COUNTING OCCURRENCES OF 231 IN AN INVOLUTION

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Abstract

We study the generating function for the number of involutions on n letters containing exactly $r \geqslant 0$ occurrences of 231. It is shown that finding this function for a given r amounts to a routine check of all involutions of length at most $2r + 2$.

Key words: Restricted involutions, Generating functions.

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1. **Introduction**

1.1. **Permutations.** Suppose that S_n is the set of permutations of $[n] = \{1, \dots, n\}$, written in one-line notation. Let $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$ and $\tau = \tau_1\tau_2 \dots \tau_k \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leqslant n$ and $\pi_{i_s} < \pi_{i_t}$ if and only if $\tau_s < \tau_t$ for any $1 \leqslant s, t \leqslant k$. In such a context, τ is usually called a *pattern*. We denote the number of occurrences of τ in π by $\tau(\pi)$ and the number of permutations $\pi \in S_n$ with $\tau(\pi) = r$ by $S_r^\tau(n)$.

Most of the results in this area investigate only the case $r = 0$, $S_0^\tau(n)$, the number of permutations of length n *avoiding* the pattern τ (for example, see [1, 2, 3, 6, 13, 16, 17, 18, 19, 20]). Only a few results investigate the case of $r > 0$ and τ of length 3. Noonan [15] showed that $S_1^{123}(n) = \frac{3}{n}\binom{2n}{n+3}$. This result was also proved by Noonan and Zeilberger [16]. Bóna [5] proved that $S_1^{132}(n) = \binom{2n-3}{n-3}$ and Fulmek [10] showed that

$$S_2^{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4},$$

which were conjectured by Noonan and Zeilberger [16]. Bóna [4] pointed out that it is a hard question to give an explicit expression for $S_r^\tau(n)$, $\tau \in S_3$, for any given r. Mansour and Vainshtein [14] suggested a new approach to this problem in the case $\tau = 132$, which allows one to get an explicit expression for $S_r^{132}(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geqslant 0} S_r^{132}(n)x^n$ for any $r \geqslant 0$. It is shown that finding this function for a given r amounts to a routine check of all permutations of length at most $2r$.

1.2. **Involutions.** An *involution* π is a permutation in S_n such that $\pi = \pi^{-1}$; let $\mathcal{I}_n$ be the set of all the involutions in S_n. We denote by $I^{\tau}_{r,n}$ the number of involutions $\pi \in \mathcal{I}_n$ with $\tau(\pi) = r$, and $I^{\tau}_r(x)$ the corresponding generating function, that is, $I^{\tau}_r(x) = \sum_{n\geqslant 0} I^{\tau}_{r,n} x^n$.

Again, most authors considered the case $r = 0$, namely involutions avoiding a given pattern τ (see [7, 9, 11, 12] and references therein). For the case $r > 0$ there exist only few results. Guibert and Mansour [12] gave an explicit expression for $I^{132}_{1,n}$, namely $I^{132}_{1,n} = \binom{n-2}{[(n-3)/2]}$. Egge and Mansour [8] proved that $I^{231}_{1,n} = (n-1)2^{n-6}$ for $n \geqslant 5$.

In the present paper we give a complete answer for this problem in the case of $\tau = 231$, which allows one to get an explicit expression for $I^{231}_{r,n}$ for any given r. More precisely, we present an algorithm that computes the generating function $I^{231}_r(x)$ for any $r \geqslant 0$. To get the result for a given r, the algorithm performs certain routine checks for each element in $\bigcup_{k=1}^{2r+2} I_k$. The algorithm has been implemented in C, and yielded explicit results for $0 \leqslant r \leqslant 7$.

2. Preliminary results

For any involution $\pi \in \mathcal{I}_n$, we can define a bipartite graph G_π in the following way which is similar to [14].

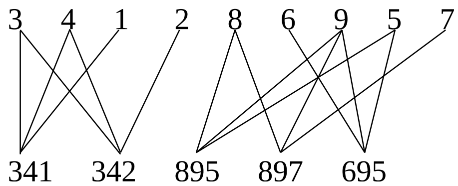

Figure 1. The graph $G_{341286957}$

The vertices in one part of G_π, denoted V_1, are the entries of π, and the vertices of the second part, denoted V_3, are the occurrences of 231 in π. Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_3$ if i enters j. For example, let $\pi = 341286957$, then π contains 5 occurrences of 231, and the graph G_π is presented in Figure 1.

Let $\widetilde{G}$ be an arbitrary connected component of G_π, let $\widetilde{V}$ be its vertex set, and set $\widetilde{V}_1 = \widetilde{V} \bigcap V_1$, $\widetilde{V}_3 = \widetilde{V} \bigcap V_3$, $t_1 = |\widetilde{V}_1|$, and $t_3 = |\widetilde{V}_3|$. Denote by G^n_π the connected component of G_π containing entry n.

For any $\pi \in \mathcal{I}_n$ where $\pi_j = n$ and $|V_1(G^n_\pi)| > 1$, suppose that i_k is the minimal index such that $\pi_{i_k} > j$ and $i_k < j$. If there exists an index m with $i_k < m < j$ such that $\pi_m < i_k$, then let i_1 be the minimal index such that π contains a subsequence

$$(\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \ldots, i_h, \pi_{i_{h+2}}, i_{h+1}, \ldots, \pi_{i_k}, i_{k-1}, \pi_j, i_k, j)$$

where $i_1 < i_2 < i_3 < \ldots < i_k < j$. We call this subsequence the *connected sequence* of π. Otherwise, $i_1 = i_k$ and the connected sequence reduces to $(\pi_{i_1}, \pi_j, i_1, j)$. For our convenience, we call i_1 the *initial index*.

Definition 2.1. *For any $\pi \in \mathcal{I}_n$ and $\pi_j = n$, we define the* 231-tail *by*

$$\chi_\pi = \begin{cases} (n, \pi_{j+1}, \ldots, \pi_{n-1}, j), & \text{if} \quad |V_1(G_\pi^n)| = 1, \\ (\pi_{i_1}, \pi_{i_1+1}, \ldots, \pi_n), & \text{if} \quad |V_1(G_\pi^n)| > 1, \end{cases}$$

where i_1 is the initial index of π.

For example, the 231-tail of the involution 216483957 is 6483957. Denote by l_π and c_π the length of π and the number of occurrences of 231 in π.

For the simplicity of the notation, we denote by $\{n; \lambda\}$ the permutation $(n, n-1, \ldots, n-s+1, \lambda)$, where λ is a nonempty permutation of $\{j+1, j+2, \ldots, n-s\}$ and does not begin with $n-s$.

In fact, for any $\pi \in \mathcal{I}_n$ with $|V_1(G_\pi^n)| = 1$, $\pi_j = n$, and $\chi_\pi = \{n; \lambda\}$. For example, if $\pi = 213986754 \in \mathcal{I}_9$ then $\chi_\pi = 986754$, $s = 2$, and $\lambda = 6754$. The following lemma holds by the definition of the 231-tail and the initial index of π.

Lemma 2.2. *Let $\pi \in \mathcal{I}_n$. Then χ_π is order-isomorphic to an involution, and there exists an involution π' such that $\pi = (\pi', \chi_\pi)$.*

Lemma 2.3. *Let $\pi \in I_n$ with $\chi_\pi = \{n; \lambda\}$ and $c_{\chi_\pi} = r$ $(r > 0)$, then $l_{\chi_\pi} \leq 2r + 2$.*

Proof. From Lemma 2.2 and the hypothesis about the form of χ_π, we have $\chi_\pi = (n, n-1, \ldots, n-s+1, \mu, n+s-l_{\chi_\pi}, \ldots, n+1-l_{\chi_\pi})$, where μ is an involution and does not begin with $n-s$. Note that each free rise (a free rise is a subsequence order-isomorphic to the pattern 12) contributes s occurrences of 231 and $c_{\chi_\pi} = r$. Hence $s \leq r$ and the maximum length of χ_π occurs when μ has just one free rise. Since χ_π is an involution, we have that μ is the appropriate sequence order-isomorphic to 12 and χ_π is the sequence order-isomorphic to $(2r+2, 2r+1, \ldots, r+3, r+1, r+2, r, r-1, \ldots, 1)$. □

Lemma 2.4. *For any $\pi \in \mathcal{I}_n$ with $|V_1(G_\pi^n)| > 1$, we have $G_{\chi_\pi} = G_\pi^n$.*

Proof. Let $(\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \ldots, i_h, \pi_{i_{h+2}}, i_{h+1}, \ldots, \pi_{i_k}, i_{k-1}, \pi_j, i_k, j)$ be the connected sequence of π, where i_1 is the initial index of π and $\pi_j = n$. It is obvious that π_{i_t} and i_t are the vertices of the connected component G_π^n for $1 \leq t \leq k+1$ with the assumption $i_{k+1} = j$. In order to prove $G_{\chi_\pi} = G_\pi^n$, we should prove that all the entries of χ_π are contained in the connected component G_π^n.

For $i_1 < s < i_2$, we have either $i_1 < \pi_s < \pi_{i_1}$ or $\pi_s > \pi_{i_1}$. In the first case, $\pi_s \pi_{i_2} i_1$ is order-isomorphic to the pattern 231. While in the second case, $\pi_{i_1} \pi_s i_1$ is order-isomorphic to the pattern 231. Hence, π_s is connected with i_1 in G_{χ_π}, which implies that π_s is a vertex of the connected component G_π^n.

For $2 \leq t \leq k+1$ and $i_t < s < \pi_{i_t}$, we have either $\pi_s > \pi_{i_{t-1}}$ or $\pi_s < \pi_{i_{t-1}}$. In the first case, $\pi_{i_{t-1}} \pi_s i_t$ is order-isomorphic to the pattern 231. While in the second case $\pi_{i_{t-1}} \pi_{i_t} \pi_s$ is order-isomorphic to the pattern 231. Hence, π_s is connected with $\pi_{i_{t-1}}$ in G_{χ_π}, which implies that π_s is a vertex of the connected component G_π^n.

The above arguments imply that all the vertices of χ_π are the vertices of the connected component G_π^n. According to the definition of the initial index, we have $\pi_s < i_1$ for $1 \leq s < i_1$, which implies that π_s is not contained in G_π^n. Hence, the bipartite graph corresponding to χ_π is G_π^n. □

By [14, Lemma 2.1], for any permutation π and any connected component $\tilde{G}$ of G_π we have $t_1 \le 2t_3 + 1$. Since the involutions form a subset of the permutations, we have the following result.

Lemma 2.5. *For any connected component $\widetilde{G}$ of G_π, one has $t_1 \le 2t_3 + 1$.*

The above lemmas lead to the following result.

Corollary 2.6. *For any $\pi \in I_n$ with $c_{\chi_\pi} = r$ $(r > 0)$, if $|V_1(G_\pi^n)| = 1$, then $l_{\chi_\pi} \le 2r+2$; otherwise $l_{\chi_\pi} \le 2r + 1$.*

3. Main Theorem and explicit results

Denote by K_t the subset of $\bigcup_{k\leqslant 2t+2} \mathcal{I}_k$ whose elements can be represented as $\{n;\lambda\}$ with $c_\lambda \le t$. Let H_t be the subset of $\bigcup_{k\leqslant 2t+1} \mathcal{I}_k$ such that the corresponding bipartite graph of each element is connected and each element contains at most t occurrences of 231. It is obvious that $K_t \cap H_t = \emptyset$. Then the main result of this paper can be formulated as follows.

Theorem 3.1. *For any $r \geqslant 1$,*

$$I_r^{231}(x) = \frac{x}{1-x} I_r^{231}(x) + \sum_{\mu \in K_r \cup H_r} x^{l_\mu} I_{r-c_\mu}^{231}(x). \qquad (*)$$

Proof. Denote by $F_r^\mu(x)$ the generating function for the number of involutions $\pi \in \mathcal{I}_n$ that contain 231 exactly r times such that χ_π is order-isomorphic to μ. We discuss three cases to find $F_r^\mu(x)$:

If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (n, n-1, \dots, n-s+1)$, then $l_\mu = s$ and $\mu = (s, s-1, \dots, 1)$. So we have

$$F_r^\mu(x) = x^s I_r^{231}(x).$$

If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = \{n;\lambda\}$ and $c_{\chi_\pi} = r$, then $\mu \in K_r$ by Lemma 2.3. Thus we have

$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$

If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (\pi_{i_1}, \pi_{i_1+1}, \dots, \pi_n)$ where i_1 is the initial index of π, then Lemma 2.5 and Lemma 2.4 yield $\mu \in H_r$ and

$$F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$

Hence, summing over all $\mu \in \{(s, s-1, s-2, \dots, 2, 1) | s \geqslant 1\} \cup K_r \cup H_r$ we get the desired result. □

Theorem 3.1, Lemma 2.3, and Lemma 2.5 provide a finite algorithm for finding $I_r^{231}(x)$ for any given $r > 0$, since we have to consider all involutions in I_k, where $k \leqslant 2r + 2$, and to perform certain routine operations with all 231-tails found so far.

Remark 3.2. *In fact, according to the Lemma 2.3, it is sufficient to check all involutions in I_k, where $k \leq 2r+1$, since the involution of length $2r+2$ containing r occurrences of the pattern 231 is $(2r+2, 2r+1, \dots, r+3, r+1, r+2, r, r-1, \dots, 1)$. As a consequence, Formula $(*)$ can be reduced as follows:*

$$I_r^{231}(x) = \frac{x}{1-x} I_r^{231}(x) + x^{2r+2} I_0^{231}(x) + \sum_{\mu \in K_r^* \cup H_r} x^{l_\mu} I_{r-c_\mu}^{231}(x),$$

where K_r^* is the set of all involutions of the form $(n, n-1, \dots, n-s+1, \lambda)$ in I_k where $k \leq 2r+1$ and λ is nonempty.

Now, we show how our results apply for some small values of r. Let us start with the case $r=0$. Observe that $(*)$ remains valid for $r=0$, provided that the left hand side is replaced by $I_0^{231}(x) - 1$; subtracting 1 here accounts for the empty permutation. Note that when $r=0$, the set $K_0 \cup H_0$ is empty. Hence we get $I_0^{231}(x) - 1 = \frac{x}{1-x} I_0^{231}(x)$, equivalently

$$I_0^{231}(x) = \frac{1-x}{1-2x}, \qquad (**)$$

which is the result of Simion and Schmidt (see [17, Proposition 6]).

Now let $r=1$. Observe that $K_1 \cup H_1 = \{4231\}$. Therefore, $(*)$ amounts to

$$I_1^{231}(x) = \frac{x}{1-x} I_1(x) + x^4 I_0^{231}(x),$$

and we get the following result from Formula $(**)$.

Corollary 3.3. (see Egge and Mansour [8, Theorem 4.3]) *The generating function $I_1^{231}(x)$ for the number of involutions containing exactly one occurrence of the pattern 231 is given by*

$$I_1^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^2};$$

equivalently, for $n \geq 5$,

$$I_{1,n}^{231} = (n-1)2^{n-6}.$$

Now let $r=2$. Exhaustive search adds four new elements to the previous list; these are 653421, 52431, 53241, and 3412, therefore we get

Corollary 3.4. *The generating function $I_2^{231}(x)$ is given by*

$$I_2^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^3}\left(1 - 3x^2 - 2x^3 + x^4 - x^5\right);$$

equivalently, for $n \geqslant 9$,

$$I_{2,n}^{231} = (n^2 + 137n - 234)2^{n-12}.$$

We have carried out exhaustive searches in $\mathcal{I}_{2r+2}$ with $r = 3, 4, 5, 6, 7$ and found that $\mathcal{I}_8$, $\mathcal{I}_{10}$, $\mathcal{I}_{12}$, $\mathcal{I}_{14}$, $\mathcal{I}_{16}$ contain 13, 24, 41, 69, 103 elements, respectively, which leads to the following corollary.

Corollary 3.5. *Let* $3 \leqslant r \leqslant 7$*, then*

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} Q_r(x),$$

where

$$\begin{aligned}
Q_3(x) &= x^5(4 - 14x + 8x^2 + 11x^3 - 6x^4 - 2x^5 + 2x^6 + 5x^7 - 2x^8 + x^9);\\
Q_4(x) &= x^6(6 - 32x + 49x^2 + 7x^3 - 73x^4 + 40x^5 + 30x^6 - 37x^7 + 2x^8 + 4x^{10}\\
&\quad -9x^{11} + 3x^{12} - x^{13});\\
Q_5(x) &= x^6(8 - 58x + 146x^2 - 120x^3 - 40x^4 - 24x^5 + 290x^6 - 184x^7 - 197x^8\\
&\quad +228x^9 + 30x^{10} - 132x^{11} + 62x^{12} + 13x^{14} - 16x^{15} + 14x^{16} - 4x^{17} + x^{18});\\
Q_6(x) &= x^6(4 - 31x + 80x^2 - 56x^3 + 4x^4 - 384x^5 + 1097x^6 - 830x^7 - 483x^8\\
&\quad +660x^9 + 685x^{10} - 1091x^{11} - 59x^{12} + 722x^{13} - 195x^{14} - 338x^{15}\\
&\quad +285x^{16} - 92x^{17} + 20x^{18} - 45x^{19} + 35x^{20} - 20x^{21} + 5x^{22} - x^{23});\\
Q_7(x) &= x^7(17 - 199x + 969x^2 - 2502x^3 + 3642x^4 - 3274x^5 + 3324x^6 - 4714x^7\\
&\quad +1874x^8 + 6326x^9 - 8262x^{10} - 231x^{11} + 5474x^{12} - 637x^{13} - 4022x^{14}\\
&\quad +1933x^{15} + 1340x^{16} - 1129x^{17} - 518x^{18} + 982x^{19} - 498x^{20} + 166x^{21}\\
&\quad -92x^{22} + 105x^{23} - 62x^{24} + 27x^{25} - 6x^{26} + x^{27}).
\end{aligned}$$

Equivalently, for $n \geq 4r + 1$*,*

$$I_{r,n}^{231} = 2^n \cdot Q_r(n),$$

where

$$\begin{aligned}
Q_3(n) &= \frac{1}{3 \cdot 2^{17}}(n^3 + 414n^2 + 12227n - 30762)\\
Q_4(n) &= \frac{1}{3 \cdot 2^{24}}(n^4 + 830n^3 + 108275n^2 + 476710n - 3117432)\\
Q_5(n) &= \frac{1}{3 \cdot 5 \cdot 2^{29}}(n^5 + 1385n^4 + 416765n^3 + 20952295n^2 + 85955874n - 544257360)\\
Q_6(n) &= \frac{1}{3^2 \cdot 5 \cdot 2^{35}}\big(n^6 + 2079n^5 + 1124515n^4 + 158232165n^3 + 3797599444n^2 +\\
&\quad +1475950836n - 72974470320\big)\\
Q_7(n) &= \frac{1}{3^2 \cdot 5 \cdot 7 \cdot 2^{40}}\big(n^7 + 2912n^6 + 2476306n^5 + 685388480n^4 + 51462119569n^3 +\\
&\quad +764352578528n^2 - 3749997108516n - 326124489600\big)
\end{aligned}$$

4. **Further results**

As an easy consequence of Theorem 3.1 we get the following result.

Corollary 4.1. *For any* $r \geq 1$ *there exists a polynomial* $P_{5r-1}(x)$ *of degree* $5r - 1$ *with integer coefficients such that*

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} P_{5r-1}(x).$$

In other words, for $n \to \infty$

$$I_{r,n}^{231} \approx \frac{2^{n-5r-1} n^r}{r!}.$$

Proof. Immediately, by the above cases we have the corollary holds for $1 \leq r \leq 7$. Let us assume by induction that the corollary holds for $1, 2, \ldots, r-1$; for r the equation $(*)$ give

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} \sum_{\rho \in K_r \cup H_r} x^{l_\rho} \frac{(1-2x)^r}{1-x} I_{r-c_\rho}^{231}(x).$$

By the induction assumption and $I_0^{231}(x) = \frac{1-x}{1-2x}$ we have that $x^{l_\rho} \frac{(1-2x)^r}{1-x} I_{r-c_\rho}(x)$ is a polynomial with integer coefficients of degree a. So Lemma 2.3 and Lemma 2.5 yield

$$a = \max\{b_j | j = 1, \ldots, r\},$$

where $b_j = 2j + 2 + r - (r - j + 1) + 1 + 5(r - j) - 1 = 5r - 2j + 1$, which means $a = 5r - 1$, as claimed. □

Another direction would be to match the approach of this paper with the previous results on restricted 231-avoiding involutions. Let $\Phi_r(x; k)$ be the generating function for the number of involutions in $\mathcal{I}_n$ containing r occurrences of 231 and avoiding the pattern $12 \ldots k \in \mathfrak{S}_k$. Our new approach allows to get a recursion for $\Phi_r(x; k)$ for any given $r \geqslant 0$.

We denote by e_λ the length of the longest increasing subsequence of any involution λ. For example, let $\lambda = 3412$, then $e_\lambda = 2$. We denote by $K_t(k) \cup H_t(k)$ the set of all involutions $\lambda \in K_t \cup H_t$ such that $e_\lambda \leq k - 1$.

Theorem 4.2. *For any $r \geqslant 1$ and $k \geqslant 3$,*

$$\Phi_r(x; k) = \frac{x}{1-x} \Phi_r(x; k-1) + \sum_{\mu \in K_r(k) \cup H_r(k)} x^{l_\mu} \Phi_{r-c_\mu}(x; k - e_\mu).$$

Besides, $\Phi_r(x; 1) = \Phi_r(x; 2) = 0$, and $\Phi_0(x; 1) = 1$ and $\Phi_0(x; 2) = \frac{1}{1-x}$.

Proof. If we replace the sets K_r and H_r with the sets $K_r(k)$ and $H_r(k)$ in the proof of Theorem 3.1, then we arrive to the recurrence relation of the generating function $\Phi_r(x; k)$. The initial conditions hold directly from the definitions. □

Similar to the case of $I_r^{231}(x)$, the statement of the above theorem remains valid for $r = 0$, provided the left hand side is replaced by $\Phi_r(x; k) - 1$. This allows to recover known explicit expressions for $\Phi_r(x; k)$ for $r = 0, 1$, as follows.

Corollary 4.3. (see Egge and Mansour [8]) *For all $k \geq 1$,*

$$\Phi_0(x; k) = \sum_{j=0}^{k-1} \left(\frac{x}{1-x}\right)^j;$$

$$\Phi_1(x; k) = x^4 \sum_{j=0}^{k-3} (j+1) \left(\frac{x}{1-x}\right)^j.$$

Proof. Observe that Theorem 4.2 remains valid for $r = 0$, provided the left hand side is replaced by $\Phi_0(x; k) - 1$; subtracting 1 here accounts for the empty involution. Note that when $r = 0$, the set $K_0(k) \cup H_0(k)$ is empty. Hence, we obtain that $\Phi_0(x; k) = 1 + \frac{x}{1-x} \Phi_0(x; k-1)$. Using induction on k with the initial condition $\Phi_0(x; 1) = 1$ we get that $\Phi_0(x; k) = \sum_{j=0}^{k-1} \left(\frac{x}{1-x}\right)^j$.

Now, note that when $r = 1$ and $k \geq 3$, we have that $K_1(k) \cup H_1(k) = \{4231\}$ and $e_{4231} = 2$. So, Theorem 4.2 gives $\phi_1(x;k) = \frac{x}{1-x}\Phi_1(x;k-1) + x^4\Phi_0(x;k-2)$, which is equivalent to $\phi_1(x;k) = \frac{x}{1-x}\Phi_1(x;k-1) + x^4\sum_{j=0}^{k-3}\left(\frac{x}{1-x}\right)^j$. Hence, using the initial conditions $\Phi_1(x;1) = \Phi_1(x;2) = 0$ and induction on k we get the desired result. □

This approach can be extended even further to cover involutions containing r occurrences of 231 and avoiding another pattern in $\mathfrak{S}_k$, for example $k \dots 321$.

We denote a_λ be the length of maximal decreasing subsequence in an involution λ. For example, if $\lambda = 87645321$, then $a_\lambda = 7$. Let $\Omega_r(x;k)$ be the generating function for the number of involutions containing exactly r occurrences 231 and avoiding $k \dots 321$.

Theorem 4.4. *For any $r \geqslant 1$ and $k \geqslant 1$,*

$$\Omega_r(x;k) = \frac{1}{1-x-\cdots-x^{k-1}}\left(\sum_{\mu\in K_r\cup H_r \text{ and } a_\mu\leq k-1} x^{l_\mu}\Omega_{r-c_\mu}(x;k)\right).$$

Proof. Denote by $V_r^\mu(x;k)$ the generating function for the number of involutions $\pi \in \mathcal{I}_n$ that avoid $k \dots 321$ and contain 231 exactly r times such that χ_π is order-isomorphic to μ. We discuss three cases to find $V_r^\mu(x;k)$:

If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (n, n-1, \dots, n-s+1)$, then $l_\mu = s$ and $\mu = (s, s-1, \dots, 1)$ with $s \leq k-1$, so we have

$$V_r^\mu(x;k) = x^s I_r^{231}(x).$$

If π is an involution in $\mathcal{I}_n$ such that $\chi_\pi = \{n;\lambda\}$ and $c_{\chi_\pi} = r$, then $\mu \in K_r$ by Lemma 2.3; thus if $a_\mu \leq k-1$ then we have

$$V_r^\mu(x;k) = x^{l_\mu}\Omega_{r-c_\mu}(x).$$

If π is an involution in $\mathcal{I}_n$ with $\chi_\pi = (\pi_{i_1}, \pi_{i_1+1}, \dots, \pi_n)$ where i_1 is the initial index of π, then Lemma 2.5 and Lemma 2.4 yield $\mu \in H_r$, and if $a_\mu \leq k-1$ then

$$V_r^\mu(x;k) = x^{l_\mu} I_{r-c_\mu}(x;k).$$

Hence, if we sum over all $\mu \in \{(s, s-1, \dots, 1) | s \leq k-1\} \cup \{\mu \in K_r \cup H_r \mid a_\mu \leq k-1\}$, then we get the desired result. □

The final direction would be to match the approach of this note with the previous results on restricted 231-avoiding even or odd involutions. We say π an even (resp; odd) involution if the number of inversions in π, namely $21(\pi)$ is even (resp; odd). We define the sign h_λ of any permutation λ as $(-1)^{21(\lambda)}$. For example, if $\lambda = (6,5,3,4,2,1)$ then $h_\lambda = 1$. We denote by $K_r^+ \cup H_r^+$ the set of all the involutions $\lambda \in K_r \cup H_r$ such that $h_\lambda = 1$ and denote by $K_r^- \cup H_r^-$ the set of all involutions $\lambda \in K_r \cup H_r$ such that $h_\lambda = -1$.

Let $I_r^+(x)$ (resp; $I_r^-(x)$) be the generating function for the number of even (resp; odd) involutions in $\mathcal{I}_n$ containing r occurrences of 231. Our new approach allows us to get an explicit expression for $I_r^+(x)$ (or $I_r^-(x)$) for any given $r \geqslant 0$.

Theorem 4.5. *For all $r \geqslant 1$,*

$$I_r^+(x) = \frac{x+x^4}{1-x^4} I_r^+(x) + \frac{x^2+x^3}{1-x^4} I_r^-(x) + \sum_{\mu \in K_r^+ \cup H_r^+} x^{l_\mu} I_{r-c_\mu}^+(x) + \sum_{\mu \in K_r^- \cup H_r^-} x^{l_\mu} I_{r-c_\mu}^-(x);$$

$$I_r^-(x) = \frac{x+x^4}{1-x^4} I_r^-(x) + \frac{x^2+x^3}{1-x^4} I_r^+(x) + \sum_{\mu \in K_r^+ \cup H_r^+} x^{l_\mu} I_{r-c_\mu}^-(x) + \sum_{\mu \in K_r^- \cup H_r^-} x^{l_\mu} I_{r-c_\mu}^+(x).$$

In particular, we have

$$I_0^+(x) - 1 = \frac{x+x^4}{1-x^4} I_0^+(x) + \frac{x^2+x^3}{1-x^4} I_0^-(x)$$

and

$$I_0^-(x) = \frac{x+x^4}{1-x^4} I_0^-(x) + \frac{x^2+x^3}{1-x^4} I_0^+(x).$$

Proof. Here we only prove the result of $I_r^+(x)$ for any $r \geq 1$. By the same method, we can obtain the formula for $I_r^-(x)$. Denote by $Q_r^\mu(x)$ the generating function for the number of involutions in $\pi \in I_n$ such that χ_π is order-isomorphic to μ and $h_\pi = 1$.

To find $Q_r^\mu(x)$, we recall six cases. If π is an involution in I_n with $\chi_\pi = (n, n-1, \ldots, n-s+1)$ and $h_\mu = \pm 1$, then we have $Q_r^\mu(x) = x^s I_r^\pm(x)$, where $\mu = (s, s-1, \ldots, 1)$.

If π is an involution in I_n such that $\chi_\pi = \{n; \lambda\}$ and $\mu \in K_r^\pm$, then Lemma 2.3 yields $Q_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^\pm(x)$.

If π is an involution in I_n with $\chi_\pi = (\pi_{i_1}, \pi_{i_1+1}, \ldots, \pi_n)$ where i_1 is the initial index of π and $h_\mu = \pm 1$, then Lemma 2.4 and Lemma 2.5 yield $Q_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^\pm(x)$, where $\mu \in H_r^\pm$.

Hence, if we sum over all $\mu \in K_r \cup H_r \cup \{(s, s-1, s-2, \ldots, 2, 1) | s \geq 1\}$ then we get the desired result. When $r = 0$, subtracting 1 here accounts for the empty permutation. □

As an example of the above theorem we get

Corollary 4.6. *For $0 \leqslant r \leqslant 2$,*

$$I_r^+(x) = \frac{E_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}}, \quad I_r^-(x) = \frac{O_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}};$$

where

$E_0(x) = 1 - 2x + 2x^2 - 2x^3$;

$E_1(x) = 2x^6(1 - 2x + 2x^2 - 2x^3)$;

$E_2(x) = x^4(1 - 5x + 11x^2 - 15x^3 + 10x^4 + 5x^5 - 11x^6 - 5x^7 + 47x^8 - 94x^9 + 86x^{10} - 62x^{11} + 16x^{12})$;

$O_0(x) = x^2$;

$O_1(x) = x^4(1 - 4x + 8x^2 - 12x^3 + 13x^4 - 8x^5 + 4x^6)$;

$O_2(x) = x^6(2 - 6x + 6x^2 - 2x^3 - 9x^4 + 4x^5 + 20x^6 - 36x^7 + 53x^8 - 24x^9 + 8x^{10})$.

Again, as an easy consequence of Theorem 4.5 we get the following result.

Corollary 4.7. *Let $r \geq 0$. Then there exist number m_r and r_r and polynomials $p_{m_r}(x)$ and $q_{n_r}(x)$ of degree m_r and n_r respectively such that*

$$I_r^+(x) = \frac{p_{m_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}}, \quad I_r^-(x) = \frac{q_{n_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}}.$$

It can be proved by induction on r as the proof of Corollary 4.1. Here we delete its proof.

As a remark we can derive other results from Theorem 4.5. For example, the generating function for the number of even or odd involutions containing exactly r occurrences of the pattern 231 and avoiding $12\dots k$ (or avoiding $k\dots 21$).

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