A Question of Muir and Littlewood's Products

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Abstract We answer a question of Muir, relating it to different determinantal expressions for the products $\prod_{i<j}(y-x_ix_j)$ and $\prod_{i\le j}(y-x_ix_j)$, and for the products of these functions by an arbitrary Schur function.

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1 Introduction

Muir [6] writes determinantal expressions for $\prod_{1\le i<j\le 3}(y-x_ix_j)$ and for $\prod_{1\le i\le j\le 3}(y-x_ix_j)$ but makes the comment that the method followed is not consistent. [1]

In this paper, we give determinantal expressions for the products $\prod_{i<j}(y-x_ix_j)$ (Theorem 2.5) and $\prod_{i\le j}(y-x_ix_j)$ (Theorem 2.6), and give another type of determinants for the the product of these functions by any Schur function (Theorem 2.7). The first determinants are related to the expression of a Schur function in terms of hook Schur functions, the last ones are related to the symplectic or orthogonal characters.

For example, for $n=3$, we write

$$-\prod_{1\le i<j\le 3}(y-x_ix_j)=\begin{vmatrix} s_{11}-y & s_{21} \\ s_{111} & s_{211}+y^2 \end{vmatrix},$$

$$\prod_{1\le i\le j\le 3}(y-x_ix_j)=\begin{vmatrix} s_2-y & s_3 & s_4 \\ s_{21} & s_{31}+y^2 & s_{41} \\ s_{211} & s_{311} & s_{411}-y^3 \end{vmatrix}$$

[1] [6]: *Still more curious is the fact that a similar quest, – the search for the equation whose roots are the binary products without repetition, namely,*

$$x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$$

– has actually been successful. Unfortunately the method followed – a quasi-dialytic – is not logically consistent ...

and

$$s_{221}\prod_{1\le i\le j\le 3}(y-x_ix_j)=\begin{vmatrix}h_6-yh_4 & h_7-y^2h_3 & h_8-y^3h_2\\ h_5-yh_3 & h_6-y^2h_2 & h_7-y^3h_1\\ h_3-yh_1 & h_4-y^2 & h_5\end{vmatrix}.$$

We first recall some basic definitions about symmetric functions, following the conventions of Macdonald [4].

Let $\mathbb{X}=\{x_1,x_2,\dots,x_n\}$ be a set of n indeterminates. The complete symmetric functions h_k and the elementary symmetric functions e_k are defined by the generating functions:

$$\sum_{k=0}^{\infty}h_kt^k=\prod_{i=1}^{n}(1-x_it)^{-1},\qquad \sum_{k=0}^{\infty}e_kt^k=\prod_{i=1}^{n}(1+x_it).$$

It follows that

$$\sum_{i=0}^{n}(-1)^ie_ih_{n-i}=0.$$

Given a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ and its conjugate λ', the Schur function s_λ has the following two expressions:

$$s_\lambda=\det\big(h_{\lambda_i-i+j}\big)_{1\le i,j\le n}=\det\big(e_{\lambda'_i-i+j}\big)_{1\le i,j\le n}.$$

Hook Schur functions $s_{j,1^i}$ can also be written as

$$s_{j,1^i}=\sum_{k=1}^{j}(-1)^{k+1}e_{i+k}h_{j-k}. \tag{1}$$

Another notation, due to Frobenius, can be used for a partition λ. If r be the biggest number such that $\lambda_i\ge i$ then

$$F(\lambda)=(\alpha_1,\dots,\alpha_r|\beta_1,\dots,\beta_r),$$

where $\alpha_i=\lambda_i-i$ and $\beta_i=\lambda'_i-i$, $1\le i\le r$.

This notation is associated to still another determinantal expression of a Schur function, as a determinant of hook Schur functions ([4, Ex.9 p.47]):

$$s_\lambda=\det\big(s_{\alpha_i|\beta_j}\big)_{1\le i,j\le r}. \tag{2}$$

In this paper, we use the following two generalizations of the Vandermonde determinant:

$$M(\mathbb{X},y)=\begin{vmatrix}x_1^{n-1} & x_1^n+x_1^{n-2}y & \dots & x_1^{2n-2}+y^{n-1}\\ x_2^{n-1} & x_2^n+x_2^{n-2}y & \dots & x_2^{2n-2}+y^{n-1}\\ \dots & \dots & \dots & \dots\\ x_n^{n-1} & x_n^n+x_n^{n-2}y & \dots & x_n^{2n-2}+y^{n-1}\end{vmatrix}$$

and

$$N(\mathbb{X},y) = \begin{vmatrix} x_1^{n+1} - x_1^{n-1}y & x_1^{n+2} - x_1^{n-2}y^2 & \dots & x_1^{2n} - y^n \\ x_2^{n+1} - x_2^{n-1}y & x_2^{n+2} - x_2^{n-2}y^2 & \dots & x_2^{2n} - y^n \\ \dots & \dots & \dots & \dots \\ x_n^{n+1} - x_n^{n-1}y & x_n^{n+2} - x_n^{n-2}y^2 & \dots & x_n^{2n} - y^n \end{vmatrix}.$$

One can factor out from these two determinants the Vandermonde $\Delta(\mathbb{X}) = \prod_{i<j}(x_j - x_i)$, obtaining two symmetric functions:

$$SY_{(n-1)^n} = M(\mathbb{X},y)\,\Delta(\mathbb{X})^{-1}, \quad OY_{(n+1)^n} = N(\mathbb{X},y)\,\Delta(\mathbb{X})^{-1},$$

where, more generally, for any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, the determinants $SY_{(n-1)^n+\lambda}$ and $OY_{(n+1)^n+\lambda}$ are the following expressions:

$$\begin{vmatrix} h_{n-1+\lambda_1} & h_{n+\lambda_1} + yh_{n-2+\lambda_1} & \dots & h_{2n-2+\lambda_1} + y^{n-1}h_{\lambda_1} \\ h_{n-2+\lambda_2} & h_{n-1+\lambda_2} + yh_{n-3+\lambda_2} & \dots & h_{2n-3+\lambda_2} + y^{n-1}h_{\lambda_2-1} \\ \dots & \dots & \dots & \dots \\ h_{\lambda_n} & h_{1+\lambda_n} + yh_{\lambda_n-1} & \dots & h_{n-1+\lambda_n} + y^{n-1}h_{\lambda_n+1-n} \end{vmatrix} \quad (3)$$

and

$$\begin{vmatrix} h_{n+1+\lambda_1} - yh_{n-1+\lambda_1} & h_{n+2+\lambda_1} - y^2h_{n-2+\lambda_1} & \dots & h_{2n+\lambda_1} - y^nh_{\lambda_1} \\ h_{n+\lambda_2} - yh_{n-2+\lambda_2} & h_{n+1+\lambda_2} - y^2h_{n-3+\lambda_2} & \dots & h_{2n-1+\lambda_2} - y^nh_{\lambda_2-1} \\ \dots & \dots & \dots & \dots \\ h_{2+\lambda_n} - yh_{\lambda_n} & h_{3+\lambda_n} - y^2h_{\lambda_n-1} & \dots & h_{n+1+\lambda_n} - y^nh_{\lambda_n+1-n} \end{vmatrix}. \quad (4)$$

These last two functions are homogeneous versions of Weyl's determinants for the symplectic and orthogonal characters of respective index $(n-1)^n + \lambda$ and $(n+1)^n + \lambda$ ([8, p.219, p.228], [1]).

In fact, as the referee points out, the Vandermonde in the variables $x_i + y/x_i$, $i = 1 \dots n$, which is equal to

$$\prod_{i<j}(x_j - x_i + y/x_j - y/x_i) = \prod_{i<j}(x_j - x_i)(1 - y/x_ix_j)$$

is represented by the determinant $\det\left((x_i + yx_i^{-1})^{j-1}\right)_{1\le i,j\le n}$. By linear combination of columns, one transforms it into the determinant

$$\begin{vmatrix} 1 & x_1 + yx_1^{-1} & x_1^2 + y^2x_1^{-2} & \dots & x_1^{n-1} + y^{n-1}x_1^{1-n} \\ 1 & x_2 + yx_2^{-1} & x_2^2 + y^2x_2^{-2} & \dots & x_2^{n-1} + y^{n-1}x_2^{1-n} \\ & \dots & \dots & \dots & \dots \\ 1 & x_n + yx_n^{-1} & x_n^2 + y^2x_n^{-2} & \dots & x_n^{n-1} + y^{n-1}x_n^{1-n} \end{vmatrix}.$$

This new determinant is proportional to $M(\mathbb{X},y)$, and therefore $M(\mathbb{X},y)$ can already be considered as an answer to the question of Muir, except for the parasitic factor $(-1)^{\binom{n}{2}}\Delta(\mathbb{X})$. A similar manipulation shows that $N(\mathbb{X},y)$ is equal to $(-1)^{\binom{n+1}{2}}\Delta(\mathbb{X})\prod_{i\le j}(y - x_ix_j)$.

2 Littlewood's Products

Littlewood has given the expansion, in terms of Schur functions, of the two products $\prod_{1\le i<j\le n}(1-x_ix_j)$ and $\prod_{1\le i\le j\le n}(1-x_ix_j)$, which appear in the Weyl character formula for the types B and C.

Littlewood's formulas [4, Ex. 9 p.78] read as follows:

$$\prod_{1\le i<j\le n}(1-x_ix_j)=\sum_{\lambda}(-1)^{|\lambda|/2}s_\lambda(\mathbb{X}), \tag{5}$$

summed over all partitions $\lambda:\ F(\lambda)=\{\alpha_1-1,\dots,\alpha_r-1|\alpha_1,\dots,\alpha_r\}$ with $\alpha_1\le n-1$;

$$\prod_{1\le i\le j\le n}(1-x_ix_j)=\sum_{\lambda}(-1)^{|\lambda|/2}s_\lambda(\mathbb{X}), \tag{6}$$

summed over all partitions $\lambda:\ F(\lambda)=\{\alpha_1+1,\dots,\alpha_r+1|\alpha_1,\dots,\alpha_r\}$ with $\alpha_1\le n-1$.

In this section, we obtain new determinantal expressions for these two products, and show that they imply Littlewood's sums. For considerations about generalizations of Littlewood's formulas, see [2].

Let $\delta_{i,j}$ denote the usual Kronecker delta function ($=1$ if $i=j$ and 0 otherwise).

The following easy lemma is a key ingredient.

Lemma 2.1 *We have*

$$SY_{(n-1)^n} = \det\big(s_{j,1^i}+(-y)^i\delta_{i,j}\big)_{1\le i,j\le n-1}, \tag{7}$$

$$OY_{(n+1)^n} = \det\big(s_{j+1,1^{i-1}}+(-y)^i\delta_{i,j}\big)_{1\le i,j\le n}. \tag{8}$$

Proof. Multiplying to the left of the matrix corresponding to the determinant $SY_{(n-1)^n}$ by the triangular matrix $\big[(-1)^{j-i}e_{j-i}\big]_{1\le i,j\le n}$ with determinant 1, we get a matrix with first column $[0,\dots,0,1]$, the cofactor of its non-zero entry being the determinant

$$\det\left(\sum_{k=1}^{j-1}(-1)^{n-i+k+1}e_{n-i+k}h_{j-1-k}+y^{j-1}\delta_{i,n-j+1}\right)_{1\le i,j-1\le n-1}.$$

Reordering rows and columns, using (1), we obtain (7).

Similarly, the determinant $OY_{(n+1)^n}$ equals

$$\det\left(\sum_{k=0}^{j}(-1)^k e_{i+k}h_{j-k}+(-y)^i\delta_{i,j}\right)_{1\le i,j\le n}=\det\left(s_{j+1,1^{i-1}}+(-y)^i\delta_{i,j}\right)_{1\le i,j\le n}.$$

∎

Lemma 2.2 *We have*

$$SY_{(n-1)^n} = \sum_{\mu} (-y)^{(n^2-n-|\mu|)/2} \, s_\mu, \tag{9}$$

sum over all μ *such that* $F(\mu) = \{\alpha_1 - 1, \ldots, \alpha_r - 1 | \alpha_1, \ldots, \alpha_r\}$ *with* $\alpha_1 \leq n-1$*;*

$$OY_{(n+1)^n} = \sum_{\mu} (-y)^{(n^2+n-|\mu|)/2} \, s_\mu, \tag{10}$$

sum over all μ *such that* $F(\mu) = \{\alpha_1 + 1, \ldots, \alpha_r + 1 | \alpha_1, \ldots, \alpha_r\}$*, where* $\alpha_1 \leq n-1$*.*

Proof. Expanding the determinants (7) and (8) according to all possible choices of the occurrences of powers of y, we get, as cofactors, determinants of hook Schur functions, in which we recognize, thanks to formula (2), the two expressions in the lemma. ∎

By multilinearity of the determinants (3) and (4), we obtain the following sums of 2^{n-1} (resp. 2^n) Schur functions (notice that the indexing of a Schur function can be any integral vector, and not only a partition).

Lemma 2.3 *We have*

$$SY_{(n-1)^n} = \sum_{\epsilon} y^{k(\epsilon)} \, s_{0,2\epsilon_2,\ldots,(2n-2)\epsilon_n} \,, \tag{11}$$

$$OY_{(n+1)^n} = \sum_{\epsilon} (-y)^{k(\epsilon)} \, s_{2\epsilon_1,\ldots,2n\epsilon_n} \tag{12}$$

sum over all ϵ_i *in* $\{0,1\}$*, the power* $k(\epsilon)$ *being such that the total degree be* $n(n-1)$ *in the first formula,* $n(n+1)$ *in the second,* y *being of degree* 2*.*

The following lemma will allow us to extract the factor $\prod_{i<j}(y - x_i x_j)$ or $\prod_{i\leq j}(y - x_i x_j)$ from the determinants that we are considering.

Lemma 2.4 *Let* f*,* g *be functions of one variable. We have*

$$\det\left(f(x_i)^{n-j} g(x_i)^{j-1}\right)_{1\leq i,j\leq n} = \prod_{i=1}^{n} f(x_i)^{n-1} \, \Delta\left(\frac{g(x_1)}{f(x_1)}, \ldots, \frac{g(x_n)}{f(x_n)}\right). \tag{13}$$

Proof. This is the Vandermonde determinant in the variables $g(x_i)/f(x_i)$, $1 \leq i \leq n$. ∎

Combining Lemmas 2.1, 2.2, 2.4, we obtain the following two theorems.

Theorem 2.5 *We have*

$$\prod_{1\leq i<j\leq n} (y - x_i x_j) = (-1)^{\binom{n}{2}} \det\left(s_{j,1^i} + (-y)^i \delta_{i,j}\right)_{1\leq i,j\leq n-1}$$

$$= y^{\binom{n}{2}} \sum_{\lambda} (-y)^{-|\lambda|/2} s_\lambda \tag{14}$$

summed over all partitions $F(\lambda) = \{\alpha_1 - 1, \ldots, \alpha_r - 1 | \alpha_1, \ldots, \alpha_r\}$ *with* $\alpha_1 \leq n-1$*.*

Theorem 2.6 *We have*

$$\prod_{1\le i\le j\le n} (y - x_i x_j) = (-1)^{\binom{n+1}{2}} \det\big(s_{j+1,1^{i-1}} + (-y)^i \delta_{i,j}\big)_{1\le i,j\le n}$$
$$= y^{\binom{n+1}{2}} \sum_{\lambda} (-y)^{-|\lambda|/2} s_\lambda \quad (15)$$

summed over all partitions $F(\lambda) = \{\alpha_1+1,\ldots,\alpha_r+1|\alpha_1,\ldots,\alpha_r\}$ *with* $\alpha_1 \le n-1$.

Proof. As already used at the end of the introduction, take $f(x) = x$, $g(x) = y + x^2$ in (13). We have

$$g(x_j)/f(x_j) - g(x_i)/f(x_i) = \frac{x_i - x_j}{x_i x_j}(y - x_i x_j).$$

Lemma 2.4 specializes into

$$\det\left(x_i^{n-j}(y + x_i^2)^{j-1}\right)_{1\le i,j\le n} = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i<j} (y - x_i x_j) \ .$$

This leads to the following identity:

$$\Delta(\mathbb{X}) SY_{(n-1)^n} = M(\mathbb{X}, y) = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i<j} (y - x_i x_j).$$

Multiplying both sides of above equation by $\prod_i (y - x_i^2)$, one gets

$$(-1)^n \Delta(\mathbb{X}) OY_{(n+1)^n} = (-1)^n N(\mathbb{X}, y) = (-1)^{\binom{n}{2}} \Delta(\mathbb{X}) \prod_{i\le j} (y - x_i x_j).$$

By Lemma 2.1 and Lemma 2.2, we complete the proof. ∎

Setting $y = 1$ in (14) and (15), we get Littlewood's formulas (5) and (6).

Let us now show that, having recourse to (3) and (4), we can obtain a determinantal expression for the product of the functions in the preceding two theorems by an arbitrary Schur function.

Theorem 2.7 *Let* n *be an integer,* λ *be a partition of length* $\le n$*. Then*

$$SY_{(n-1)^n+\lambda} = (-1)^{\binom{n}{2}} s_\lambda \prod_{1\le i<j\le n} (y - x_i x_j)$$

and

$$OY_{(n+1)^n+\lambda} = (-1)^{\binom{n+1}{2}} s_\lambda \prod_{1\le i\le j\le n} (y - x_i x_j)$$

Proof. The function $SY_{(n-1)^n+\lambda}$ is a weighted sum of skew Schur functions $s_{\clubsuit/\mu}$ with $\clubsuit = (n-1)^n + \lambda$ and $F(\mu) = (\alpha_1 - 1, \ldots, \alpha_r - 1 | \alpha_1, \ldots, \alpha_r)$ with $\alpha_1 \leq n-1$. Interpreting a Schur function in terms of Young tableaux, we remark that the tableaux of shape $\clubsuit/\mu$ in n letters are the concatenation of an arbitrary skew tableau of shape $(n-1)^n/\mu$ and an arbitrary tableau of shape λ. Hence we have the factor s_λ.

For example, for $n = 4$, and a tableau of shape $([3,3,3,3] + [2,2,1,1])/[2,1,1]$, one has the decomposition

$$\begin{array}{|c|c|c|c|c|} a & b & c & \alpha & \\ & d & e & \beta & \\ & f & g & \gamma & \delta \\ & & h & \epsilon & \iota \end{array} \Leftrightarrow \left(\begin{array}{ccc} a & b & c \\ & d & e \\ & f & g \\ & & h \end{array} \quad \& \quad \begin{array}{cc} \alpha & \\ \beta & \\ \gamma & \delta \\ \epsilon & \iota \end{array} \right).$$

The remaining factor corresponds to the case $\lambda = 0$ which has been described in Theorem 2.5. This proves the first assertion of the theorem. The second part can be proved in the same manner with a summation over partitions contained in $(n+1)^n$. ∎

For example, for $n = 3$, we have

$$SY_{332} = \begin{vmatrix} h_3 & h_4 + yh_2 & h_5 + y^2 h_1 \\ h_2 & h_3 + yh_1 & h_4 + y^2 \\ 1 & h_1 & h_2 \end{vmatrix} = s_{11}\, SY_{222}$$
$$= -(x_3x_1 + x_3x_2 + x_2x_1)(y - x_2x_1)(y - x_3x_1)(y - x_3x_2).$$

Notice that the theorem implies, for $y = 1$, a factorization property of symplectic and orthogonal Schur functions in n variables, indexed by partitions with parts $\geq n-1$ or $\geq n+1$ respectively. For more informations about symplectic and orthogonal characters, we refer to [1], and for relevant tableau considerations, to [7].

3 Remarks

Given a polynomial of degree 3, with roots x_1, x_2, x_3, the polynomial $(y - x_1^2)(y - x_2^2)(y - x_3^2)$ can be interpreted as the resultant of $(x^2 - y)$ and $(x - x_1)(x - x_2)(x - x_3)$. Using the determinantal expression of the resultant ascribed to Sylvester, one can therefore write

$$-(y - x_1^2)(y - x_2^2)(y - x_3^2) = \begin{vmatrix} 1 & 0 & -y & 0 & 0 \\ 0 & 1 & 0 & -y & 0 \\ 0 & 0 & 1 & 0 & -y \\ 1 & -e_1 & e_2 & -e_3 & 0 \\ 0 & 1 & -e_1 & e_2 & -e_3 \end{vmatrix}.$$

Farkas (1881) [5, p. 335] found that one could derive similar determinants for expressing the polynomial whose roots are $x_i x_j : 1 \leq i < j \leq 3$, or $x_i x_j : 1 \leq$

$i \leq j \leq 3$. Returning to this problem in 1927, Muir [6] was still puzzled by these determinants, finding no other proof than direct verification for degree 3. Farkas determinants are

$$-(y-x_1x_2)(y-x_1x_3)(y-x_2x_3) = \begin{vmatrix} 1 & 0 & 0 & 0 & y^2 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & e_1 & e_2 & e_3 & 0 \\ 0 & 1 & e_1 & e_2 & e_3 \end{vmatrix}$$

and

$$\prod_{1\leq i\leq j\leq 3}(y-x_ix_j) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -y^3 \\ 0 & 1 & 0 & 0 & 0 & -y^2 & 0 \\ 0 & 0 & 1 & 0 & -y & 0 & 0 \\ 1 & e_1 & e_2 & e_3 & 0 & 0 & 0 \\ 0 & 1 & e_1 & e_2 & e_3 & 0 & 0 \\ 0 & 0 & 1 & e_1 & e_2 & e_3 & 0 \\ 0 & 0 & 0 & 1 & e_1 & e_2 & e_3 \end{vmatrix}.$$

However, if we expand the following determinant of order $2n-1$ along the first n rows,

$$\begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & y^{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \iddots & 0 \\ 0 & & 1 & 0 & y & & 0 & \vdots \\ 0 & & 0 & 1 & 0 & & 0 & 0 \\ 1 & \cdots & e_{n-2} & e_{n-1} & e_n & \cdots & 0 & 0 \\ 0 & \cdots & e_{n-3} & e_{n-2} & e_{n-1} & \ddots & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & \cdots & 1 & e_1 & e_2 & \cdots & e_{n-1} & e_n \end{vmatrix},$$

the cofactors are Schur functions, up to sign.

More precisely, we have 2^{n-1} choices of non vanishing $n \times n$ minors in the first n rows, which are equal to some power of y, up to sign. Writing the bottom part

$$\begin{vmatrix} e_0 & e_1 & e_2 & \cdots & e_{2n-4} & e_{2n-3} & e_{2n-2} \\ 0 & e_0 & e_1 & \cdots & e_{2n-5} & e_{2n-4} & e_{2n-3} \\ \vdots & & \ddots & \ddots & & & \ddots \end{vmatrix},$$

one recognizes that the cofactors are Schur functions (expressed in the e_i) of indices $[2\epsilon_1, 4\epsilon_2, \ldots, (2n-2)\epsilon_{n-1}]$, with $\epsilon_i \in \{0,1\}$. These are the indices occurring in Lemma 2.3, taking into account that the involution exchanging elementary and symmetric functions conjugate partitions.

Therefore, the above determinant equals $(-1)^{\binom{n}{2}} \prod_{1\leq i<j\leq n}(y-x_ix_j)$.

Similarly, we obtain

$$(-1)^{\binom{n+1}{2}} \prod_{1\le i\le j\le n} (y - x_i x_j) = \begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & -y^{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \iddots & 0 \\ 0 & & 1 & 0 & -y & & 0 & \vdots \\ 1 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & e_{n-2} & e_{n-1} & e_n & \ddots & 0 & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & e_1 & \cdots & e_{n-1} & e_n \end{vmatrix}.$$

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