

A Note on $(3, 1)^*$ -Choosable Toroidal Graphs [†]

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Abstract

An $(L, d)^*$ -coloring is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that at most d neighbors of v receive color $\phi(v)$. A graph is called $(m, d)^*$ -choosable, if G admits an $(L, d)^*$ -coloring for every list assignment L with $|L(v)| \geq m$ for all $v \in V(G)$. In this note, it is proved that every toroidal graph, which contains no adjacent triangles and contains no 6-cycles and l -cycles for some $l \in \{5, 7\}$, is $(3, 1)^*$ -choosable.

Key words: Triangle, choosability, toroidal graph

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1 Introduction

Graphs considered in this paper are finite, simple and undirected. A *toroidal* graph $G = (V, E, F)$ is a graph embedded on the torus, where V , E and F denote the set of vertices, edges and faces of G , respectively.

A face of an embedded graph is said to be *incident* with the edges and vertices on its boundary. Two faces are *adjacent* if they share a common edge. In particular, two adjacent 3-faces are often referred as *adjacent triangles*. The *degree* of a face f of G , denoted by $d_G(f)$, is the number

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of edges incident with it. Note that each cut-edge is counted twice in the degree. A k -vertex (or k -face) is a vertex (or a face) of degree k , a k^- -vertex (or k^- -face) is a vertex (or a face) of degree *at most* k , and a k^+ -vertex (or k^+ -face) is defined similarly. An n -face f is called an (l_1, l_2, \dots, l_n) -face if the vertices incident with f have degree l_1, l_2, \dots, l_n sequentially. A cycle is called an m -cycle if it is of length m . Undefined terms and notion follow [1].

For each vertex $v \in V(G)$, we assign a set of colors, $L(v)$ (called it *list*), to v . An L -coloring with impropriety d for non-negative integer d , or simply $(L, d)^*$ -coloring, is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that at most d neighbors of v receive color $\phi(v)$. For integers $m \geq d \geq 0$, a graph is called $(m, d)^*$ -choosable, if G admits an $(L, d)^*$ -coloring for every list assignment L with $|L(v)| \geq m$ for all $v \in V(G)$. An $(m, 0)^*$ -choosable graph is simply called m -choosable.

It is a hard problem to decide if a plane graph is 3-choosable, even for triangle-free plane graphs. Thomassen proved that every plane graph of girth at least 5 is 3-choosable [7]. In [8], Voigt and Wirth constructed a family of triangle-free plane graphs that is not 3-choosable. In [10], it was proved that every triangle-free plane graph containing no 8- and 9-cycles is 3-choosable. However, the 3-choosability of triangle-free plane graph without 6- and 7-cycles is still open.

The concept of list improper coloring was first introduced by Škrekovski [4], and Eaton and Hull [2], independently. They proved that every plane graph is $(3, 2)^*$ -choosable and every outerplanar graph is $(2, 2)^*$ -choosable. Škrekovski [5, 6] investigated the relationship between $(m, d)^*$ -choosability and the girth in plane graphs. For instance, he proved every plane graph G is $(3, 1)^*$ -choosable if its girth, $g(G)$, is at least 4, and is $(2, d)^*$ -choosable if $g(G) \geq 5$ and $d \geq 4$. In [3], it was showed that every plane graph without 4-cycles and l -cycles for some $l \in \{5, 6, 7\}$ is $(3, 1)^*$ -choosable.

For toroidal graphs, Xu and Zhang [9] proved that every toroidal graph without adjacent triangles is $(4, 1)^*$ -choosable. In this note, we make the further restriction $(3, 1)^*$ -choosability on toroidal graphs to improve Xu and Zhang's result to $(3, 1)^*$ -choosable.

Let \mathcal{G} denote the family of toroidal graphs containing no adjacent triangles and containing no 6-cycles and l -cycles for $l \in \{5, 7\}$. The main result is to show that every graph in \mathcal{G} is $(3, 1)^*$ -choosable. In order to prove the main theorem, we use the technique of "discharging" to obtain several forbidden configurations for the graphs in \mathcal{G} and state as a theorem below.

Theorem 1. *For every graph $G \in \mathcal{G}$, one of the following must hold:*

- (1) $\delta(G) < 3$.
- (2) G contains two adjacent 3-vertices.
- (3) G contains a $(3, 4, 4)$ -face.

(4) G contains a $(3, 4, 3, 4)$ -face.

As a consequence of the above result, we can prove the following

Theorem 2. *Every graph in \mathcal{G} is $(3, 1)^*$ -choosable.*

2 Proofs of the theorems

In the proof of Theorem 1, we use the technique of discharging. In the beginning, each vertex v is assigned a charge $\frac{d_G(v)}{3} - 1$ and each face f is assigned a charge $\frac{d_G(f)}{6} - 1$. By following the rules stated in the proof of the theorem, we will redistribute the charges for the vertices and faces so that the new charges are nonnegative and the sum of the new charges is still the same as before, which leads to a contradiction to Euler's formula.

Proof of Theorem 1: Assume to the contrary that the theorem does not hold. Let G be a connected toroidal graph in \mathcal{G} satisfying $\delta(G) \geq 3$, every 3-vertex is adjacent to only 4^+ -vertices, and G contains neither $(3, 4, 4)$ -faces nor $(3, 4, 3, 4)$ -faces.

Recall that we can rewrite Euler's formula $|V| + |F| - |E| = 0$ for toroidal graphs as

$$\sum_{v \in V(G)} \left\{ \frac{d_G(v)}{3} - 1 \right\} + \sum_{f \in F(G)} \left\{ \frac{d_G(f)}{6} - 1 \right\} = 0 \quad (1)$$

Defining a charge function ω on $V(G) \cup F(G)$ by letting $\omega(v) = \frac{d_G(v)}{3} - 1$ if $v \in V(G)$ and $\omega(f) = \frac{d_G(f)}{6} - 1$ if $f \in F(G)$. Then the total sum of the charges, $\sum_{x \in V(G) \cup F(G)} \omega(x)$, is zero.

For two elements x and y of $V(G) \cup F(G)$, we use $W(x \rightarrow y)$ to denote the charge transferred from x to y .

Case 1. $G \in \mathcal{G}$ contains neither 5- nor 6-cycles.

By the choice of G , it is easy to have the following observation.

$(O_{1,1})$ G contains no 5- and 6-faces, no adjacent 4^- -faces.

Let v be a k -vertex of G and f a 3- or 4-face incident with v . Denote the number of 3- or 4-faces incident with v by r_v . Then it is not hard to see $r_v \leq \lfloor \frac{k}{2} \rfloor$.

The new charge function $\omega'(x)$ is obtained by following discharging rules given below:

$(R_{1,1})$ For all v and f , $W(v \rightarrow f) = \frac{1}{6}$ if $k = 4$; $W(v \rightarrow f) = \frac{1}{3}$ if $k \geq 5$.

($R_{1,2}$) Each 7^+ -face transfers $\frac{1}{42}$ to each of its adjacent 4^- -faces.

Reader is reminded that a face may be adjacent to another face *multiple* times. Now, we ought to prove that $\omega'(x) \geq 0$ for any $x \in V(G) \cup F(G)$.

If $k = 3$, then $\omega'(v) = \omega(v) = 0$.

If $k = 4$, then $r_v \leq 2$ and $\omega'(v) \geq \omega(v) - \frac{r_v}{6} = \frac{2-r_v}{6} \geq 0$.

If $k \geq 5$, then $\omega'(v) \geq \omega(v) - \frac{r_v}{3} = \frac{k-3-r_v}{3} \geq 0$ (note that $r_v \leq 2$ if $k = 5$).

Let f be an h -face of G . If $h \geq 7$, then $\omega'(f) \geq \omega(f) - h \cdot \frac{1}{42} = \frac{6h-42}{42} \geq 0$.

If $h = 3$, by ($O_{1,1}$), f is adjacent to three 7^+ -faces. Since G contains no adjacent 3-vertices and contains no $(3, 4, 4)$ -face, f is either incident with a 5^+ -vertex and a 4^+ -vertex or incident with three 4^+ -vertices. In the former case, f receives at least $\frac{1}{3}$ from the 5^+ -vertex and receives at least $\frac{1}{6}$ from another 4^+ -vertex, and hence

$$\omega'(f) \geq \frac{-1}{2} + \frac{1}{3} + \frac{1}{6} + 3 \cdot \frac{1}{42} > 0. \quad (2)$$

In the latter case, f receives at least $\frac{1}{6}$ from each of the vertices incident with it, and hence

$$\omega'(f) \geq \frac{-1}{2} + 3 \cdot \frac{1}{6} + 3 \cdot \frac{1}{42} > 0. \quad (3)$$

If $h = 4$, then f is adjacent to four 7^+ -faces. Since G contains neither adjacent 3-vertices nor $(3, 4, 3, 4)$ -faces, f is incident with at least two 4^+ -vertices. Furthermore, if f is incident with a 3-vertex, then f is either incident with a 5^+ -vertex or incident with three 4^+ -vertices. Therefore,

$$\omega'(f) \geq \omega(f) + \frac{1}{3} + \frac{1}{6} + 4 \cdot \frac{1}{42} > 0. \quad (4)$$

Thus, $\omega'(x) \geq 0$ for each $x \in V(G) \cup F(G)$. By (2), (3) and (4), $\omega'(f) > 0$ if $d_G(f) = 3, 4$. If G contains no 3- and 4-faces, then $\omega'(f) = \omega(f) > 0$ for any face f . Therefore, $0 < \sum_{x \in V(G) \cup F(G)} \omega'(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = 0$, a contradiction.

Case 2. $G \in \mathcal{G}$ contains neither 6- nor 7-cycles.

By the choice of G , we have the following observations.

($O_{2,1}$) G contains no 6- and 7-faces, no adjacent 3-faces, and no adjacent 4-faces.

($O_{2,2}$) No 5-face is adjacent to 3- or 4-faces.

($O_{2,3}$) Each 3-face is adjacent to at most one 4-face and each 4-face is adjacent to at most one 3-face.

Let v be a k -vertex and f an l -face incident with v . Denote the numbers of 4^- -faces and 5^- -faces incident with v by r_1 and r_2 , respectively. By $(O_{2,2})$ and $(O_{2,3})$, we can see that $r_1 \leq \lfloor \frac{2k}{3} \rfloor$ and $3\lceil \frac{r_1}{2} \rceil + r_2 \leq k + 1$.

The discharging rules are as follows:

- $(R_{2,1})$ For $k = 4$, $W(v \rightarrow f) = \frac{1}{6}$ if $3 \leq l \leq 4$; $W(v \rightarrow f) = \frac{1}{18}$ if $l = 5$.
- $(R_{2,2})$ For $k \geq 5$, $W(v \rightarrow f) = \frac{1}{4}$ if $l = 3$; $W(v \rightarrow f) = \frac{1}{6}$ if $l = 4$;
 $W(v \rightarrow f) = \frac{1}{18}$ if $l = 5$.
- $(R_{2,3})$ An 8^+ -face transfers $\frac{1}{24}$ to each of its adjacent 5^- -faces.

We now verify that $\omega'(x) \geq 0$ for any $x \in V(G) \cup F(G)$.

If $k = 3$, then $\omega'(v) = \omega(v) = 0$.

If $k = 4$, then $r_1 \leq 2$. From $(O_{2,2})$, $r_1 = 2$ implies $r_2 = 0$ and $r_2 \geq 2$ implies $r_1 = 0$. Hence $3r_1 + r_2 \leq 6$ and $\omega'(v) \geq \omega(v) - \frac{r_1}{6} - \frac{r_2}{18} = \frac{1}{3} - \frac{1}{18}(3r_1 + r_2) \geq 0$.

If $k = 5$, then $r_1 \leq 3$. If $r_1 = 3$, then $r_2 = 0$ and v is incident with a 4-face. Hence $\omega'(v) \geq \omega(v) - 2 \cdot \frac{1}{4} - \frac{1}{6} = 0$. If $r_1 = 2$, then $r_2 \leq 1$ and thus $\omega'(v) \geq \omega(v) - 2 \cdot \frac{1}{4} - \frac{1}{18} > 0$. If $r_1 \leq 1$, then $r_2 \leq 2$ and thus $\omega'(v) \geq \omega(v) - \frac{1}{4} - \frac{2}{18} > 0$.

If $k \geq 6$, since G contains no adjacent triangles, v is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces. If $r_1 \leq \lfloor \frac{k}{2} \rfloor$, then

$$\begin{aligned} \omega'(v) &\geq \omega(v) - \frac{r_1}{4} - \frac{r_2}{18} = \frac{k-3}{4} - \frac{9r_1+2r_2}{36} \\ &= \frac{k-3}{4} - \frac{3(3\lceil \frac{r_1}{2} \rceil + r_2) + 9\lfloor \frac{r_1}{2} \rfloor - r_2}{36} \\ &\geq \frac{12k-36-3(k+1)-9\lfloor \frac{r_1}{2} \rfloor + r_2}{36} \geq \frac{9k-33-9\lfloor \frac{k}{4} \rfloor}{36} \geq 0. \end{aligned}$$

If $r_1 > \lfloor \frac{k}{2} \rfloor$, then v is incident with at least $r_1 - \lfloor \frac{k}{2} \rfloor$ 4-faces and thus

$$\begin{aligned} \omega'(v) &\geq \omega(v) - \frac{1}{4}\lfloor \frac{k}{2} \rfloor - \frac{1}{6}(r_1 - \lfloor \frac{k}{2} \rfloor) - \frac{r_2}{18} \\ &= \frac{k-3}{4} - \frac{9\lfloor \frac{k}{2} \rfloor + 6r_1 - 6\lfloor \frac{k}{2} \rfloor + 2r_2}{36} \\ &= \frac{k-3}{4} - \frac{3\lfloor \frac{k}{2} \rfloor + 2(3\lceil \frac{r_1}{2} \rceil + r_2) + 6\lfloor \frac{r_1}{2} \rfloor}{36} \\ &\geq \frac{12k-36-3\lfloor \frac{k}{2} \rfloor - 2(k+1) - 6\lfloor \frac{r_1}{2} \rfloor}{36} \geq \frac{10k-38-3\lfloor \frac{k}{2} \rfloor - 6\lfloor \frac{k}{3} \rfloor}{36} \geq 0. \end{aligned}$$

Let f be an h -face of G .

If $h \geq 8$, then, by $(R_{2,3})$, $\omega'(f) \geq \omega(f) - \frac{h}{24} = \frac{3h-24}{24} \geq 0$.

If $h = 5$, then f is incident with at least three 4^+ -vertices (note that G contains no adjacent 3-vertices) and, by $(R_{2,1})$ and $(R_{2,2})$, each of these 4^+ -vertices transfers at least $\frac{1}{18}$ to f and hence $\omega'(f) \geq \omega(f) + \frac{3}{18} = 0$ if f is not adjacent to 8^+ -faces and

$$\omega'(f) = \omega(f) + \frac{3}{18} + \frac{1}{24} > 0 \quad \text{if } f \text{ is adjacent to at least one } 8^+\text{-face.} \quad (5)$$

If $h = 4$, f is incident with at least two 4^+ -vertices and is adjacent to at least three 8^+ -faces. Thus

$$\omega'(f) \geq \omega(f) + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{24} > 0. \quad (6)$$

If $h = 3$, f is adjacent to at least two 8^+ -faces of which each transfers $\frac{1}{24}$ to f . Therefore, $\omega'(f) \geq \frac{-1}{2} + \frac{1}{4} + \frac{1}{6} + 2 \cdot \frac{1}{24} = 0$ if f is incident with a 5^+ -vertex and another 4^+ -vertex, and $\omega'(f) \geq \frac{-1}{2} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{24} > 0$ if f is incident with three 4^+ -vertices.

By (6), $\omega'(f) > 0$ for every 4-face f . From (5), $\omega'(f) > 0$ for every 5-face f adjacent to some 8^+ -faces. If G contains no 4-faces, then every 3-face is adjacent to three 8^+ -faces that yields $\omega'(f) > 0$ for any 3-face f . If G contains no 5^- -face adjacent to 8^+ -faces, then $\omega'(f) > 0$ for any 8^+ -face f . So, $0 < \sum_{x \in V(G) \cup F(G)} \omega'(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = 0$. This contradiction leads to the proof of Case 2 and thus Theorem 1. ■

Proof of Theorem 2: Assume to the contrary. Let G be a counterexample with the fewest vertices, i.e., there exists a list assignment L with $|L(v)|=3$ for all $v \in V(G)$ such that G is not $(L, 1)^*$ -choosable, but any proper subgraph of G is.

If $\delta(G) < 3$, let v be a 2-vertex of G . Then, $G - v$ is $(3, 1)^*$ -choosable by the choice of G . Since in any $(L, 1)^*$ -coloring of $G - v$, there must exist a color in $L(v)$ that is not used by any neighbors of v , any $(L, 1)^*$ -coloring of $G - v$ can be extended to a $(L, 1)^*$ -coloring of G , a contradiction. So we assume that $\delta(G) \geq 3$.

If G contains two adjacent 3-vertices, say u and v , then by the choice of G , $G - \{u, v\}$ is $(3, 1)^*$ -choosable. In any $(L, 1)^*$ -coloring of $G - \{u, v\}$, there exists a color in $L(u)$ that is not used by any neighbors of u in $G - \{u, v\}$, and the same holds for v . Applying the same argument as the above, we see that G is $(L, 1)^*$ -choosable, a contradiction.

Suppose that G contains a $(3, 4, 4)$ -face f with the boundary $xyzx$, say, $d_G(x) = 3$ and $d_G(y) = d_G(z) = 4$. Let $H = G - \{x, y, z\}$. By the choice of G , H admits an $(L, 1)^*$ -coloring ϕ . For $w \in \{x, y, z\}$, let $L'(w) = L(w) \setminus \{\phi(u) | u \in N_H(w)\}$. Then, $|L'(x)| \geq 2$, $|L'(y)| \geq 1$ and $|L'(z)| \geq 1$. If $L'(y) = L'(z)$, then color y and z with a same color γ in $L'(y)$ and color x with a color in $L'(x) \setminus \{\gamma\}$. If $L'(y) \neq L'(z)$, then color y with a color $\alpha \in L'(y) \setminus L'(z)$, color z with a color in $L'(z)$, and color x with an arbitrary color in $L'(x)$. In either case, we obtain an $(L, 1)^*$ -coloring of G , a contradiction.

By Theorem 1, we may assume that G contains a $(3, 4, 3, 4)$ -face f with the boundary $xyzux$. By symmetry, we assume that $d_G(x) = d_G(z) = 3$ and $d_G(y) = d_G(u) = 4$. Let $F = G - \{x, y, z, u\}$. By the choice of G , F admits an $(L, 1)^*$ -coloring ψ . For $w \in \{x, y, z, u\}$, let $L'(w) = L(w) \setminus \{\psi(v) | v \in N_F(w)\}$. Then, $|L'(x)| \geq 2$, $|L'(z)| \geq 2$, $|L'(y)| \geq 1$ and $|L'(u)| \geq 1$. It is easy to verify that $xyzux$ admits an $(L', 1)^*$ -coloring. This together with ψ yields an $(L, 1)^*$ -coloring of G . This contradiction completes the proof of Theorem 2. ■

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