On (n, k)-extendable graphs and induced subgraphs

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Abstract

Let G be a graph with vertex set $V(G)$. Let n and k be non-negative integers such that $n + 2k \leq |V(G)| - 2$ and $|V(G)| - n$ is even. If when deleting any n vertices of G the remaining subgraph contains a matching of k edges and every k-matching can be extended to a 1-factor, then G is called an $(n, k)-$**extendable graph**. In this paper we present several results about (n, k)-extendable graphs and its subgraphs. In particular, we proved that if $G - V(e)$ is (n, k)-extendable graph for each $e \in F$ (where F is a fixed 1-factor in G), then G is (n, k)-extendable graph.

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Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A **matching** M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A matching of size k is called a k-**matching**. If M is a matching so that every vertex (or except one) of G is incident with an edge of M, then M is called 1-**factor** (or **near** 1-**factor**).

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Let S be a subset of $V(G)$. Denote by $G[S]$ the induced subgraph of G by S and we write $G-S$ for $G[V(G)\setminus S]$. $E(S,T)$ denotes the edges between two vertex sets S and T. The number of odd components of G is denoted by $o(G)$.

Let M be a matching of G. If there is a matching M' of G such that $M \subseteq M'$, then we say that M can be extended to M' or M' is an extension of M. If each k-matching can be extended to a 1-factor, then G is called **k-extendable**. A graph G is called **n-factor-critical** if after deleting any n vertices the remaining subgraph of G has a 1-factor. The properties of 2-factor-critical and k-extendable graphs were studied extensively by Lovász and Plummer. The history and applications of these topics can be found in [2] and [5]. Liu and Yu [1] have introduced new concept, (n,k)-extendable graph, to combine the n-factor-criticality and the k-extendability.

Let n and k be non-negative integers such that $n+2k \leq |V(G)|-2$ and $|V(G)|-n$ is even. If when deleting any n vertices from G the remaining subgraph of G contains a k-matching and each k-matching in the subgraph can be extended to 1-factor, then G is called a **(n,k)-extendable graph**. Clearly, a graph is $(0,0)$-extendable if and only if it has a 1-factor. Similarly, $(0,k)$-extendable graphs are precisely those k-extendable graphs and $(n,0)$-extendable graphs are exactly n-critical graphs. A characterization and basic properties of (n,k)-extendable graphs were discussed in [1].

Nishimura and Saito [3] and Yu [7] studied the relationships between k-extendable graphs and its subgraphs and proved the followings

Theorem A. (Nishimura and Saito [3]) Let G be a graph with a 1-factor. If $G-V(e)$ is k-extendable for each $e \in E(G)$, then G is k-extendable.

Theorem B. (Yu [7]) A graph G is k-extendable if and only if for any matching M of size i $(1 \leq i \leq k)$ the graph $G-V(M)$ is (k-i)-extendable.

Based on Theorem B, Theorem A can be improved to the following:

Theorem 1. Let G be a graph with a 1-factor. If $G-V(e)$ is k-extendable for each $e \in E(G)$ and $|V(G)| \geq 2k+4$, then G is $(k+1)$-extendable.
Proof: Let $i=1$ in Theorem B, then the result follows. $\square$

In fact, the reverse of Theorem 1 is also true from Theorem B. Next we generalize this result to (n,k)-extendable graphs.

Theorem 2. If $G-V(e)$ is an (n,k)-extendable graph for each $e \in E(G)$, then G is $(n,k+1)$-extendable graph but may not be an $(n,k+2)$-extendable or $(n+2,k)$-extendable graph.
Proof: Consider any vertex set S and $(k+1)$-matching M with $|S|=n$ and

$V(M) \cap S = \emptyset$. Let e be an edge of M. Since $G - V(e)$ is (n,k)-extendable, there exists a 1-factor in $(G - V(e)) - (S \cup V(M - \{e\}) = G - (S \cup V(M))$. Therefore, G is an $(n, k+1)$-extendable graph.

To see that G may not be $(n, k+2)$-extendable, we consider the graph

$$H_1 = (2K_{2n+1}) + (K_n \cup (k+2)K_2)$$

Then H_1 is not an $(n, k+2)$-extendable graph by considering $S = V(K_n)$ and $(k+2)$-matching $(k+2)K_2$. In the mean time, it is not hard to verify that for any $e \in E(H_1)$ $H_1 - V(e)$ is an (n,k)-extendable graph.

Similarly, to see that G may not be $(n+2, k)$-extendable, we consider the graph

$$H_2 = (2K_{2n+1}) + (K_{n+2} \cup kK_2)$$

Then H_2 is not an $(n+2, k)$-extendable graph but for any $e \in E(H_2)$ $H_2 - V(e)$ is an (n,k)-extendable graph. □

Before proceeding further, we quote two results from [1] as lemmas.

Lemma 1. Let G be an (n,k)-extendable graph. Then it is also a $(n-2, k+1)$-extendable graph.

Lemma 2. If G is an (n,k)-graph, then

(1) G is also $(n-2, k)$-extendable for $n \geq 2$;

(2) G is also $(n, k-1)$-extendable for $k \geq 1$.

For the convenience of the future arguments, we introduce one more term. Let S be a vertex set and M a k-matching with $S \cap V(M) = \emptyset$. If $G - S - V(M)$ has a 1-factor, then we say that G has a (S, M)-**extension**.

Since an $(n+2, k)$-extendable or an $(n, k+2)$-extendable graph must be $(n, k+1)$-extendable, Theorem 2 indicates that $(n, k+1)$-extendability is the best possible under the general conditions. But by introducing an additional condition on the size of graph in Theorem 2, we can improve it to the following:

Theorem 3. If $G - V(e)$ is an (n,k)-extendable graph $(n > 1)$ for each $e \in E(G)$ and $V(G) \leq 2k + 3n + 4$, then G is an $(n+2, k)$-extendable graph.

Proof: Suppose that G is not an $(n+2, k)$-extendable graph. By the definition, there exists a vertex set S with $|S| = n+2$ and k-matching M so that $G - S - V(M)$ has no 1-factor.

Let $G' = G - S - V(M)$. From Tutte's Theorem, there exists a vertex set $S' \subseteq V(G')$ such that $o(G' - S') \geq |S'| + 2$.

Claim 1. $G' - S'$ has exactly $|S'| + 2$ odd components.

Otherwise, if $o(G'-S') \neq |S'|+2$, by parity, then we have $o(G'-S') \geq |S'|+4$. Set $S_1 = S - \{a,b\}$ (where a,b are any two vertices of S) and $S_1' = S' \cup \{a,b\}$. Then

$$o(G-S_1-V(M)-S_1') = o(G-S-V(M)-S') = o(G'-S') \geq |S'|+4 = |S_1'|+2$$

That is, $G-S_1-V(M)$ has no 1-factor or G has no (S_1,M)-extension. But $|S_1| = n$ and $|M| = k$, so it contradicts to that G is (n,k)-extendable.

Claim 2. S and S' are independent sets.

If S is not independent, let e be an edge of $G[S]$ and $S_1 = S - V(e)$, then $G - V(e)$ has no (S_1,M)-extension. This contradicts to the fact that $G - V(e)$ is an (n,k)-extendable graph.

Similarly, if S' is not independent, let e be an edge of $G[S']$, $S_1 = S - \{a,b\}$ (where a, b are any two vertices of S) and $S_1' = S' - V(e) \cup \{a,b\}$, then $o(G-V(e)-S_1-V(M)-S_1') = o(G-V(e)-S-V(M)-S') = o(G'-S') \geq |S_1'|+2$ or $G-V(e)$ has no (S_1,M)-extension. This contradicts to that $G-V(e)$ is an (n,k)-extendable graph.

Claim 3. $E(S,S') = \emptyset$.

Otherwise, let $e = xy \in E(S,S')$ and $x \in S$, $y \in S'$. Replacing the vertex y by a vertex of $S - \{x\}$ and moving y to S, then the new pair still have all of the properties of the old pair S and S' have but the new pair is against Claim 2, a contradiction.

Claim 4. No vertex in an even component is adjacent to $S \cup S'$.

If there is an edge $e = xy$ so that $x \in S'$ and y is in an even component. Set $S_1' = S' \cup \{y\}$. Then

$$o(G-S-V(M)-S_1') = o(G-S-V(M)-S')+1 = o(G'-S')+1 \geq |S'|+2+1 = |S_1'|+2$$

But $e = xy \in S_1'$, a contradiction to Claim 2.

Similarly, if there is an edge $e = xy$ so that $x \in S$ and y is in an even component. Set $S_1' = S' - \cup\{y\}$. Then

$$o(G-S-V(M)-S_1') = o(G-S-V(M)-S')+1 = o(G'-S')+1 \geq |S'|+2+1 = |S_1'|+2$$

But $e = xy \in E(S,S_1')$, a contradiction to Claim 3.

With the preparation above, we can proceed to the proof of the theorem now.

From Theorem 2, G is $(n,k+1)$-extendable. Applying Lemma 1 repeatedly we see that G is $(\epsilon,(k+1+\lfloor n/2 \rfloor))$-extendable, where $\epsilon = 0$ or 1. When k-matching M is extended to a 1-factor (or near 1-factor) then $S \cup S'$ has to match to the vertices of odd components $\cup O_i$. As $o(G'-S') = |S'|+2$ and $n \geq 2$, so at least one of O_i's has at least 3 vertices. Choose an edge e_1 from such an odd component, say O_1, now we can extend $(k+1)$-matching $M \cup \{e_1\}$ to a 1-factor (or near 1-factor). Thus $S \cup S'$ has to match to the vertices of $\cup O_i - V(e_1)$ and there exists an edge in $\cup O_i - V(e_1)$. If this process is repeated, we can find $\lfloor n/2 \rfloor + 1$ disjoint edges in $\cup O_i$, namely, $\{e_1, e_2, \cdots, e_l\}$ (where $l = \lfloor n/2 \rfloor + 1$). Since G is $(\epsilon, k+l)$-extendable, $M \cup \{e_1, e_2, \cdots, e_l\}$ can be extended to a 1-factor (or near 1-factor), and thus $S \cup S'$ has to match to some vertices of $\cup O_i - V(e_1) - V(e_2) - \cdots - V(e_l)$. Therefore, we have

$$|V(G)| \geq 2|S \cup S'| + 2k + 2(\lfloor n/2 \rfloor + 1)$$

$$\geq 2(n+2) + 2k + (n-1) + 2 = 2n + 4 + 2k + n + 1 = 3n + 2k + 5$$

which contradicts to the given condition. Hence, G is an $(n+2, k)$-extendable graph. □

Recently, Nishimura improved Theorem A by reducing the conditions required in the theorem. Instead of checking the k-extendability of $G - V(e)$ for every edge e in G, now one needs only checking the k-extendability of $G - V(e)$ for the edges belonging to a 1-factor of G.

Theorem C. (Nishimura [4]) Let G be a graph with 1-factors and let F be an arbitrary 1-factor of G. If $G - V(e)$ is k-extendable graph (or n-factor-critical) for each $e \in F$, then G is k-extendable (or n-factor-critical) graph.

We will generalize the above result to (n, k)-extendable graphs.

Theorem 4. Let G be a graph with 1-factors and let F be an arbitrary 1-factor of G. If $G - V(e)$ is (n, k)-extendable graph for each $e \in F$, then G is (n, k)-extendable graph.

Proof: We may assume that $n > 0$ and $k > 0$.

We proceed to prove the theorem by contradiction. Suppose that there exists a 1-factor F of G such that $G - V(e)$ is (n, k)-extendable for any $e \in F$ but G is not (n, k)-extendable. Then there exists a k-matching M and a vertex set S of size n, where $V(M) \cap S = \emptyset$, such that $G - V(M) - S$ has no 1-factor. Let $G' = G - V(M) - S$. Applying Tutte's 1-Factor Theorem, there exists $S' \subseteq V(G')$ so that $o(G'-S') > |S'|$. By the parity, $o(G'-S') \geq |S'|+2$. Our aim is to find an edge $e \in F$ so that $G - V(e)$ is not (n, k)-extendable and thus leads to a contradiction.

At first, we show that 1-factor F can only match vertices from $V(M)$ to rest by the next claim.

Claim 1. For the given F, S and G', we have

(i) $F \cap E[S] = \emptyset$;

(ii) $F \cap E(S') = \emptyset$;

(iii) $F \cap E(S, S') = \emptyset$;

To see (i), if $e \in F \cap E(S)$, then $|S - V(e)| = n$ - 2 and $G - V(e)$ is not $(n-2, k)$-extendable. Thus, G is not (n, k)-extendable, a contradiction.

To see (ii), if $e \in F \cap E(S')$, then $G' - V(e)$ has no 1-factor or $G - V(e)$ is not (n, k)-extendable, a contradiction.

To see (iii), if $e \in F \cap E(S, S')$, where $e = ab$ and $a \in S$, $b \in S'$, choosing a vertex c from an odd component of $G' - S'$ and then $S - \{a\} \cup \{c\}$ and M can not be extended to a 1-factor as $o(G' - V(e) - S') > |S'| + 2 - 1$.

From (i) - (iii), it follows that a 1-factor F is in $E(S \cup V(M), G')$ or $E(S, V(M))$ or $E(G[V(M)])$.

Claim 2. G' has no even components.

Otherwise, let D be an even component and let $e = ab$ be an edge of F, where $a \in V(D)$.

If $b \in S$, choose $c \in V(D) - \{a\}$, then $T = S - \{b\}$ and M can not extended to a 1-factor in $G - \{a, b\}$ as $o((G' - V(e) - T - V(M)) - S') \geq |S'| + 2$, a contradiction.

If $b \in V(M)$, consider an alternating path of $M \cup F$ with end-vertex a. If another end-vertex c of this alternating path is in S. Similarly to the previous case, let $T = S - \{c\} \cup \{x\}$ (where $x \in V(D) - \{a\}$ and $M' = M - \{bc'\} \cup \{ab\}$. Then $G - \{c, c'\}$ (where $cc' \in F$) has no (T, M')-extension, a contradiction.

If c is in S', it is similar.

If c is in a component (either odd or even), let $T = S$ and $M' = M - \{bc'\} \cup \{ab\}$, then $G - \{c, c'\}$ has no (T, M')-extension as $G' - \{a, c\} - S'$ has at least $|S'| + 2$ odd components.

Claim 3. $S' = \emptyset$.

If $S' \neq \emptyset$, let $a \in S'$, then a is matched to a vertex b in the 1-factor F and b must be in $V(M)$. Consider an alternating path of $M \cup F$, say $abb' \cdots dd'c$.

If $c \in S'$, let $T = S$ and $M' = M - \{bb', dd'\} \cup \{ab, b'd\}$, then $G - \{d', c\}$ has no (T, M')-extension as $G' - \{a, c\}$ has no 1-factor.

If $c \in S$, let $T = S - \{c\} \cup \{x\}$ (where x is a vertex of a component) and $M' = M - \{bb', dd'\} \cup \{ab, b'd\}$, then $G - \{d', c\}$ has no (T, M')-extension as $G' - \{a, c\} - (S' - \{a\})$ has $o(G' - S')$ - 1 odd components, a contradiction.

If $c \in C$ (where C is any component), using the same argument we can see that $G' - \{a, c\} - (S' - \{a\})$ loses at most one odd component and obtain a contradiction.

Claim 4. $o(G' - S') = o(G') = 2$.

Suppose $o(G') > 2$ (i.e., $o(G') \geq 4$). If there exists an edge $e \in F$ and $e \in E(S, C_1)$, choose c from an odd component C_2, let $T = S - \{b\} \cup \{c\}$ and $M' = M$, then $o(G' - \{a, c\}) \geq 2$ or $G - \{a, b\}$ has no (T, M')-extension, a contradiction.

Otherwise, all vertices in $\cup C_i$ are matched into $V(M)$. Consider the alternating paths of $F \cup M$, there exists such a path starting with C_i and ending C_j. Let $c_i x_1 y_1 x_2 y_2 \cdots x_m y_m c_j$ be the alternating path, where $c_i \in C_i$, $c_j \in C_j$ and $c_i x_1$, $y_1 x_2$, $\cdots$, $y_m c_j \in F$, $x_1 y_1$, $x_2 y_2$, $\cdots$, $x_m y_m \in M$.

Let $T = S$ and $M' = M - \{x_1 y_1, \cdots, x_m y_m\} \cup \{y_1 x_2, \cdots, y_m c_j\}$. Then $G - \{c_i, x_1\}$ has no (T, M')-extension as $o(G' - \{c_i, c_j\}) \geq 2$, a contradiction.

Claim 5. $F \cap E(S, V(M)) = \emptyset$.

Consider the alternating path $ab \cdots c$ of $F \cup M$ with end-vertex a. If $c \in S$, let $T = S - \{a, c\}$ and $M' = M - \{bb'\} \cup \{cb'\}$, then $G - \{a, b\}$ does not have (T, M')-extension, that is $G - \{a, b\}$ is not $(n-2, k)$-extendable, a contradiction. If $c \in C_1$ (where C_1 is an odd component) and $|C_1| \geq 3$, choose $d \in V(C_1) - \{c\}$ and let $T = S - \{a\} \cup \{d\}$ and $M' = M - \{bb'\} \cup \{b'c\}$. Then $G - \{a, b\}$ (where $ab \in F$) has no (T, M')-extension as $o(G' - \{c, d\}) \geq 2$.

If $c \in C_1$ but $|C_1| = 1$, then we have $|C_2| \geq 3$ because G' has only two odd components, no even component and $|G'| \geq 4$. Suppose that $F \cap E(S, C_2) \neq \emptyset$. Let $e = gh \in F \cap E(S, C_2)$, where $g \in V(C_2)$ and $h \in S$. Choose $y \in V(C_2) - \{g\}$ and set $T = S - \{h\} \cup \{y\}$ and $M' = M$, then $G - \{g, h\}$ has no (T, M')-extension as $o(G' - \{g, y\}) \geq 2$, a contradiction.

So we may assume $F \cap E(S, C_2) = \emptyset$. In this case, all vertices of C_2 are matched to $V(M)$ in F. Considering $F \cup M$, there must be an alternating path with both end-vertices in $V(C_2)$ or an alternating path starting in $V(C_2)$ and ending in S. In either case, it yields a contradiction.

Now we are ready to conclude the proof.

Since $|S| \geq 1$ and $F \cap E(S, V(M)) = \emptyset$, there exists an edge $e = ab \in F$ from S to an odd component C_1 (where $a \in S$, $b \in V(C_1)$). If $|C_1| \geq 3$, let $c \in V(C_1) - \{c\}$ and set $T = S - \{a\} \cup \{c\}$ and $M' = M$, then $G - \{a, b\}$ has no (T, M')-extension, a contradiction. If $|C_1| = 1$, then $|C_2| \geq 3$. Without loss of generality, we assume $F \cap E(S, C_2) = \emptyset$. Thus, all vertices of $V(C_2)$ are matched to $V(M)$ in F. Considering $F \cup M$, there exists an alternating path P with both of ends in C_2 or an alternating path P from C_2 to S.

Let $P = c x_1 y_1 d$, where $c x_1, y_1 d \in F$ and $x_1 y_1 \in M$. If $c, d \in V(C_2)$, let $T = S$ and $M' = M - \{x_1 y_1\} \cup \{d y_1\}$, then $G - \{c, x_1\}$ has no (T, M')-extension as $o(G' - \{c, d\}) \geq 2$. If $c \in V(C_2)$ and $d \in S$, let $T = S - \{d\} \cup \{g\}$ (where $g \in V(C_2) - \{e\}$) and $M' = M - \{x_1 y_1\} \cup \{d y_1\}$, then $G - \{c, x_1\}$ has no (T, M')-extension, a contradiction.

The proof is completed. □

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