalities:

$$b \le v,\ r \le k,\ v + r \ge b + k,\ (2k - v)(v - 1) \le k(k - 1). \tag{6}$$

Since $vr = bk$, the second inequality is equivalent to each of the following:

$$v^2 + r^2 \ge b^2 + k^2,\ (v - 1)(r - 1) \le (b - 1)(k - 1). \tag{7}$$

Applying the 3-arc graph construction [24] and the covering graph construction [4, Chapter 19] successively, we obtain the following construction of imprimitive symmetric graphs $(\Gamma, G, \mathcal{B})$ satisfying the conditions of Theorem 3.7.

Construction 3.8 Let Σ be an $(H, 2)$-arc transitive graph of valency $v_\Sigma \ge 3$ such that a self-paired H-orbit Δ on $\mathtt{Arc}_3(\Sigma)$ exists. Let $\Xi := \Xi(\Sigma, \Delta)$ be the 3-arc graph of Σ with respect to Δ. Then Ξ is an H-symmetric graph which admits $\mathcal{B}(\Sigma)$ as an H-invariant partition such that $\Xi_{\mathcal{B}(\Sigma)} \cong \Sigma$ and $k_{\mathcal{B}(\Sigma)} = v_{\mathcal{B}(\Sigma)} - 1 = v_\Sigma - 1 \ge 2$ ([24, Theorem 1]), where as before the blocks of $\mathcal{B}(\Sigma)$ are $B(\sigma) := \{\sigma\tau : \tau \in \Sigma(\sigma)\}$, $\sigma \in V(\Sigma)$. (Here we write an arc as $\sigma\tau$ instead of (σ, τ).) Let us take a covering graph $\Gamma := \tilde{\Xi}(K, \phi)$ of Ξ, where K is a group and ϕ is a K-chain on Ξ such that H acts as a group of automorphisms of K and that ϕ is compatible with the actions of H on K and $\mathtt{Arc}(\Xi)$. Let $G := K.H$ (semi-direct product of K by H). Then from [4, Proposition 19.4] it follows that Γ is a G-symmetric graph which admits $\mathcal{P}(\Xi) := \{P(\sigma\tau) : \sigma\tau \in \mathtt{Arc}(\Sigma)\}$ as a natural G-invariant partition such that $\Gamma_{\mathcal{P}(\Xi)} \cong \Xi$, where $P(\sigma\tau) := \{(g, \sigma\tau) : g \in K\}$ for $\sigma\tau \in \mathtt{Arc}(\Sigma)$. (Note that the group G in [4, Proposition 19.4] is not necessarily the full automorphism group of the underlying graph. It can be any subgroup of the full automorphism group which is transitive on the t-arcs of the underlying graph.) Define $\mathcal{B} = \{A(\sigma) : \sigma \in V(\Sigma)\}$, where $A(\sigma) := \bigcup_{\tau \in \Sigma(\sigma)} P(\sigma\tau)$. Then $\mathcal{B}$ is a G-invariant partition of $V(\Gamma) = K \times V(\Xi) = K \times \mathtt{Arc}(\Sigma)$. For this partition $\mathcal{B}$ one can check that the blocks of $\overline{\mathcal{D}}(A(\sigma))$ are precisely $P(\sigma\tau)$ for $\tau \in \Sigma(\sigma)$, and that $(\Gamma_{\mathcal{P}(\Xi)})_{\mathcal{B}(\Sigma)} \cong \Gamma_{\mathcal{B}}$. Hence $(\Gamma, G, \mathcal{B})$ satisfies the conditions of Theorem 3.7 with $\mathcal{P}(\Xi)$ and $\mathcal{B}(\Sigma)$ playing the roles of $\mathcal{P}$ and $\hat{\mathcal{B}}$ (defined in (1) and (2)) respectively.

The graph Γ in Construction 3.8 is constructed through two "lifts": we first "lift" Σ to the 3-arc graph Ξ and then "lift" Ξ to the covering graph Γ. Note that not all imprimitive G-symmetric graphs $(\Gamma, G, \mathcal{B})$ satisfying the conditions of Theorem 3.7 can be obtained this way. For a discussion on the existence of Δ needed in lifting Σ to $\Xi(\Sigma, \Delta)$, see [24, Remark 4] and also [37] when Σ is trivalent. From [4, Chapter 19] a pair (K, ϕ) required in the construction above always exists.

Example 3.9 Let K_{b+1} be the complete graph with vertex set $[b+1] := \{1, 2, \dots, b+1\}$, where $b \geq 3$. Let H be a 3-transitive permutation group on $[b+1]$. From the classification [6, pp.8] of 3-transitive groups, H is one of the following: (i) S_{b+1} ($b \geq 3$); (ii) A_{b+1} ($b \geq 4$); (iii) M_{b+1} ($b = 10, 11, 21, 22, 23$) or M_{11} ($b = 11$); (iv) $\mathrm{AGL}(d, 2)$ ($b = 2^d - 1 \geq 3$); (v) $\mathbb{Z}_2^4.A_7$ ($b = 15$); (vi) H is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, b)$ ($b \geq 3$ a prime power) as given explicitly in [18, Theorem 2.1]. For each possibility, all self-paired H-orbits Δ on $\mathtt{Arc}_3(K_{b+1})$ have been determined in [35]. Let $\Xi := \Xi(K_{b+1}, \Delta)$, so that Ξ is an H-symmetric graph with vertices ij (where $i, j \in [b+1], i \neq j$) in which two vertices ij and $i'j'$ are adjacent if and only if $(j, i, i', j') \in \Delta$. Let $V(b+1, 2)$ be the $(b+1)$-dimensional linear space over GF(2), so that its additive group is $\mathbb{Z}_2^{b+1}$. Let ϵ_i be the unit vector of $V(b+1, 2)$ with the ith coordinate 1 and all other coordinates 0. Then H acts on $\{\epsilon_1, \dots, \epsilon_{b+1}\}$ by $(\epsilon_i)^h := \epsilon_{i^h}$ for $h \in H$ and $1 \leq i \leq b+1$, and this action can be extended to $\mathbb{Z}_2^{b+1}$ in the obvious way so that H acts on $\mathbb{Z}_2^{b+1}$ as a group of automorphisms. Let K be the subgroup of $\mathbb{Z}_2^{b+1}$ generated by $\{\epsilon_i + \epsilon_{i'} : 1 \leq i < i' \leq b+1\}$. It is easily shown that K is H-invariant and $K \cong \mathbb{Z}_2^b$. Define

$$\phi_1 : \ \mathtt{Arc}(\Xi) \to K, \quad (ij, i'j') \mapsto \epsilon_i + \epsilon_{i'};$$
$$\phi_2 : \ \mathtt{Arc}(\Xi) \to K, \quad (ij, i'j') \mapsto \epsilon_i + \epsilon_{i'} + \epsilon_j + \epsilon_{j'}.$$

Then both ϕ_1 and ϕ_2 are K-chains on Ξ. Moreover, one can check that ϕ_1 and ϕ_2 are compatible with the actions of H on K and on $\mathtt{Arc}(\Xi)$. Thus, for each self-paired H-orbit Δ on $\mathtt{Arc}_3(K_{b+1})$, from Construction 3.8 we obtain two imprimitive $(\mathbb{Z}_2^b.H)$-symmetric graphs, namely $\tilde{\Xi}(\mathbb{Z}_2^b, \phi_1)$ and $\tilde{\Xi}(\mathbb{Z}_2^b, \phi_2)$, which satisfy the conditions of Theorem 3.7.

For the self-paired H-orbit Δ_0 on $\mathtt{Arc}_3(K_{b+1})$ consisting of all 3-cycles (j, i, i', j) of K_{b+1}, we can construct a third graph using Construction 3.8. In fact, the 3-arc graph of K_{b+1} with respect to this orbit is $\Xi_0 \cong (b+1) \cdot K_b$. One can check that

$$\phi_0 : \ \mathtt{Arc}(\Xi_0) \to \mathbb{Z}_2^{b+1}, \quad (ij, i'j) \mapsto \epsilon_j$$

defines a $\mathbb{Z}_2^{b+1}$-chain on Ξ_0 which is compatible with the actions of H on $\mathbb{Z}_2^{b+1}$ and on $\mathtt{Arc}(\Xi_0)$. Hence $\tilde{\Xi}_0(\mathbb{Z}_2^{b+1}, \phi_0)$ is a $(\mathbb{Z}_2^{b+1}.H)$-symmetric graph satisfying the conditions of Theorem 3.7.

For each of (i)-(vi) above, all 3-arc graphs of the $(H, 2)$-arc transitive graph K_{b+1} have been determined in [35]. In the case where H is 4-transitive, we have either $H = S_{b+1}$

$(b \geq 3)$, or $H = A_{b+1}$ $(b \geq 5)$, or $H = M_{b+1}$ $(b = 10, 11, 22, 23)$; in this case there are exactly two 3-arc graphs of K_{b+1}, namely Ξ_0 above and the 3-arc graph with respect to $\mathtt{Arc}_3(K_{b+1}) \setminus \Delta_0$ (which is self-paired by the 4-transitivity of H). The latter graph is isomorphic to $(K(b+1,2))[\overline{K}_2]$ (see [35, Example 3.15]), the lexicographic product of the Kneser graph $K(b+1,2)$ by the empty graph $\overline{K}_2$. (For integers m, n with $2 \leq 2m < n$, the *Kneser graph* $K(n,m)$ is the graph with vertices the m-subsets of a given n-set such that two vertices are adjacent if and only if they have no common element.)

4 Two constructions

This section is devoted to the general case, Case 3, for which $\lambda \geq 1, \overline{\lambda} \geq 1$. By Theorem 3.2 in this case $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}}^*(B)$ are 2-designs, and they admit G_B as a group of automorphisms acting 2-transitively on the points and transitively on the blocks. The well-known Fisher's inequality applied to $\mathcal{D}^*(B)$ gives:

$$b \leq v, \quad r \leq k. \tag{8}$$

Also, applying [26, Theorem 2] to $\mathcal{D}(B)$ we get:

$$\lambda(v-1) \leq k(k-1), \tag{9}$$

which is, by (4), equivalent to

$$v + r \geq b + k. \tag{10}$$

The inequalities (8), (9) and (10) are the same as the ones in (6), noting by (3) that $\lambda = 2k - v$ when $\overline{\lambda} = 0$.

Many interesting and natural problems arise in Case 3. For example, we may ask the following general questions about the relationship between Γ, $\Gamma_{\mathcal{B}}$ and $\mathcal{D}^*(B)$ (or $\overline{\mathcal{D}}^*(B)$ equivalently).

Question 4.1 When can a 2-point-transitive, flag-transitive and block-transitive 2-design occur as $\mathcal{D}^*(B)$? When can a symmetric 2-point-transitive 2-design occur as $\mathcal{D}^*(B)$? And what can we say about the structure of Γ and $\Gamma_{\mathcal{B}}$ if $\mathcal{D}^*(B)$ is known?

Perhaps a complete solution to these questions for general symmetric graphs with 2-arc transitive quotients is not accessible. In the following we will focus on the case where $\mathcal{D}^*(B)$ or $\overline{\mathcal{D}}^*(B)$ is the trivial Steiner system with block size 2, that is, a complete graph. In other words, either $\lambda = 1$ and $r = 2$, or $\overline{\lambda} = 1$ and $b - r = 2$. In each of these cases we will give a construction which can be used to construct Γ from $\Gamma_{\mathcal{B}}$.

For an integer $s \geq 1$, an *s-path* in a graph is an s-arc identified with its reverse s-arc. A 2-path with mid-vertex σ and end-vertices τ, τ' will be denoted by $\tau\sigma\tau'$, with the understanding that $\tau'\sigma\tau$ represents the same 2-path. Thus, when we write $\tau\sigma\tau' = \eta\varepsilon\eta'$ we mean $\sigma = \varepsilon$ and $\{\tau, \tau'\} = \{\eta, \eta'\}$. For each vertex σ of Σ, let $B_2(\sigma)$ denote the set of 2-paths of Σ with mid-vertex σ, that is,

$$B_2(\sigma) := \{\tau\sigma\tau' : \tau, \tau' \in \Sigma(\sigma), \tau \neq \tau'\}.$$

Obviously,

$$\mathcal{B}_2(\Sigma) := \{B_2(\sigma) : \sigma \in V(\Sigma)\}$$

is a partition of the 2-paths of Σ.

4.1 Construction 1: $\lambda = 1$ and $r = 2$

Let us first give the following construction in a general setting.

Construction 4.2 Let Σ be a regular graph with $\mathtt{val}(\Sigma) \geq 2$. Let Δ be a self-paired subset of $\mathtt{Arc}_3(\Sigma)$. Define $\Gamma_2(\Sigma, \Delta)$ to be the graph with the set of 2-paths of Σ as vertex set such that two distinct "vertices" $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent if and only if they have a common edge (that is, $\sigma \in \{\eta, \eta'\}$ and $\varepsilon \in \{\tau, \tau'\}$) and moreover the 3-arc formed by "gluing" the common edge is in Δ. See Figure 1 below for an illustration.

For instance, if $\sigma = \eta', \varepsilon = \tau'$, then the 3-arc thus formed is $(\tau, \sigma, \varepsilon, \eta)$, which should be in Δ if $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are to be adjacent in $\Gamma_2(\Sigma, \Delta)$. The self-parity of Δ ensures that $\Gamma_2(\Sigma, \Delta)$ is a well-defined undirected graph.

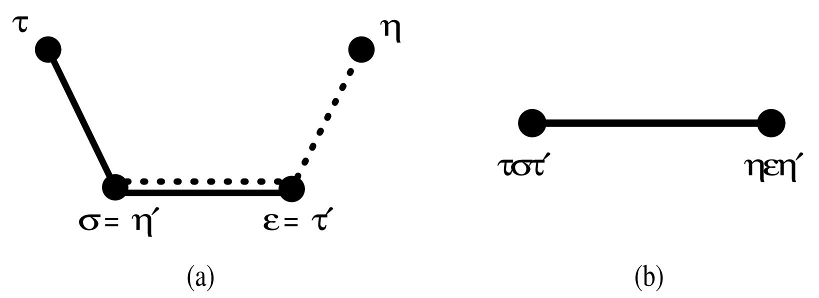


Figure 1: (a) 2-paths $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$; (b) $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent in $\Gamma_2(\Sigma, \Delta)$.

The main results of this subsection can be summarised in the next theorem, which follows from Theorems 4.4 and 4.10 immediately.

Theorem 4.3 *Let Σ be a $(G,2)$-arc transitive graph with valency ≥ 3 and Δ a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. Then $\Gamma := \Gamma_2(\Sigma, \Delta)$ is a G-symmetric graph admitting $\mathcal{B} := \mathcal{B}_2(\Sigma)$ as a G-invariant partition such that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive and not multi-covered by Γ, and $\lambda = 1, r = 2$.*

Conversely, any imprimitive G-symmetric graph $(\Gamma, \mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive and not multi-covered by Γ, and $\lambda = 1, r = 2$ is isomorphic to $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ for a self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$.

A regular graph Σ with $\mathtt{val}(\Sigma) \geq 2$ is said [10] to be $(G,2)$-*path transitive* if it admits G as a group of automorphisms such that G is transitive on the set of 2-paths. Note that such a graph Σ is necessarily G-vertex transitive because each vertex is a mid-vertex of at least one 2-path of Σ and the elements of G permute the mid-vertices while permuting the 2-paths. Note also that a $(G,2)$-path transitive graph must be G-symmetric by [10, Theorem 1].

Theorem 4.4 *Let Σ be a regular graph of valency $b \geq 2$, and let G be a group acting on $V(\Sigma)$ as a group of automorphisms of Σ. Let Δ be a self-paired G-invariant subset of $\mathtt{Arc}_3(\Sigma)$. Then $\Gamma := \Gamma_2(\Sigma, \Delta)$ admits G as a group of automorphisms, and $\mathcal{B} := \mathcal{B}_2(\Sigma)$ is a G-invariant partition of the vertex set of Γ with block size $v = b(b-1)/2$. Moreover, if G is faithful on the vertex set of Σ, then it is faithful on the vertex set of Γ. Furthermore, the following (a)-(b) hold.*

- (a) *Γ is G-vertex transitive if and only if Σ is $(G,2)$-path transitive; and in this case $\Sigma \cong \Gamma_{\mathcal{B}}$ with respect to the bijection $\sigma \mapsto B_2(\sigma)$, $\sigma \in V(\Sigma)$.*
- (b) *Γ is G-symmetric if and only if Σ is $(G,2)$-path transitive and Δ is a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$; in this case Σ must be $(G,2)$-arc transitive, the parameters with respect to $(\Gamma, \mathcal{B})$ satisfy $k = b-1$, $r = 2$, $\lambda = 1$ and $\overline{\lambda} = ((b-1)(b-4)/2) + 1$, and moreover $\Gamma[B_2(\sigma), B_2(\varepsilon)] \cong K_{b-1,b-1}$ for adjacent blocks $B_2(\sigma), B_2(\varepsilon)$ of $\mathcal{B}$ if and only if Σ is $(G,3)$-arc transitive.*

Proof Since Δ is G-invariant, the induced action of G on the 2-paths of Σ preserves the adjacency of Γ, and hence Γ admits G as a group of automorphisms. It is readily seen that $\mathcal{B}$ is a G-invariant partition of the 2-paths of Σ with block size $v = b(b-1)/2$. An element of G which fixes every 2-path of Σ must fix every vertex of Σ. Thus, if G is faithful on $V(\Sigma)$, then it must be faithful on $V(\Gamma)$.

(a) Clearly, Γ is G-vertex transitive if and only if Σ is $(G,2)$-path transitive. Let $B_2(\sigma)$ and $B_2(\varepsilon)$ be blocks of $\mathcal{B}$ which are adjacent in $\Gamma_{\mathcal{B}}$. Then there exist $\tau\sigma\tau' \in B_2(\sigma)$,

$\eta\varepsilon\eta' \in B_2(\varepsilon)$ such that $\tau\sigma\tau'$, $\eta\varepsilon\eta'$ are adjacent in Γ. That is, $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ share an edge, and hence σ and ε are adjacent in Σ. Conversely, suppose Σ is $(G,2)$-path transitive, and let σ, ε be adjacent vertices of Σ. Let $(\tau_0, \sigma_0, \varepsilon_0, \eta_0)$ be a 3-arc in Δ. Since Σ is $(G,2)$-path transitive, it is G-symmetric and hence there exists $g \in G$ such that $(\sigma_0, \varepsilon_0)^g = (\sigma, \varepsilon)$. Denote $\tau = \tau_0^g$ and $\eta = \eta_0^g$. Then $(\tau, \sigma, \varepsilon, \eta) = (\tau_0, \sigma_0, \varepsilon_0, \eta_0)^g \in \Delta$ as Δ is G-invariant. Thus, $\tau\sigma\varepsilon$ and $\sigma\varepsilon\eta$ are adjacent in Γ. Hence $B_2(\sigma)$ and $B_2(\varepsilon)$ are adjacent in $\Gamma_{\mathcal{B}}$. Therefore, $\sigma \mapsto B_2(\sigma)$ for $\sigma \in V(\Sigma)$ defines an isomorphism from Σ to $\Gamma_{\mathcal{B}}$, provided that Σ is $(G,2)$-path transitive. This completes the proof of (a).

(b) From construction 4.2 it is clear that Γ is G-symmetric if and only if Σ is $(G,2)$-path transitive and Δ is a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. Let us now prove that Σ is $(G,2)$-arc transitive if Γ is G-symmetric. Suppose Γ is G-symmetric, so that Σ is $(G,2)$-path transitive and Δ is a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. Then Σ is G-vertex transitive. From construction 4.2 it follows that, for each "vertex" $\tau\sigma\varepsilon$ of Γ, any vertex of Γ adjacent to $\tau\sigma\varepsilon$ must have mid-vertex τ or ε, and furthermore both cases can occur since Σ is $(G,2)$-path transitive. Thus, there are exactly two blocks, namely $B_2(\tau)$ and $B_2(\varepsilon)$, which contain neighbours of $\tau\sigma\varepsilon$ and hence $r := |\Gamma_{\mathcal{B}}(\text{“}\tau\sigma\varepsilon\text{”})| = 2$. Since $\Gamma_{\mathcal{B}} \cong \Sigma$ as shown above, we have $\mathtt{val}(\Gamma_{\mathcal{B}}) = b$; hence $vr = bk$, which implies $k = b-1$ by noting $v = b(b-1)/2$. Also, since $\Gamma_{\mathcal{B}} \cong \Sigma$, $\Gamma_{\mathcal{B}}$ is G-vertex transitive and 2-path transitive, and hence $G_{B_2(\sigma)}$ is 2-homogeneous on $\Gamma_{\mathcal{B}}(B_2(\sigma))$. In particular, $G_{B_2(\sigma)}$ acts primitively on $\Gamma_{\mathcal{B}}(B_2(\sigma))$. Thus, from Lemma 2.4 the multiplicity m of $\mathcal{D}(B_2(\sigma))$ is equal to 1 or b.

If $m = b$, then we must have $b = m = 2$ since m is a divisor of $r = 2$. In this case, we have $v = k = 1$, $\lambda = 1$, $\overline{\lambda} = 0$, and $\Gamma \cong \Sigma \cong \ell \cdot C_n$ for some integers $\ell \geq 1$ and $n \geq 3$. Thus, since Γ is G-symmetric, Σ is G-symmetric and hence $(G,2)$-arc transitive as it consists of cycles.

If $m = 1$, then we have $b > 2$ for otherwise we would have $v = 1$ and and $m = r = 2$, a contradiction. Thus, $k = b - 1 \geq 2$, and this implies that the b blocks of $\mathcal{D}(B_2(\sigma))$ do not form a partition of $B_2(\sigma)$ as $|B_2(\sigma)| = b(b-1)/2$. Thus, there exist $B_2(\tau), B_2(\varepsilon) \in \mathcal{B}$ such that $\Gamma(B_2(\tau)) \cap \Gamma(B_2(\varepsilon)) \cap B_2(\sigma) \neq \emptyset$. It follows that there exist ξ and η such that $(\xi, \tau, \sigma, \varepsilon, \eta)$ is a 4-arc of Σ with $(\xi, \tau, \sigma, \varepsilon), (\tau, \sigma, \varepsilon, \eta) \in \Delta$. Since Δ is a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$, there exists $g \in G$ such that $(\xi, \tau, \sigma, \varepsilon)^g = (\eta, \varepsilon, \sigma, \tau)$. Thus, $(\tau, \sigma, \varepsilon)^g = (\varepsilon, \sigma, \tau)$. This together with the $(G,2)$-path transitivity of Σ implies that Σ is $(G,2)$-arc transitive. Thus, $\Gamma_{\mathcal{B}}$ ($\cong \Sigma$) is $(G,2)$-arc transitive and hence Corollary 3.3 applies. Since $v = b(b-1)/2$, $k = b-1$ and $r = 2$, from (4) and (3) we get $\lambda = 1$ and $\overline{\lambda} = ((b-1)(b-4)/2) + 1$.

Let $B_2(\sigma)$ and $B_2(\varepsilon)$ be adjacent blocks of $\mathcal{B}$. Suppose that Σ is $(G,3)$-arc transitive. Then $\Delta = \mathtt{Arc}_3(\Sigma)$ since Δ is a G-orbit on $\mathtt{Arc}_3(\Sigma)$. Thus, for any $\tau \in \Sigma(\sigma) \setminus \{\varepsilon\}$

and $\eta \in \Sigma(\varepsilon) \setminus \{\sigma\}$, we have $(\tau, \sigma, \varepsilon, \eta) \in \Delta$ and hence "$\tau\sigma\varepsilon$" $\in B_2(\sigma)$ and "$\sigma\varepsilon\eta$" $\in B_2(\varepsilon)$ are adjacent. Therefore, $\Gamma[B_2(\sigma), B_2(\varepsilon)] \cong K_{b-1,b-1}$. Suppose conversely that $\Gamma[B_2(\sigma), B_2(\varepsilon)] \cong K_{b-1,b-1}$. Then for any $\tau \in \Sigma(\sigma) \setminus \{\varepsilon\}$ and $\eta \in \Sigma(\varepsilon) \setminus \{\sigma\}$, "$\tau\sigma\varepsilon$" and "$\sigma\varepsilon\eta$" are adjacent in Γ, and hence $(\tau, \sigma, \varepsilon, \eta) \in \Delta$. Since Σ is G-symmetric and Δ is a G-orbit on $\mathtt{Arc}_3(\Sigma)$, this implies $\Delta = \mathtt{Arc}_3(\Sigma)$ and hence Σ is $(G, 3)$-arc transitive. $\Box$

Remark 4.5 (a) In Theorem 4.4(b) we have $\overline{\lambda} = 0$ when $b = 2$ or 3, and $\overline{\lambda} \geq 1$ when $b \geq 4$. As shown above, in the case where $b = 2$, the partition $\mathcal{B}_2(\Sigma)$ is trivial and $\Gamma \cong \Sigma$ is a union of disjoint cycles. In the case where $b = 3$ (that is, Σ is trivalent), $\Gamma_2(\Sigma, \Delta)$ is isomorphic to the 3-arc graph $\Xi(\Sigma, \Delta')$, where $\Delta' := \{(\tau', \sigma, \varepsilon, \eta') : \tau' \in \Sigma(\sigma) \setminus \{\tau, \varepsilon\}, \eta' \in \Sigma(\varepsilon) \setminus \{\eta, \sigma\}$ for some $(\tau, \sigma, \varepsilon, \eta) \in \Delta\}$, via the bijection $\tau\sigma\varepsilon \mapsto (\tau', \sigma)$, for 2-paths $\tau\sigma\varepsilon$ of Σ. See Example 4.7 below for such graphs $\Gamma_2(\Sigma, \Delta)$ when $b = 3$.

(b) In the case where Σ is $(G, 3)$-arc transitive, the valency of $\Gamma_2(\Sigma, \Delta)$ is $2(b-1)$ by Theorem 4.4(b); and also $\Gamma_2(\Sigma, \Delta)$ is connected if Σ is connected. If Σ is $(G, 2)$-arc but not $(G, 3)$-arc transitive, then the valency of $\Gamma_2(\Sigma, \Delta)$ is equal to $2|\eta^{G_{\tau\alpha\varepsilon}}|$, where $(\tau, \sigma, \varepsilon, \eta) \in \Delta$.

(c) Construction 4.2 was inspired by a general construction, the flag graph construction, introduced in [34, 35] by the second-named author. After discovering construction 4.2, we found that a related construction exists in a completely different setting [5], where the *path graph* $P_{s+1}(\Sigma)$ is defined to be the graph with "strict s-paths" of Σ as vertices such that two vertices are adjacent if and only if they overlap on an $(s-1)$-path. Here by a *strict s-path* we mean an s-path containing no repeated vertex. (Such a path is called a s-path in pure graph theory.) In the case where $\Delta = \mathtt{Arc}_3(\Sigma)$, our graph $\Gamma_2(\Sigma, \Delta)$ is precisely the path graph $P_3(\Sigma)$.

Example 4.6 A $(G, 2)$-arc transitive graph with girth 3 must be a complete graph K_{b+1}. In this case the set Δ of 3-cycles of K_{b+1} is a self-paired G-orbit on $\mathtt{Arc}_3(K_{b+1})$. For this Δ we have $\Gamma_2(K_{b+1}, \Delta) \cong ((b-1)b(b+1)/6) \cdot K_3$ and the bipartite subgraph between any two blocks of $\mathcal{B}_2(K_{b+1})$ is a matching of $b-1$ edges. In fact, each 3-cycle $(\tau, \sigma, \varepsilon, \tau)$ of K_{b+1} induces a 3-cycle of $\Gamma_2(K_{b+1}, \Delta)$ with vertices $\tau\sigma\varepsilon$, $\sigma\varepsilon\tau$ and $\varepsilon\tau\sigma$.

Example 4.7 It is well-known [9, 13] that a connected trivalent G-symmetric graph Σ is of one of seven types, G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, with subscript s denoting (G, s)-arc regularity and superscript indicating whether or not G contains an involution flipping an edge. It is known [37] that Σ has a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$ if and only if it is not of type G_2^2. A graph Σ of type G_1 must be $(G, 2)$-path transitive [10, Section 3], and it has [37] exactly two self-paired G-orbits Δ_1, Δ_2 on $\mathtt{Arc}_3(\Sigma)$. From Theorem 4.4(b), the corresponding graphs $\Gamma_2(\Sigma, \Delta_1)$ and $\Gamma_2(\Sigma, \Delta_2)$ are both G-symmetric with valency 2. For $\Sigma \neq K_4$ of type G_2^1, there are also exactly two G-orbits Δ_1', Δ_2' on $\mathtt{Arc}_3(\Sigma)$ [37], and by Theorem 4.4(b) the corresponding graphs $\Gamma_2(\Sigma, \Delta_1')$, $\Gamma_2(\Sigma, \Delta_2')$ are both G-symmetric with valency 2. In the case where Σ is of type G_3, G_4^1, G_4^2 or G_5, $\Delta := \mathtt{Arc}_3(\Sigma)$ is the unique self-paired G-orbit on 3-arcs of Σ, and $\Gamma_2(\Sigma, \Delta)$ is a connected 4-valent G-symmetric but not $(G, 2)$-arc transitive graph.

In the case when Σ is $(G, 3)$-arc transitive of valency $b \geq 3$, by Theorem 4.4(b) we have $\Gamma[B_2(\sigma), B_2(\varepsilon)] \cong K_{b-1,b-1}$ (where $\Gamma = \Gamma_2(\Sigma, \Delta)$), which is not a matching, and hence Γ cannot be $(G, 2)$-arc transitive. Note that in this case Γ has valency $2(b-1)$ and it must be connected if Σ is connected. Thus, we have the following corollary, of which the second assertion follows from the fact that there are infinitely many 7-arc transitive graphs (see e.g. [11]). Note that there exists no s-arc transitive graph of valency ≥ 3 if $s \geq 8$ [31].

Corollary 4.8 *Every connected $(G, 3)$-arc transitive graph Σ of valency $b \geq 3$ is a quotient graph of at least one connected G-symmetric but not $(G, 2)$-arc transitive graph Γ of valency $2(b-1)$. In particular, there are infinitely many connected symmetric but not 2-arc transitive graphs which have at least one 7-arc transitive quotient.*

The same results also follow from [24, Theorem 2] and the 3-arc graph construction, with Γ having valency $b^2 - 1$ instead of $2(b-1)$. They suggest that the level of s-arc transitivity of the quotient graph can be much higher than that of the original graph, although on the other hand the quotient may not even inherit 2-arc transitivity from the original. It would be interesting to understand when an imprimitive symmetric graph admits a highly arc-transitive quotient. The reader is referred to [20] for related questions and discussion.

From [10, Theorem 2], if a $(G, 2)$-path transitive graph Σ is not $(G, 2)$-arc transitive, then G_σ has odd order and is 2-homogenous but not 2-transitive on $\Sigma(\sigma)$, and $b \equiv 3 \pmod 4$ is a prime power, where $b = \mathtt{val}(\Sigma)$. For this case Theorem 4.4 implies the following result, which will be used in the proof of Theorem 4.12.

Corollary 4.9 *Let Σ be a $(G, 2)$-path but not $(G, 2)$-arc transitive graph. Then no element of G can reverse a 3-arc of Σ; in other words, there exists no self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$.*

The following theorem shows that any imprimitive G-symmetric graph $(\Gamma, \mathcal{B})$ with $\lambda = 1$, $r = 2$ and $(G, 2)$-arc transitive quotient $\Gamma_{\mathcal{B}}$ can be constructed from $\Gamma_{\mathcal{B}}$ by using construction 4.2. By Example 4.6, if $\Gamma_{\mathcal{B}}$ has girth 3, then $\Gamma \cong ((b-1)b(b+1)/6) \cdot K_3$.

Theorem 4.10 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive with valency $b \geq 2$. Suppose further that $\lambda = 1$ and $r = 2$. Then $\Gamma \cong \Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ for a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. Moreover, $b \geq 3$, $v = b(b-1)/2$, and $k = b-1$.*

Proof Since $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, we have $m = 1$. Also, since $\lambda = 1$ and $r = 2$, by (4) we have $k = b-1$ and hence $v = b(b-1)/2$. Note that $b \geq 3$ for otherwise we would have $k = v = 1$ and thus $\mathcal{B}$ is a trivial partition, a contradiction.

Since $r = 2$, for each $\alpha \in V(\Gamma)$ there exist precisely two blocks $C(\alpha), D(\alpha)$ of $\mathcal{B}$ which contain neighbours of α, that is, $\Gamma_{\mathcal{B}}(\alpha) = \{C(\alpha), D(\alpha)\}$. Let $B(\alpha)$ denote the block of $\mathcal{B}$ containing α. Then $C(\alpha)B(\alpha)D(\alpha)$ is a 2-path of $\Gamma_{\mathcal{B}}$. The $(G,2)$-arc transitivity of $\Gamma_{\mathcal{B}}$ implies that any 2-path CBD of $\Gamma_{\mathcal{B}}$ is of the form $C(\alpha)B(\alpha)D(\alpha)$, and moreover there is a unique vertex α such that $CBD = C(\alpha)B(\alpha)D(\alpha)$ since $\lambda = 1$. Therefore,

$$\phi : \alpha \mapsto C(\alpha)B(\alpha)D(\alpha),\ \alpha \in V(\Gamma)$$

defines a bijection from $V(\Gamma)$ to the set of 2-paths of $\Gamma_{\mathcal{B}}$.

Let

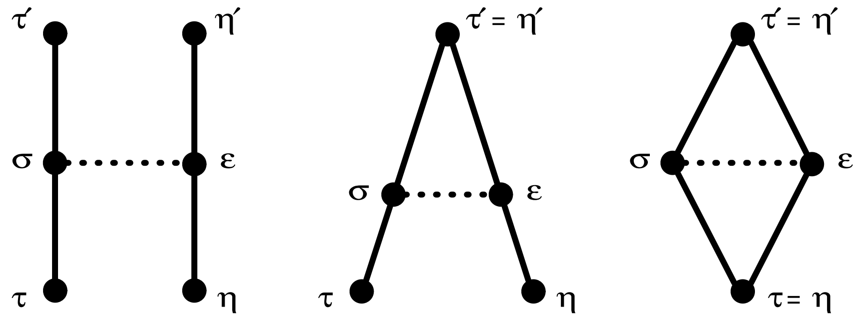
$$\Delta := \{(C, B(\alpha), B(\beta), D) : (\alpha, \beta) \in \mathtt{Arc}(\Gamma), C \in \Gamma_{\mathcal{B}}(\alpha) \setminus \{B(\beta)\}, D \in \Gamma_{\mathcal{B}}(\beta) \setminus \{B(\alpha)\}\}.$$

Then obviously Δ is a self-paired subset of $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. Since $r = 2$, the mapping $(\alpha, \beta) \mapsto (C, B(\alpha), B(\beta), D)$ is a one-to-one correspondence between the arcs of Γ and the 3-arcs in Δ. Since Γ is G-symmetric, it follows that Δ is a self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$, and hence $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$ is G-symmetric by Theorem 4.4(b). Note that, for $(\alpha, \beta) \in \mathtt{Arc}(\Gamma)$, we have $\phi(\alpha) = CB(\alpha)B(\beta)$, $\phi(\beta) = B(\alpha)B(\beta)D$, and $\phi(\alpha), \phi(\beta)$ are adjacent in $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$. Conversely, if $C(\alpha)B(\alpha)D(\alpha)$ and $C(\beta)B(\beta)D(\beta)$ are adjacent in $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$, then $B(\alpha) \in \{C(\beta), D(\beta)\}$ and $B(\beta) \in \{C(\alpha), D(\alpha)\}$. Without loss of generality we may assume that $B(\alpha) = C(\beta)$ and $B(\beta) = D(\alpha)$. Then $(C(\alpha), B(\alpha), B(\beta), D(\beta)) \in \Delta$ by the definition of $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$. Thus, there exist $(\alpha', \beta') \in \mathtt{Arc}(\Gamma)$, $C \in \Gamma_{\mathcal{B}}(\alpha') \setminus \{B(\beta')\}$ and $D \in \Gamma_{\mathcal{B}}(\beta') \setminus \{B(\alpha')\}$ such that $(C(\alpha), B(\alpha), B(\beta), D(\beta)) = (C, B(\alpha'), B(\beta'), D)$. This implies $\alpha, \alpha' \in \Gamma(C(\alpha)) \cap \Gamma(B(\beta)) \cap B(\alpha)$. However, $\lambda = 1$, so we must have $\alpha = \alpha'$. Similarly, $\beta = \beta'$. Thus, α and β are adjacent in Γ. Therefore, ϕ is an isomorphism between Γ and $\Gamma_2(\Gamma_{\mathcal{B}}, \Delta)$. $\square$

Using Theorem 4.3 the second-named author has classified all imprimitive G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $\lambda = 1$, $r = 2$, and $\Gamma_{\mathcal{B}}$ is complete and $(G,2)$-arc transitive. To keep the present paper in a reasonable length, this classification will appear in a subsequent paper [38].

4.2 Construction 2: $\overline{\lambda} = 1$ and $b - r = 2$

Two 2-paths $\tau\sigma\tau'$, $\eta\varepsilon\eta'$ of Σ are called *joined* if σ, ε are adjacent in Σ and $\sigma, \varepsilon \notin \{\tau, \tau', \eta, \eta'\}$. As illustrated in Figure 2, such a pair of joined 2-paths together with the edge between σ and ε form an H-shape subgraph if $\{\tau, \tau'\} \cap \{\eta, \eta'\} = \emptyset$, an A-shape subgraph if $|\{\tau, \tau'\} \cap \{\eta, \eta'\}| = 1$, and a θ-shape subgraph if $\{\tau, \tau'\} = \{\eta, \eta'\}$, and we say that $(\tau\sigma\tau', \eta\varepsilon\eta')$ is of *type* H, A, θ accordingly. The last two types occur only when the girth of Σ is 3. Denote by $\mathbf{J}(\Sigma)$ the set of ordered pairs of joined 2-paths of Σ. Note that any group of automorphisms of Σ induces an action on $\mathbf{J}(\Sigma)$.



(a) (b) (c)

Figure 2: A pair $(\tau\sigma\tau', \eta\varepsilon\eta')$ of 2-paths: (a) type H; (b) type A; (c) type θ.

Construction 4.11 Let Σ be a regular graph with valency $\mathtt{val}(\Sigma) \geq 3$. Let Δ be a self-paired subset of $\mathbf{J}(\Sigma)$. Define $\Psi_2(\Sigma, \Delta)$ to be the graph with vertex set the set of 2-paths of Σ such that two distinct "vertices" $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent if and only if $(\tau\sigma\tau', \eta\varepsilon\eta') \in \Delta$.

We say that Δ is of *type* H, A or θ if all pairs of joined 2-paths in Δ are of type H, A or θ, respectively. Note that, if Δ is a self-paired G-orbit on $\mathbf{J}(\Sigma)$, then all members of Δ have the same type, which is the type of Δ.

The main results in this subsection are the following theorem and Theorem 4.14.

Theorem 4.12 *Let Σ be a regular graph of valency $b \geq 3$, and let G be a group acting on $V(\Sigma)$ as a group of automorphisms of Σ. Let Δ be a self-paired G-invariant subset of $\mathbf{J}(\Sigma)$. Then $\Psi := \Psi_2(\Sigma, \Delta)$ admits G as a group of automorphisms, and $\mathcal{B} := \mathcal{B}_2(\Sigma)$ is a G-invariant partition of the vertex set of Γ with block size $v = b(b-1)/2$. Moreover, if G is faithful on the vertex set of Σ, then it is faithful on the vertex set of Ψ. Furthermore, the following (a)-(b) hold.*

(a) Ψ *is* G*-vertex transitive if and only if* Σ *is* $(G,2)$*-path transitive; and in this case* $\Sigma \cong \Psi_{\mathcal{B}}$ *with respect to the bijection* $\sigma \mapsto B_2(\sigma)$*,* $\sigma \in V(\Sigma)$*.*

(b) Ψ *is* G*-symmetric if and only if* Σ *is* $(G,2)$*-path transitive and* Δ *is a self-paired* G*-orbit on* $\mathbf{J}(\Sigma)$*; in this case* Σ *must be* $(G,2)$*-arc transitive if* Δ *is of type* H *or* A*. Moreover, if* G_σ *is* 3*-transitive on* $\Sigma(\sigma)$*, then the parameters with respect to* $(\Psi,\mathcal{B})$ *satisfy* $k=(b-1)(b-2)/2$*,* $r=b-2$*,* $\lambda=(b-2)(b-3)/2$ *and* $\overline{\lambda}=1$*.*

Proof Similar to Theorem 4.4, the statements before (a) can be verified easily, and hence their proofs are omitted.

(a) From construction 4.11 it is clear that Ψ is G-vertex transitive if and only if Σ is $(G,2)$-path transitive. Let $B_2(\sigma)$ and $B_2(\varepsilon)$ be blocks of $\mathcal{B}$ which are adjacent in $\Psi_{\mathcal{B}}$. Then there exist $\tau\sigma\tau' \in B_2(\sigma)$, $\eta\varepsilon\eta' \in B_2(\varepsilon)$ such that $(\tau\sigma\tau',\eta\varepsilon\eta') \in \Delta$. In particular, this implies that σ and ε are adjacent in Σ. Now suppose Σ is $(G,2)$-path transitive, and let σ, ε be adjacent vertices of Σ. Let $(\tau_0\sigma_0\tau_0',\eta_0\varepsilon_0\eta_0')$ be a member of Δ. Since Σ is $(G,2)$-path transitive, it must be G-symmetric [10, Theorem 1]. Hence there exists $g \in G$ such that $(\sigma_0,\varepsilon_0)^g = (\sigma,\varepsilon)$. Let $\tau = \tau_0^g$, $\tau' = (\tau_0')^g$, $\eta = \eta_0^g$ and $\eta' = (\eta_0')^g$. Then $(\tau\sigma\tau',\eta\varepsilon\eta') = (\tau_0\sigma_0\tau_0',\eta_0\varepsilon_0\eta_0')^g \in \Delta$ since Δ is G-invariant. Thus, $\tau\sigma\tau'$ and $\eta\varepsilon\eta'$ are adjacent in Ψ. Hence $B_2(\sigma)$ and $B_2(\varepsilon)$ are adjacent in $\Gamma_{\mathcal{B}}$. Therefore, if Σ is $(G,2)$-path transitive, then $\sigma \mapsto B_2(\sigma)$ for $\sigma \in V(\Sigma)$ defines an isomorphism from Σ to $\Psi_{\mathcal{B}}$.

(b) Clearly, Ψ is G-symmetric if and only if Σ is $(G,2)$-path transitive and Δ is a self-paired G-orbit on $\mathbf{J}(\Sigma)$. Now let us prove that in this case Σ must be $(G,2)$-arc transitive provided that Δ is of type H or A. Let $(\tau\sigma\tau',\eta\varepsilon\eta') \in \Delta$ be an arc of Ψ. Since Δ is self-paired, there exists $g \in G$ such that $(\tau\sigma\tau',\eta\varepsilon\eta')^g = (\eta\varepsilon\eta',\tau\sigma\tau')$, that is, g interchanges σ and ε, and also interchanges $\{\tau,\tau'\}$ and $\{\eta,\eta'\}$. Thus, g^2 fixes σ and ε, and fixes each of $\{\tau,\tau'\}$ and $\{\eta,\eta'\}$ setwise. In the case where Δ is of type H, the six vertices involved in $(\tau\sigma\tau',\eta\varepsilon\eta')$ are pairwise distinct and hence without loss of generality we may assume $\tau^g = \eta$ and $(\tau')^g = \eta'$. Then either (i) $\eta^g = \tau$ and $(\eta')^g = \tau'$, or (ii) $\eta^g = \tau'$ and $(\eta')^g = \tau$. In the former case, g reverses the 3-arc $(\tau,\sigma,\varepsilon,\eta)$ of Σ, and hence Σ must be $(G,2)$-arc transitive by Corollary 4.9. In the latter case (ii), we have $\tau^{g^2} = \eta^g = \tau'$ and $(\tau')^{g^2} = (\eta')^g = \tau$, and hence g^2 reverses the 2-arc (τ,σ,τ') of Σ. Since Σ is G-vertex transitive and $(G,2)$-path transitive, it follows that Σ must be $(G,2)$-arc transitive. In the case where Δ is of type A, without loss of generality we may assume $\tau' = \eta'$, so that τ,τ',η are distinct. In this case g exchanges $\{\tau,\tau'\}$ and $\{\eta,\tau'\}$, which implies that $(\tau')^g = \tau$ or τ'. Hence $\tau^g = \eta$, $(\tau')^g = \tau'$ $(= \eta')$, and the argument above for type H applies. Thus, for Δ of type A, Σ must be $(G,2)$-arc transitive as well.

Now suppose G_σ is 3-transitive on $\Sigma(\sigma)$. Then for any $\varepsilon_0 \in \Sigma(\sigma) \setminus \{\tau, \tau'\}$ there exists $g \in G_{\sigma\tau\tau'}$ such that $\varepsilon^g = \varepsilon_0$. Let $\eta_0 := \eta^g$ and $\eta_0' := (\eta')^g$. Then $(\tau\sigma\tau', \eta_0\varepsilon_0\eta_0') = (\tau\sigma\tau', \eta\varepsilon\eta')^g \in \Delta$, and hence $\tau\sigma\tau'$ is adjacent to $\eta_0\varepsilon_0\eta_0' \in B_2(\varepsilon_0)$ in Ψ. Since this is true for any $\varepsilon_0 \in \Sigma(\sigma) \setminus \{\tau, \tau'\}$ and since $\tau\sigma\tau'$ is not adjacent to any vertex in $B_2(\tau)$ or $B_2(\tau')$, we conclude that $\tau\sigma\tau'$ has neighbours in exactly $b-2$ neighbouring blocks of $B_2(\sigma)$, that is, $r = b-2$. Recall that $v = b(b-1)/2$ and $vr = bk$. Hence $k = (b-1)(b-2)/2$, and $\lambda = (b-2)(b-3)/2$ by (4). Finally, we have $\overline{\lambda} = 1$ by (3). $\square$

Example 4.13 Consider the complete graph K_{b+1} on $b+1 \geq 6$ vertices. The set Δ of H-type pairs of joined 2-paths of K_{b+1} is a self-paired S_{b+1}-orbit on $\mathbf{J}(K_{b+1})$. By Theorem 4.12(b), $\Psi_2(K_{b+1}, \Delta)$ is an S_{b+1}-symmetric graph with valency $(b-2)(b-3)(b-4)$. This graph has vertex set $\{(i, \{j, j'\}) : i, j, j' \in [b+1]$ pairwise distinct$\}$ in which $(i, \{j, j'\})$ and $(k, \{\ell, \ell'\})$ are adjacent if and only $\{j, i, j'\} \cap \{\ell, k, \ell'\} = \emptyset$. Note that $(i, \{j, j'\}) \mapsto \{j, i, j'\}$ defines a 3-to-1 mapping from the vertex set of $\Psi_2(K_{b+1}, \Delta)$ to the set of 3-subsets of $[b+1]$. Thus, $\Psi_2(K_{b+1}, \Delta)$ is isomorphic to $(K(b+1, 3))[\overline{K}_3]$, the lexicographic product of the Kneser graph $K(b+1, 3)$ with the empty graph $\overline{K}_3$ on three vertices. (See Example 3.9 for the definition of a Kneser graph.) Two more S_{b+1}-symmetric graphs can be constructed by considering the sets of pairs of type-A and type-θ, respectively.

The only 2-arc transitive graphs with girth 3 are complete graphs. Thus, in the following theorem the self-paired G-orbit Δ is of type A or θ only when $\Gamma_{\mathcal{B}} \cong K_{b+1}$. Note that the proof below applies to all types unanimously.

Theorem 4.14 *Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is not a multicover of $\Gamma_{\mathcal{B}}$. Suppose that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive with valency $b \geq 3$. Suppose further that $\overline{\lambda} = 1$ and $r = b-2$. Then $\Gamma \cong \Psi_2(\Sigma, \Delta)$ for a self-paired G-orbit Δ on $\mathbf{J}(\Gamma_{\mathcal{B}})$, and the parameters with respect to $(\Gamma, \mathcal{B})$ satisfy $v = b(b-1)/2$, $k = (b-1)(b-2)/2$ and $\lambda = (b-2)(b-3)/2$.*

Proof Since $r = b-2 \geq 1$, for each $\alpha \in V(\Gamma)$ there are exactly two blocks in $\Gamma_{\mathcal{B}}(B(\alpha))$ but not in $\Gamma_{\mathcal{B}}(\alpha)$, where $B(\alpha)$ is the block of $\mathcal{B}$ containing α. Let us denote these two blocks by $C(\alpha)$ and $D(\alpha)$. Then $C(\alpha)B(\alpha)D(\alpha)$ is a 2-path of $\Gamma_{\mathcal{B}}$. Since $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, any 2-path CBD of $\Gamma_{\mathcal{B}}$ is of the form $C(\alpha)B(\alpha)D(\alpha)$. Moreover, since $\overline{\lambda} = 1$, there is a unique vertex $\alpha \in V(\Gamma)$ such that $CBD = C(\alpha)B(\alpha)D(\alpha)$. Thus,

$$\phi : \alpha \mapsto C(\alpha)B(\alpha)D(\alpha),\ \alpha \in V(\Gamma)$$

defines a bijection from $V(\Gamma)$ to the set of 2-paths of $\Gamma_{\mathcal{B}}$. From (3) and $\overline{\lambda} = 1$, we have $\lambda = 2k - v + 1$. Plugging this into (4), and $r = b - 2$ into $vr = bk$, we get:

$$\begin{cases} (b-1)v & - & (b+1)k & = & b-1 \\ (b-2)v & - & bk & = & 0. \end{cases}$$

Solving, we obtain $v = b(b-1)/2$ and $k = (b-1)(b-2)/2$, and hence $\lambda = (b-2)(b-3)/2$. From the bijection ϕ and the value of v it follows that, for each block $B \in \mathcal{B}$, the restriction of ϕ to B (that is, $\phi|_B : \alpha \mapsto C(\alpha)BD(\alpha)$, $\alpha \in B$) is a bijection from B to the set of 2-paths of $\Gamma_{\mathcal{B}}$ with mid-vertex B.

Define

$$\Delta := \{(C(\alpha)B(\alpha)D(\alpha), C(\beta)B(\beta)D(\beta)) : (\alpha, \beta) \in \mathtt{Arc}(\Gamma)\}.$$

Since Σ is G-symmetric, Δ is a self-paired G-orbit on $\mathbf{J}(\Gamma_{\mathcal{B}})$. Hence $\Psi_2(\Gamma_{\mathcal{B}}, \Delta)$ is G-symmetric by Theorem 4.12(b). Clearly, for adjacent vertices α, β of Γ, $\phi(\alpha)$ and $\phi(\beta)$ are adjacent in $\Psi_2(\Gamma_{\mathcal{B}}, \Delta)$. Conversely, if $\phi(\alpha)$ and $\phi(\beta)$ are adjacent in $\Psi_2(\Gamma_{\mathcal{B}}, \Delta)$ for $\alpha, \beta \in V(\Gamma)$, then $(C(\alpha)B(\alpha)D(\alpha), C(\beta)B(\beta)D(\beta)) \in \Delta$. Since α is the unique vertex in $B(\alpha)$ with $\phi(\alpha) = C(\alpha)B(\alpha)D(\alpha)$ and β is the unique vertex in $B(\beta)$ with $\phi(\beta) = C(\beta)B(\beta)D(\beta)$, from the definition of Δ it follows that α and β must be adjacent in Γ. Therefore, ϕ is an isomorphism between Γ and $\Psi_2(\Gamma_{\mathcal{B}}, \Delta)$. This completes the proof. $\Box$

Remark 4.15 (a) The condition that G_σ is 3-transitive on $\Sigma(\sigma)$ is sufficient but not necessary to guarantee the specific values of $k, r, \lambda, \overline{\lambda}$ in (b) of Theorem 4.12. In fact, the last paragraph of its proof shows that $k, r, \lambda, \overline{\lambda}$ achieve the same values if Σ is G-vertex transitive and the stabiliser in G of a 2-path $\tau\sigma\tau'$ is transitive on $\Sigma(\sigma) \setminus \{\tau, \tau'\}$.

(b) For an arbitrary pair (Σ, Δ) with Σ a $(G, 2)$-arc transitive graph of valency ≥ 3 and Δ a self-paired G-orbit on $\mathbf{J}(\Sigma)$, we do not know whether $(\Psi_2(\Sigma, \Delta), \mathcal{B}_2(\Sigma))$ always satisfies $\overline{\lambda} = 1$ and $r = b - 2$. It seems that the family of such graphs $(\Psi_2(\Sigma, \Delta), \mathcal{B}_2(\Sigma))$ is larger than the family of symmetric graphs $(\Gamma, \mathcal{B})$ in Theorem 4.14. Further investigation is needed regarding construction 4.11.

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