Infinite paths in planar graphs IV, dividing cycles

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Abstract

Nash-Williams conjectured that a 4-connected infinite planar graph contains a spanning 2-way infinite path if, and only if, the deletion of any finite set of vertices results in at most two infinite components. In this paper, we prove the Nash-Williams conjecture for graphs with no dividing cycles and for graphs with infinitely many vertex disjoint dividing cycles. A cycle in an infinite plane graph is called *dividing* if both regions of the plane bounded by this cycle contain infinitely many vertices of the graph.

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1 Introduction

We use the terminology in [8], [9], and [10]. For convenience we repeat some here. Let H be a (finite or infinite) subgraph of a (finite or infinite) graph G, let $v_1, \ldots, v_k \in V(G)$, and $\{u_i, w_i\} \subseteq V(H) \cup \{v_1, \ldots, v_k\}$, $i = 1, \ldots, m$. Then $H + \{v_1, \ldots, v_k, u_1w_1, \ldots, u_mw_m\}$ denotes the graph with vertex set $V(H) \cup \{v_1, \ldots, v_k\}$ and edge set $E(H) \cup \{u_1w_1, \ldots, u_kw_k\}$. For any $x \in V(H) \cup E(H)$, we write $H + x$ instead of $H + \{x\}$.

Let C be a cycle in a plane graph G and let $x, y \in V(C)$. When $x \neq y$ then xCy denotes the subpath of C from x to y in clockwise order, and when $x = y$ then xCy denotes the trivial path consisting of $x = y$ only. For a (finite or infinite) path P and $x, y \in V(P)$, we use xPy to denote the unique finite path in P between x and y.

By the Jordan curve theorem, each cycle C in a (finite or infinite) plane graph G divides the plane into two closed regions whose intersection is C. If G is infinite and exactly one of these two closed regions, say $\mathcal{D}$, contains a finite subgraph of G, then we use $I_G(C)$ to denote the subgraph of G contained in $\mathcal{D}$. If there is no danger of confusion, we use $I(C)$ instead of $I_G(C)$. Note that $C \subseteq I(C)$, and if $I(C) = C$ then C is a facial cycle.

A graph G is *k-indivisible*, where k is a positive integer, if, for every finite $X \subseteq V(G)$, $G - X$ has at most $k-1$ infinite components. Nash-Williams ([2], [3], and [7]) conjectured that a 4-connected infinite planar graph contains a spanning 2-way infinite path if, and only if, G is 3-indivisible.

In [8] and [9], the Nash-Williams conjecture is established for 2-indivisible graphs. To deal with those graphs which are 3-indivisible but not 2-indivisible, we define *dividing* cycles in an infinite plane graph G as those cycles C for which $I_G(C)$ is not defined. A *non-dividing* cycle in G is then a cycle which is not dividing. Let $\gamma(G)$ denote the maximum number of vertex disjoint dividing cycles in an infinite plane graph G. With this notation, we may divide 3-indivisible infinite plane graphs G into three classes: those with $\gamma(G) = 0$ (including all 2-indivisible graphs), those with $\gamma(G) = \infty$, and those for which $\gamma(G)$ is a positive integer. (Note that, when $\gamma(G) = 0$, the drawing of G may be modified to give a VAP-free drawing of G; see [5] and [1].) The objective of this paper is to give a proof of the following result, which establishes the Nash-Williams conjecture for two of these three classes.

(1.1) Theorem. *Let G be a 4-connected 3-indivisible infinite plane graph, and assume that $\gamma(G) = 0$ or $\gamma(G) = \infty$. Then G contains a spanning 2-way infinite path.*

Throughout the rest of the paper, graphs will be finite unless it is clear from the context or otherwise mentioned. In Section 2 we summarize those concepts and results from [8], [9] and [10] which will be used in this paper. We prove in Section 3 three lemmas concerning 2-way infinite Tutte paths in two special classes of graphs. These lemmas will serve as bases for inductive arguments. Section 4 includes results which show that certain finite sequences of non-dividing cycles guarantee the existence of a 2-way infinite Tutte path. Theorem (1.1) will be proved in Section 5 for graphs with $\gamma(G) = 0$. The proof of Theorem (1.1) will then be completed in Section 6.

2 Nets and Tutte paths

A *net* in an infinite plane graph G is a sequence $N := (C_1, C_2, \ldots)$ of cycles in G such that $I(C_i)$ is defined for all $i \geq 1$, and the following properties are satisfied:

(1) $I(C_i) \subseteq I(C_{i+1})$ for all $i \geq 1$,

(2) $\bigcup_{i=1}^{\infty} I(C_i) = G$, and

(3) either $C_i \cap C_j = \emptyset$ for all $i \neq j$, or for $i \geq 1$, $C_i \cap C_{i+1}$ is a non-trivial path, $C_i \cap C_{i+1} \subseteq C_{i+1} \cap C_{i+2}$, and neither endvertex of $C_i \cap C_{i+1}$ is an endvertex of $C_{i+1} \cap C_{i+2}$.

If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then N is called a *radial net*; otherwise, N is a *ladder net*. Let $\partial N = \emptyset$ if N is a radial net; otherwise, let $\partial N = \bigcup_{i=1}^{\infty}(C_i \cap C_{i+1})$.

Let G be a (finite or infinite) graph and H be a (finite or infinite) subgraph of G. An H-*bridge* of G is a (finite or infinite) subgraph of G which is induced by either (1) an edge of $E(G) - E(H)$ whose incident vertices are in $V(H)$ or (2) the edges contained in a component of $G - V(H)$ and the edges from that component to H. Also, we say that G is $(4, H)$-*connected* if, for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains a vertex of H. The following result is Theorem (2.1) in [10] (its 4-connected version is shown in [9]), which gives a structural description of graphs with nets.

(2.1) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph with a facial cycle C such that G is $(4, C)$-connected, and let S denote the set of vertices of G of infinite degree. Then $|S| \leq 2$, and there is a set F of edges of G such that*

(1) for any $f \in F$, f is incident with a vertex in S,

(2) $G - F$ has a net $N = (C_1, C_2, \ldots)$, $C \subseteq I(C_1)$, $S \subseteq \partial N$, and for any $f \in F$ both incident vertices of f are contained in a common infinite S-bridge of ∂N,

(3) if $|S| = 1$, then either one S-bridge of ∂N contains all vertices incident with edges in F or each S-bridge of ∂N contains infinitely many vertices incident with edges in F, and

(4) *if* $|S| = 2$, *then for any* $T \subseteq V(G) - S$ *with* $|T| \leq 3$, S *is contained in a component of* $(G - F) - T$.

For an infinite plane graph G, let ∂G denote the subgraph of G such that for each $x \in V(G) \cup E(G)$, $x \in \partial G$ if and only if $x \notin (E(I(D)) - E(D)) \cup (V(I(D)) - V(D))$ for every cycle D in G. Clearly, $\partial G = \emptyset$ when G admits a radial net. From Theorem (2.1), we can show that when G does not admit a radial net then ∂G is a path, or a 1-way infinite path, or a 2-way infinite path. The following observation will be useful.

(2.2) Lemma. *Let* G *be a 2-connected infinite plane graph and* C *be a facial cycle of* G *such that* G *is* $(4, C)$*-connected. If* G *is 2-indivisible, then all but one face of* G *are bounded by cycles, and* ∂G *is precisely the subgraph of* G *lies on the boundary of the exceptional face of* G.

Proof. Suppose G is 2-indivisible and let R be a face of G. Suppose R is incident with a vertex or edge which is not in ∂G. Then there exists some cycle D in G such that R is incident with some element of $(E(I(D)) - E(D)) \cup (V(I(D)) - V(D))$. This shows that R is a face of $I(D)$. Since $I(D)$ is a 2-connected plane graph, R is bounded by a cycle. Now assume that all vertices or edges of G incident with R are in ∂G. Then since G is 2-indivisible, it follows from Theorem (2.1) that ∂G is precisely the subgraph of G that lies on the boundary of R. □

The next result is a generalization of Lemma (2.3) in [9].

(2.3) Lemma. *Let* G *be a 2-connected infinite plane graph and let* C *be a facial cycle of* G *such that* G *is* $(4, C)$*-connected and* $\gamma(G) = 0$. *Then there is an infinite sequence* $(D_1, D_2, \ldots)$ *of cycles in* G *such that* $C \subseteq I(D_1)$ *and the following properties hold:*

(1) *for each* $i \geq 1$, $I(D_i) \subseteq I(D_{i+1})$, *and* $D_i \cap D_{i+1}$ *is minimal among all subgraphs* $D_i \cap D^*$ *arising from cycles* D^* *in* G *such that* $I(D_i) \subseteq I(D^*)$,

(2) *for each* $i \geq 1$, G *has no finite* $I(D_i)$*-bridge,*

(3) *for each* $i \geq 1$, $D_i \cap D_{i+1} \subseteq D_{i+1} \cap D_{i+2}$, *and*

(4) $\bigcup_{i \geq 1} I(D_i) = G$.

The proof of Lemma (2.3) in [9] uses two properties: (a) for any finite $X \subseteq V(G)$, $G - X$ has only finitely many components, and (b) every cycle in G is non-dividing (implied by cohesiveness). In the above lemma, (a) is guaranteed by the assumption that G is planar and $(4, C)$-connected, and (b) is guaranteed by the assumption that $\gamma(G) = 0$.

In the remainder of this section, we state several results concerning Tutte paths in finite or infinite plane graphs. Let G be a (finite or infinite) graph and H be a (finite or infinite) subgraph of G. If B is an H-bridge of G, then the vertices in $V(H \cap B)$ are called *attachments* of B (on H). The subgraph H is a *Tutte subgraph* of G if every H-bridge of G is finite and has at most three attachments. For a (finite or infinite) subgraph C of G, we say that H is a *C-Tutte subgraph* of G if H is a Tutte subgraph of G and every H-bridge of G containing an edge of C has at most two attachments. A (finite or infinite) *Tutte path* is a (finite or infinite) path which is a Tutte subgraph.

The following result is the main theorem in [7].

(2.4) Lemma. *Let G be a 2-connected plane graph with a facial cycle C. Assume that $x \in V(C)$, $e \in E(C)$, and $y \in V(G - x)$. Then G contains a C-Tutte path P from x to y such that $e \in E(P)$.*

The next result is (2.6) from [4].

(2.5) Lemma. *Let G be a 2-connected plane graph with a facial cycle C. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in clockwise order. Then G contains a vCu-Tutte path P from u to v such that $\{e, f\} \subseteq E(P)$.*

We remark here that both Lemma (2.4) and Lemma (2.5) may be applied when e or f or both are vertices. We need Lemma (3.3) from [10], which will be convenient for extending Tutte paths.

(2.6) Lemma. *Let K be a connected (finite or infinite) plane graph, C be a facial walk of K, Q be a path between p and q on C, $u \in V(C) - V(Q)$, L be a subgraph of $K - V(Q)$, and Q' be a cycle in L or a path in L or a 2-way infinite path in L. Suppose the following three conditions are satisfied:*

(1) for any $(L \cup Q)$-bridge B of K, $|V(B \cap L)| \leq 1$ and $V(B \cap L) \subseteq V(Q')$,

(2) $K - V(L)$ is finite and all vertices of $K - V(L)$ have finite degree in K, and

(3) L contains a Q'-Tutte subgraph T with $u \in V(T)$ and $|V(Q') \cap V(T)| \geq 2$.

Then $K - V(T)$ contains a path S between p and q such that $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' is also an $(S \cup T)$-bridge of K.

The following result is Corollary (3.7) in [9].

(2.7) Lemma. *Let G be a 2-connected infinite plane graph with a ladder net N, and let $x \in V(\partial N)$ and $uv \in E(\partial N)$ such that $u \in V(x\partial Nv)$. Then G contains a 1-way infinite ∂N-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.*

We also need Theorem (1.2) from [10].

(2.8) Theorem. *Let G be a 2-connected 2-indivisible infinite plane graph, let C be a facial cycle of G, let $x \in V(C)$ and $uv \in E(C)$ with $x \neq v$, and let Q denote the subpath of $C - v$ between u and x. Assume that G is $(4, C)$-connected and v is contained in the infinite component of $G - V(Q)$. Then G contains a 1-way infinite C-Tutte path P from x such that $uv \in E(P)$ and $u \in V(xPv)$.*

3 Two-way infinite Tutte paths

The goal of this section is to prove three results on 2-way infinite Tutte paths. These results will be used as bases for inductive arguments.

(3.1) Lemma. *Let G be a 2-connected 3-indivisible infinite plane graph, let C be a facial cycle of G, and let $u, v \in V(C)$ be distinct such that G is $(4, C)$-connected and $G - \{u, v\}$ has two infinite components. Then for any $e \in E(C)$, G contains a 2-way infinite C-Tutte path through e.*

Proof. Without loss of generality, we may assume that the face of G bounded by C is an open disc. Since G is $(4, C)$-connected, G has at most three $\{u, v\}$-bridges: G_1 containing vCu, G_2 containing uCv, and possibly a third $\{u, v\}$-bridge induced by uv (when $uv \in E(G)$). Since $G - \{u, v\}$ has two infinite components, G_1 and G_2 are infinite. For each $i \in \{1, 2\}$, let B_i be the infinite block of G_i. Since G is 2-connected, $B_1 \cap vCu$ and $B_2 \cap uCv$ are nontrivial paths. Let $u' \in V(B_2 \cap uCv)$ with uCu' minimal, and let $v' \in V(B_1 \cap vCu)$ with vCv' minimal. See Figure 1, where the possible edge uv is not drawn. Let $G' := G$ if $uv \notin E(G)$; otherwise, let $G' := G - uv$. Then G' has exactly two infinite $\{u', v'\}$-bridges, one containing B_1 and the other containing B_2.

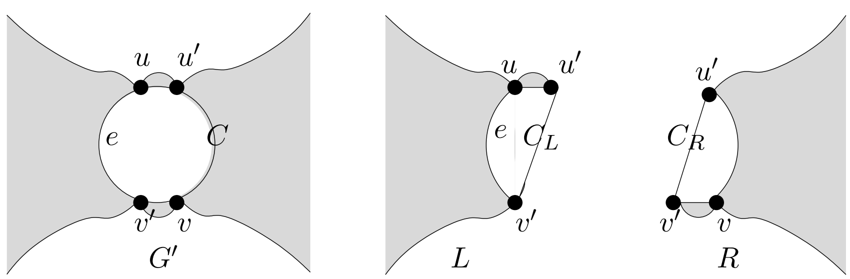


Figure 1: G, L and R

Since G is $(4, C)$-connected, neither $u'Cv$ nor $v'Cu$ is an edge; for otherwise, $G - \{u', v\}$ or $G - \{u, v'\}$ has a component containing no vertex of C, a contradiction. Let L

be obtained from G' by replacing the $\{u',v'\}$-bridge of G' containing B_2 with the edge $u'v'$, and let R be obtained from G' by replacing the $\{u',v'\}$-bridge of G' containing B_1 with the edge $v'u'$. Let $C_L := v'Cu' + u'v'$ and $C_R := u'Cv' + v'u'$. We may assume that the edges are added so that the faces of L and R bounded by C_L and C_R, respectively, are open discs. See Figure 1. Because G is 3-indivisible, both L and R are 2-indivisible. Since G is $(4,C)$-connected and since $V(C_L) \cup V(C_R) = V(C)$, L must be $(4,C_L)$-connected and R must be $(4,C_R)$-connected.

By symmetry, we may assume that $e \in E(v'Cu')$, and let $e = ab$ so that v', b, a, u' occur on C_L in clockwise order. Since $v' \in B_1$, we see that b is in the infinite component of $L - V(aC_Lu')$. Hence, by Theorem (2.8), there is a 1-way infinite C_L-Tutte path P_L from u' in L such that $e \in E(P_L)$ and $a \in V(u'P_Lb)$. By planarity, $u'v' \notin E(P_L)$ and, therefore, $u \in V(P_L)$. We claim that $v' \in V(P_L)$. For otherwise v' is contained in a P_L-bridge B of L. Clearly, $u' \in V(B \cap P_L)$. Since P_L is a C_L-Tutte path of L, $|V(B \cap P_L)| = 2$ and B is finite. Let $v'' \in V(B \cap P_L) - \{u'\}$. Then v'' lies on C and $v'Cv'' - v'' \not\subseteq B_1$, contradicting the choice of v'.

If $v = v'$ or $uv \notin E(G)$ then we use Theorem (2.8) to find a 1-way infinite C_R-Tutte path P_R in R from v' and through $v'u'$. It is easy to see that $P := P_L \cup (P_R - v')$ is a 2-way infinite C-Tutte path in G such that $e \in E(P)$.

Now assume $v \neq v'$ and $uv \in E(G)$. Suppose $u = u'$. In R we use Theorem (2.8) to find a 1-way infinite C_R-Tutte path P_R from u' and through $u'v'$. Then $v \in V(P_R)$. Let P_u denote the infinite u-bridge of P_L and P_v denote the infinite v-bridge of P_R. Clearly $e \in E(P_u)$. It is easy to verify that $P := (P_u \cup P_v) + uv$ gives the desired 2-way infinite C-Tutte path in G.

Hence we may assume $u \neq u'$. Suppose $e \in E(vCu)$. In R, we use Theorem (2.8) to find a 1-way infinite C_R-Tutte path P_R from v' and through the edge of $u'Cv'$ incident with v'. Then $v'u' \notin E(P_R)$ and $v \in V(P_R)$. By a similar argument as above for showing $v' \in V(P_L)$, we can show that $u' \in V(P_R)$. Let P_u denote the infinite u-bridge of P_L and P_v denote the infinite v-bridge of P_R. Since $e \in E(vCu)$, $e \in E(P_u)$. It is easy to verify that $P := (P_u \cup P_v) + uv$ gives the desired 2-way infinite C-Tutte path in G.

To deal with the remaining case when $e \in E(uCu')$, we view $G_i + uv$ (for each $1 \leq i \leq 2$) as a plane graph with a facial cycle C_i, where $C_1 = vCu + uv$ and $C_2 = uCv + uv$. In $G_1 + uv$ we apply Theorem (2.8) to find a 1-way infinite C_1-Tutte path P_1 from u through uv. In $G_2 + uv$ we apply Theorem (2.8) to find a 1-way infinite C_2-Tutte path P_2 from v through e. Because $\{v, u'\}$ is a 2-cut in G_2, we see that $vu \in E(P_2)$. Hence, $P := P_1 \cup P_2$ gives the desired 2-way infinite C-Tutte path in G. □

For the next two lemmas, we need additional notation. Let G be a 2-connected 3-indivisible infinite plane graph, and let C be a facial cycle of G such that G is $(4,C)$-connected. Let H be an infinite block of $G - V(C)$ and let D be the cycle of H which bounds the face of H containing C. See Figure 2. Let $w_1, \ldots, w_b$ denote the attachments

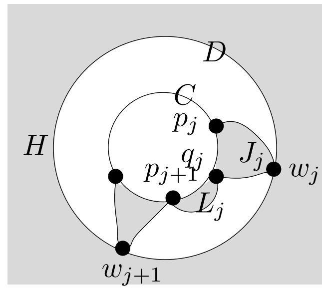


Figure 2: Illustration for Lemma (3.2)

on H of $(H \cup C)$-bridges of G which occur on D in clockwise order. Let $p_j, q_j \in V(C)$ with p_jCq_j maximal such that $\{p_j, w_j\}$ is contained in an $(H \cup C)$-bridge of G, $\{q_j, w_j\}$ is contained in an $(H \cup C)$-bridge of G, and any $(H \cup C)$-bridge of G containing some $w_l \neq w_j$ contains no vertex of $V(p_jCq_j) - \{p_j, q_j\}$. Note that p_j and q_j are well defined because G is $(4, C)$-connected. Let J_j denote the union of p_jCq_j and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(p_jCq_j) \cup \{w_j\}$. (Note that if $p_j = q_j$ then J_j is induced by a single edge.) Let L_j denote the union of q_jCp_{j+1} and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(q_jCp_{j+1})$, where $p_{b+1} = p_1$.

(3.2) Lemma. *Let G, C, H, D and w_j, J_j, L_j $(1 \leq j \leq b)$ be defined as above. Let $e \in E(C)$. Suppose there is some $1 \leq j \leq b$ such that L_j is infinite and $e \notin E(L_j)$. Then G contains a 2-way infinite C-Tutte path through e.*

Proof. Without loss of generality, we may assume that $j = 1$. Since $e \in E(p_2Cq_1)$, $e \in E(p_rCp_{r+1})$ for some $w_r \neq w_1$ or $e \in E(q_{r-1}Cq_r)$ for some $w_r \neq w_2$. Note the symmetry between clockwise and counter clockwise orientations of C, and also note the symmetry between w_1 and w_2. We may therefore assume that $e \in E(p_rCp_{r+1})$ for some $w_r \neq w_1$.

Since G is 3-indivisible and L_1 is infinite, H is 2-indivisible. Since G is $(4, C)$-connected and by planarity, H is $(4, D)$-connected. Hence by Theorem (2.8), H contains a 1-way infinite D-Tutte path P from w_1 and through w_r.

Since G is 2-connected and L_1 is infinite, $L_1 - q_1$ contains a 1-way infinite path from p_2. Let $L_1' := L_1 + q_1p_2$ such that $C_1' := q_1Cp_2 + q_1p_2$ is a facial cycle of L_1'. By Theorem (2.8), L_1' has a 1-way infinite C_1'-Tutte path Q_1 from q_1 such that $q_1p_2 \in E(Q_1)$.

In $J_1 + p_1w_1$, we apply Lemma (2.4) to find a p_1Cq_1-Tutte path P_1 from w_1 to p_1 and through q_1.

We apply Lemma (2.6) to $K := G - V((J_1 \cup L_1) - \{p_1, p_2, w_1\})$, $H, p_2Cp_1, D, P, p_2, p_1, w_1$ (as K, L, Q, Q', T, p, q, u, respectively). Note that the conditions of Lemma (2.6) are satisfied. In particular, $w_1, w_r \in V(P)$ implies that $|V(P \cap D)| \geq 2$. Hence, by Lemma (2.6), there is a path S in $K - V(P)$ between p_2 and p_1 such that $S \cup P$ is a p_2Cp_1-Tutte subgraph in K and every P-bridge of H containing no edge of D is also an $(S \cup P)$-bridge of K.

We may assume that $e \in E(S)$. This may be seen as follows. By planarity and because $w_r \in V(P)$, p_r and p_{r+1} are cut vertices of $K - V(P)$. Hence, $p_r, p_{r+1} \in V(S)$, $(J_r \cup L_r) - w_r$ is a $\{p_r, p_{r+1}\}$-bridge of $K - V(P)$, and $p_rSp_{r+1} \subseteq (J_r \cup L_r) - w_r$. In $(J_r \cup L_r) + p_{r+1}w_r$, we apply Lemma (2.5) to find a p_rCp_{r+1}-Tutte path S_r from p_r to w_r such that $p_{r+1}w_r, e \in E(S_r)$. By replacing the subpath p_rSp_{r+1} of S with $S_r - w_r$, we obtain the desired path S through e.

Now $P^* := P \cup P_1 \cup S \cup (Q_1 - q_1)$ is a 2-way infinite path through e. Note that every P^*-bridge of G is one of the following: an $(S \cup P)$-bridge of H, or a P_1-bridge of J_1, or a Q_1-bridge of L_1'. Hence, P^* is a 2-way infinite C-Tutte path in G through e. □

(3.3) Lemma. *Let G, C, H, D and w_j, J_j, L_j $(1 \leq j \leq b)$ be defined as above. Let $e \in E(C)$. Suppose there is some $1 \leq j \leq b$ such that J_j is infinite and e is not contained in the unique infinite block of $J_j - w_j$. Then G contains a 2-way infinite C-Tutte path through e.*

Proof. Without loss of generality, we may assume that $j = 1$. If $e \in E(J_1)$, we choose an arbitrary $w_r \neq w_1$. Since e is not contained in the infinite block of $J_j - w_j$, there is a vertex $v \in V(p_1Cq_1 - \{p_1, q_1\})$ such that the infinite $\{w_1, v\}$-bridge of J_1 does not contain e_1. In this case, let J_1^* denote the infinite $\{w_1, v\}$-bridge of J_1. Now assume $e \notin E(J_1)$. Then $e \in E(q_1Cp_1)$, and hence $e \in E(p_rCp_{r+1})$ for some $w_r \neq w_1$ or $e \in E(q_{r-1}Cq_r)$ for some $w_r \neq w_1$. Let $J_1^* = J_1$ and $v := p_1$.

Note the symmetry between clockwise and counter clockwise orientations of C. We may therefore assume that when $e \notin E(J_1)$ we have $e \in E(p_rCp_{r+1})$ for some $w_r \neq w_1$, and when $e \in E(J_1)$ then $q_1 \in V(J_1^*)$.

Since G is 3-indivisible and J_1 is infinite, H is 2-indivisible. Since G is $(4, C)$-connected and by planarity, H is $(4, D)$-connected. Hence by Theorem (2.8), H contains a 1-way infinite D-Tutte path P from w_1 and through w_r.

Since G is 3-indivisible and H is infinite, J_1^* must be 2-indivisible. Let X be a path in J_1 from w_1 to $V(vCq_1 - \{v, q_1\})$ such that $X \cap C$ consists of a single vertex x. Let J_1^v and J_1^q denote the subgraphs of J_1^* such that $v \in V(J_1^v)$, $q_1 \in V(J_1^q)$, $J_1^v \cap J_1^q = X$, and $J_1^v \cup J_1^q = J_1^*$. Then either J_1^v or J_1^q is finite.

Suppose J_1^q is finite. Then J_1^q contains a path Q from w_1 to q_1 such that Q is contained in the facial cycle of G which contains $\{w_1, w_2, q_1, p_2\}$. Let $J' := J_1^* + vw_1$

be the plane graph in which $C' := (Q \cup vCq_1) + vw_1$ is a facial cycle. Since G is 3-indivisible and H is infinite, $J' - V(Q)$ has a unique infinite component, denoted J''. Then $v \in V(J'')$; for otherwise, by planarity, the neighbors of J'', which are furtherest apart on Q, form a 2-cut S in G such that the component of $G - S$ containing J'' has no vertex of C, contradicting $(4, C)$-connectivity of G. By Theorem (2.8), J' contains a 1-way infinite C'-Tutte path Y from q_1 such that $vw_1 \in E(Y)$ and $w_1 \in V(q_1Yv)$. Hence, $Y - vw_1$ consists of a path P' from w_1 to q_1 and a 1-way infinite path P'' from v such that $P' \cap P'' = \emptyset$ and $P' \cup P''$ is a vCq_1-Tutte subgraph of J_1^*.

When J_1^v is finite, we may apply the same argument to $J_1^* + q_1w_1$ as in the preceding paragraph to show that J_1^* contains a path P' from w_1 to v and a 1-way infinite path P'' from q_1 such that $P' \cap P'' = \emptyset$ and $P' \cup P''$ is a vCq_1-Tutte subgraph of J_1^*.

Next we apply Lemma (2.6) to $K := G - V(J_1^* - \{v, q_1, w_1\}), H, q_1Cv, D, P, v, q_1, w_1$ (as K, L, Q, Q', T, p, q, u, respectively). Note that the conditions of Lemma (2.6) are satisfied. In particular, $w_1, w_r \in V(P)$ implies that $|V(P) \cap V(D)| \geq 2$. Hence, by Lemma (2.6), there is a path S in $K - V(P)$ between q_1 and v such that $S \cup P$ is a q_1Cv-Tutte subgraph in K and every P-bridge of H containing no edge of D is also an $(S \cup P)$-bridge of K.

Because $w_1, w_r \in V(P)$ and by the same argument as in previous lemma, we may assume that $e \in E(S)$. Let $P^* := P \cup S \cup P' \cup P''$. Then every P^*-bridge of G is either an $(S \cup P)$-bridge of H or a $(P' \cup P'')$-bridge of J_1. Hence P^* is a 2-way infinite C-Tutte path in G through e. □

4 Tight partial nets

Let G be a 2-connected infinite plane graph. A *tight partial net* in G is a sequence $(F_1, \ldots, F_n)$ of vertex disjoint non-dividing cycles in G, where n is a positive integer, such that $I(F_1) = F_1$ and for each $1 \leq i \leq n-1$, $I(F_i) \subseteq I(F_{i+1})$ and every $(I(F_i) \cup F_{i+1})$-bridge of $I(F_{i+1})$ has at most one attachment on F_{i+1}.

A *separation* of a graph G is an ordered pair (G_1, G_2) of subgraphs of G such that $E(G_i) \neq E(G)$ for $i \in \{1, 2\}$, $E(G_1 \cap G_2) = \emptyset$, and $G_1 \cup G_2 = G$. The following observation will be convenient.

(4.1) Lemma. *Let G be a 2-connected infinite plane graph and let $(F_1, \ldots, F_n)$ be a tight partial net in G. Then*

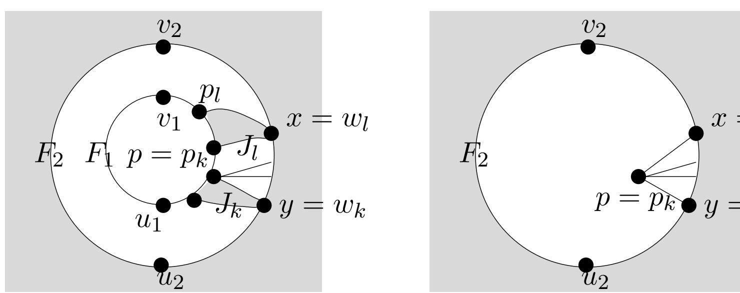
(1) there is a plane embedding of G such that $I(F_n)$ is contained in the closed disc in the plane bounded by F_n, and

(2) for any distinct $x, y \in V(F_n)$, there is a separation (M_1, M_2) of $I(F_n)$ such that $|V(M_1 \cap M_2)| \leq 2n$, $yF_nx \subseteq M_1$ and $xF_ny \subseteq M_2$.

Proof. Note that F_n is a facial cycle of $G-V(I(F_n)-V(F_n))$. Therefore, $G-V(I(F_n)-V(F_n))$ has a plane embedding in which the open disc bounded by F_n is a face. Since $I(F_n)$ is a finite plane graph, it has a plane embedding such that F_n bounds its infinite face. Hence, by combining the new embedding of $G-V(I(F_n)-V(F_n))$ and the new embedding of $I(F_n)$, we see that G has a plane embedding in which $I(F_n)$ is contained in the closed disc in the plane bounded by F_n. Thus we have (1).

To prove (2), we apply induction on n. Clearly, (2) holds when $n=1$. So assume $n \geq 2$. For convenience and by (1), let us assume without loss of generality that $I(F_n)$ is contained in the closed disc bounded by F_n. Because $(F_1,\ldots,F_n)$ is a tight partial net, every $(I(F_{n-1}) \cup F_n)$-bridge of $I(F_n)$ has at most one attachment on F_n. Thus, there exist vertices $x', y' \in V(F_{n-1})$ such that x and x' are incident with a common face of G and y and y' are incident with a common face of G. If $x'=y'$ then by planarity of $I(F_n)$ we see that $I(F_n)$ has a separation (M_1,M_2) such that $V(M_1 \cap M_2)=\{x, x'=y', y\}$, $yF_nx \subseteq M_1$, and $xF_ny \subseteq M_2$. So we may assume that $x' \neq y'$. Then by induction, $I(F_{n-1})$ has a separation (M_1', M_2') such that $|V(M_1' \cap M_2')| \leq 2(n-1)$, $y'F_{n-1}x' \subseteq M_1'$, and $x'F_{n-1}y' \subseteq M_2'$. Now by planarity of $I(F_n)$, we see that $I(F_n)$ has a separation (M_1,M_2) such that $V(M_1 \cap M_2) = V(M_1' \cap M_2') \cup \{x,y\}$, $yF_nx \subseteq M_1$, and $xF_ny \subseteq M_2$.□

The next result is a reduction lemma which shows that, when there is a certain tight partial net with two non-dividing cycles, an infinite graph can be reduced in a certain way so that the existence of a 2-way infinite Tutte paths is preserved. See Figure 3 for an illustration of the situation described in the lemma and its proof. For a subgraph H of a graph G, we use $N_G(H)$, or simply $N(H)$, to denote the set of vertices in $V(G)-V(H)$ each of which is adjacent to some vertex of H.



G $\qquad$ G'

Figure 3: Illustration for Lemma (4.2)

(4.2) Lemma. *Let G be a 2-connected infinite plane graph and (F_1, F_2) be a tight partial net in G such that G is $(4, F_1)$-connected. Suppose $I(F_2)$ is contained in the* *closed disc bounded by F_2, v_i and u_i are distinct vertices on F_i for $1 \leq i \leq 2$, and $uv \in E(v_1F_1u_1)$ with v_1, v, u, u_1 on $v_1F_1u_1$ in order. Assume there exist two vertex disjoint paths in $I(F_2)$ from v_1F_1v to $v_2F_2u_2$ or from uF_1u_1 to $v_2F_2u_2$, which are also internally disjoint from $F_1 \cup F_2$. Then there exist $p \in V(v_1F_1u_1 - \{u_1, v_1\})$, $x, y \in V(v_2F_2u_2)$, and $f \in \{px, py\}$ such that v_2, x, y, u_2 occur on $v_2F_2u_2$ in order and $N(xF_2y - \{x, y\}) \cap V(I(F_2)) \subseteq \{x, y, p\}$ and such that if $(G - V(I(F_2) - (V(F_2) \cup \{p\})) + \{px, py\}$ has a 2-way infinite $(yF_2x + \{p, px, py\})$-Tutte path through f then G contains a 2-way infinite F_1-Tutte path through uv.*

Proof. Let G_1 be the infinite block of $G - V(F_1)$, and let $w_1, \ldots, w_b$ be the attachments on G_1 of $(G_1 \cup F_1)$-bridges of G. Because (F_1, F_2) is a tight partial net, $F_2 \subseteq G_1$. By planarity, $w_i \in V(F_2)$. Without loss of generality, we may assume that $w_1, \ldots, w_b$ occur on F_2 in clockwise order. For each w_t, $1 \leq t \leq b$, let $p_t, q_t \in V(F_1)$ with $p_tF_1q_t$ maximal such that $\{p_t, w_t\}$ is contained in a $(G_1 \cup F_1)$-bridge of G, $\{q_t, w_t\}$ is contained in a $(G_1 \cup F_1)$-bridge of G, and any $(G_1 \cup F_1)$-bridge of G containing some $w_l \neq w_t$ contains no vertex from $V(p_tF_1q_t) - \{p_t, q_t\}$. See Figure 3. Note that p_t and q_t are well defined because G is $(4, F_1)$-connected.

We may assume that there are vertex disjoint paths in $I(F_2)$ from v_1F_1v to $v_2F_2u_2$ and internally disjoint from $F_1 \cup F_2$; the other case can be taken care of in the same way. Then $uv \in E(p_kF_1p_{k+1})$ for some $w_k \in V(v_2F_2u_2 - v_2)$. We choose such w_k that $w_kF_2u_2$ is minimal. Let J_k denote the union of $p_kF_1p_{k+1}$ and those $(G_1 \cup F_1)$-bridges of G whose attachments are all contained in $V(p_kF_1p_{k+1}) \cup \{w_k\}$. Then there is some $w_r \in V(v_2F_2w_k)$ such that $J_k - p_r$ contains a path from w_k to p_{k+1} and through uv. Select w_r so that $w_rF_2w_k$ is minimal. Let $w_l = w_r$ if $w_r \neq w_k$; otherwise, let $w_l = w_{k-1}$. Let J_l denote the union of $p_lF_1p_k$ and those $(G_1 \cup F_1)$-bridges of G whose attachments are all contained in $V(p_lF_1p_k) \cup \{w_l\}$.

Let $G' := G_1 + \{p_k, p_kw_j : w_j \in V(w_lF_2w_k)\}$. Since G is $(4, F_1)$-connected, we see that all $(G_1 \cup F_1)$-bridges of G containing some $w_j \in V(w_lF_2w_k - \{w_l, w_k\})$ are induced by the edge p_kw_j. Hence $G' = (G - V(I(F_2) - (V(F_2) \cup \{p_k\})) + \{p_kw_l, p_kw_k\}$. Let $F_2' := w_kF_2w_l + \{p_k, p_kw_l, p_kw_k\}$. See Figure 3.

Let $f := p_kw_l$ and assume that G' contains a 2-way infinite F_2'-Tutte path P' through f. Note that $P' - p_k$ is an F_2-Tutte subgraph of G_1 and $P' - p_k$ consists of two disjoint 1-way infinite paths. We shall show that the assertion of the lemma holds for $p := p_k$, $x := w_l$ and $y := w_k$.

First, we find a path S from p_{k+1} to p_l by applying Lemma (2.6) to $K := G - V((J_k \cup J_l) - \{p_l, p_{k+1}, w_l, w_k\})$ (with $K, G_1, p_{k+1}F_1p_l, F_2, P' - p_k, p_{k+1}, p_l, w_l$ as K, L, Q, Q', T, p, q, u, respectively). Clearly, the conditions of Lemma (2.6) hold. In particular, we note that $|V(P' - p_k) \cap V(F_2)| \geq 2$. Hence by Lemma (2.6), there is a path S from p_{k+1} to p_l in $K - V(P' - p_k)$ such that $(P' - p_k) \cup S$ is a $p_{k+1}F_1p_l$-Tutte subgraph of K, and every $(P' - p_k)$-bridge of G_1 not containing an edge of F_2 is a $((P' - p_k) \cup S)$-bridge of K.

Let w_j denote the endvertex of $P' - p_k$ other than w_l. Since P' is a 2-way infinite F_2'-Tutte path in G', we see that $w_k \in V(P')$ (for otherwise, the P'-bridge of G' containing w_k would have three attachments, namely, $p = p_k$ and two on F_2). We shall complete the desired path in G by finding a path from w_j to p_{k+1} and a path from w_l to p_l. We distinguish two cases.

Case 1. $p_k \neq p_l$.

In $J_l + w_l p_k$ we apply Lemma (2.4) to find a $p_l F_1 p_k$-Tutte path P_l' from p_l to p_k such that $w_l p_k \in E(P_l')$, and let $P_l := P_l' - p_k$. If $w_k = w_j$, then in $J_k + w_k p_{k+1}$ we apply Lemma (2.5) to find a $p_k F_1 p_{k+1}$-Tutte path P_k from w_k to p_{k+1} such that $p_k \in V(P_k)$ and $uv \in E(P_k)$. If $w_k \neq w_j$, then in $J_k + \{w_k p_k, w_k p_{k+1}\}$ we apply Lemma (2.5) to find a $p_k F_1 p_{k+1}$-Tutte path P_k' from w_k to p_{k+1} such that $w_k p_k, uv \in E(P_k')$; let $P_k := (P_k' - w_k) + \{w_j, p_k w_j\}$.

Let $P := (P' - p_k) \cup S \cup P_k \cup P_l$. Then every P-bridge of G is one of the following: a $((P' - p_k) \cup S)$-bridge of K, or a P_k-bridge of $J_k + w_k p_{k+1}$ when $w_k = w_j$, or a P_k'-bridge of $J_k + \{w_k p_k, w_k p_{k+1}\}$ when $w_k \neq w_j$, or a P_l'-bridge of $J_l + w_l p_k$, or a P'-bridge of G' containing some $w_i \in V(w_l F_2 w_j - \{w_l, w_j\})$ (which has three attachments: p_k, and two on $w_l F_2 w_j$). It is easy to see that P gives the desired 2-way infinite F_1-Tutte path in G through uv.

Case 2. $p_k = p_l$.

Then $w_l = w_{k-1}$, $w_j = w_k$, J_l is induced by the edge $w_l p_l$, and $J_k - p_k = J_k - p_l$. Since $J_k - p_l$ has a path R from w_k to p_{k+1} and through uv, we let J_k' denote the union of blocks of $J_k - p_k$ each of which contains an edge of R. Let R' denote the path from w_k to p_{k+1} containing uv such that R' is on the boundary of the face of $G - p_k$ which is not a face of G. Let $p' \in V(F_1 \cap J_k')$ with $p_k F_1 p'$ minimal. By applying Lemma (2.5) we find a R'-Tutte path P_k in J_k' from w_k to p_{k+1} such that $p' \in V(P_k)$ and $uv \in E(P_k)$. Let $P := ((P' - p_k) \cup S \cup P_k) + w_l p_l$. Then every P-bridge of G is one of the following: a $((P' - p_k) \cup S)$-bridge of K, or a P_k-bridge of J_k', or a $(J_k' \cup \{p_k\})$-bridge of J_k with attachments p_k and p', or a subgraph of J_k obtained from a P_k-bridge B of J_k' with two attachments by adding p_k and all edges from p_k to $B - V(P_k)$. Thus, P gives the desired 2-way infinite F_1-Tutte path in G through uv. □

The next lemma shows that certain tight partial nets can force the existence of a 2-way infinite Tutte path. Let G be a 2-connected infinite plane graph which is 3-indivisible but not 2-indivisible, let $(F_1, \ldots, F_n)$ be a tight partial net in G such that G is $(4, F_1)$-connected, and assume that $I(F_n)$ is drawn in the closed disc bounded by F_n. For $1 \leq i \leq n$, let u_i, v_i be distinct vertices of F_i such that any two consecutive vertices from $u_n, \ldots, u_1, v_1, \ldots, v_n$ are contained in a facial cycle of G, and assume that there is no separation (H_1, H_2) of $I(F_n)$ such that $|V(H_1 \cap H_2)| < 2n$, $\{u_n, v_n\} \subseteq V(H_1 \cap H_2)$, $v_n F_n u_n \subseteq H_1$, and $u_n F_n v_n \subseteq H_2$.

(4.3) Lemma. *Let $G, (F_1, \dots, F_n)$, and u_i, v_i, $1 \le i \le n$, be defined as above. Suppose either (1) $(G - V(I(F_n) - V(F_n))) - \{u_n, v_n\}$ has two infinite components or (2) $G - V(I(F_n))$ has two infinite blocks, say H and H', such that the face of H containing $I(F_n)$ contains H' and is bounded by a cycle, and such that no path in G from H' to $v_nF_nu_n - \{v_n, u_n\}$ is internally disjoint from $I(F_n) \cup H$. Let $uv \in E(v_1F_1u_1)$ such that v_1, v, u, u_1 occur on $v_1F_1u_1$ in order, and assume when $n \ge 2$ there exist two vertex disjoint paths in $I(F_2)$ from v_1F_1v to $v_2F_2u_2$ or from u_1F_1u to $v_2F_2u_2$, which are also internally disjoint from $F_1 \cup F_2$. Then there is a 2-way infinite F_1-Tutte path in G through uv.*

Proof. We apply induction on n. Suppose $n = 1$. If $G - \{u_1, v_1\}$ has two infinite components, then by Lemma (3.1) there is a 2-way infinite F_1-Tutte path in G through uv. So assume that $G - V(I(F_1))$ has two infinite blocks H and H' such that the face of H containing $I(F_1)$ contains H' and is bounded by a cycle D, and such that no path in G from H' to $v_1F_1u_1 - \{v_1, u_1\}$ is internally disjoint from $H \cup I(F_1)$. Then we see that every $(H \cup F_1)$-bridge of G has at most one attachment on H (which must be on D), and H' is contained in an infinite $(H \cup F_1)$-bridge of G. Let $w_1, \dots, w_b$ denote the attachments on H of $(H \cup F_1)$-bridges of G and let them occur on D in clockwise order. Let $p_j, q_j \in V(F_1)$ with $p_jF_1q_j$ maximal such that $\{p_j, w_j\}$ is contained in an $(H \cup F_1)$-bridge of G, $\{q_j, w_j\}$ is contained in an $(H \cup F_1)$-bridge of G, and any $(H \cup F_1)$-bridge of G containing some $w_l \ne w_j$ contains no vertex from $V(p_jF_1q_j) - \{p_j, q_j\}$. Because G is $(4, F_1)$-connected, p_j and q_j are well defined. Let J_j denote the union of $p_jF_1q_j$ and those $(H \cup F_1)$-bridges of G whose attachments are all contained in $V(p_jF_1q_j) \cup \{w_j\}$. Let L_j denote the union of $q_jF_1p_{j+1}$ and those $(H \cup F_1)$-bridges of G whose attachments are all contained in $V(q_jF_1p_{j+1})$, where $p_{b+1} = p_1$. Then there is some $1 \le j \le b$ such that $H' \subseteq J_j$ or $H' \subseteq L_j$. Recall the assumption that no path in G from H' to $v_1F_1u_1 - \{v_1, u_1\}$ is internally disjoint from $H \cup I(F_1)$. Thus, if $H' \subseteq L_j$ then $L_j \cap F_1 \subseteq u_1F_1v_1$, whence $uv \notin E(L_j)$; if $H' \subseteq J_j$ then uv is not in the unique infinite block of $J_j - w_j$. Hence by Lemma (3.2) and Lemma (3.3), G contains a 2-way infinite F_1-Tutte path P through uv.

So assume $n \ge 2$. Note that the conditions of Lemma (4.2) are satisfied. By Lemma (4.2), there exist $p \in V(v_1F_1u_1 - \{u_1, v_1\})$, $x, y \in V(v_2F_2u_2)$, and an edge $f \in \{px, py\}$ such that if $G' := (G - V(I(F_2) - (V(F_2) \cup \{p\})) + \{px, py\}$ has a 2-way infinite $(yF_2x + \{p, px, py\})$-Tutte path through f then G has a 2-way infinite F_1-Tutte path through uv.

Therefore it suffices to show that G' has a 2-way infinite $(yF_2x + \{p, px, py\})$-Tutte path through f. For convenience, let $F_2' := yF_2x + \{p, px, py\}$ and assume $f = px$. There is a tight partial net $(F_2', \dots, F_n')$ in G' such that $u_iF_i'v_i = u_iF_iv_i$ and $I_{G'}(F_i') - \{px, py\} \subseteq I_G(F_i)$, for all $2 \le i \le n$. (This can be shown by applying induction on i, by noting that any two consecutive vertices from $u_n, \dots, u_2, v_2, \dots, v_n$ are contained in a facial cycle of G' and that $\{u_i, v_i\} \subseteq V(I_{G'}(F_i'))$, and by taking $I_{G'}(F_i')$ minimal subject to the

condition $F_i' \cap F_{i-1}' = \emptyset$.) Note that there is no separation (M_1, M_2) of $I_{G'}(F_n')$ such that $|V(M_1 \cap M_2)| < 2(n-1)$, $\{u_n, v_n\} \subseteq V(M_1 \cap M_2)$, $v_n F_n' u_n \subseteq M_1$, and $u_n F_n' v_n \subseteq M_2$; for otherwise by planarity $I_G(F_n)$ has a separation (H_1, H_2) such that $|V(H_1 \cap H_2)| < 2n$, $\{u_n, v_n\} \subseteq V(H_1 \cap H_2)$, $v_n F_n u_n \subseteq H_1$, and $u_n F_n' v_n \subseteq H_2$, a contradiction. We claim that when $n \geq 3$ there must be two disjoint paths in $I_{G'}(F_3')$ from $v_2 F_2' x$ to $v_3 F_3' u_3$ or from $p F_2' u_2$ to $v_3 F_3' u_3$, which are internally disjoint from $F_2' \cup F_3'$. For otherwise, there exist vertices u_2', v_2' such that all paths in $I_{G'}(F_3')$ from $v_2 F_2' u_2$ to $v_3 F_3' u_3$ internally disjoint from $F_2' \cup F_3'$ intersect $\{u_2', v_2'\}$. Then $I_G(F_n)$ has a separation (H_1, H_2) such that $H_1 \cap H_2 = \{u_n, \ldots, u_3, u_2', v_2', v_3, \ldots, v_n\}$, $v_n F_n u_n \subseteq H_1$, and $u_n F_n' v_n \subseteq H_2$, a contradiction. Therefore, by induction, G' has a 2-way infinite F_2'-Tutte path through f.
□

5 Graphs with $\gamma(G) = 0$

In this section we prove Theorem (1.1) for graphs with no dividing cycles.

(5.1) Theorem. *Let G be a 4-connected 3-indivisible infinite plane graph and assume $\gamma(G) = 0$. Then G contains a spanning 2-way infinite path.*

Proof. We may assume that G is not 2-indivisible, for otherwise the assertion of this theorem follows from [8] and [9]. Since G is 4-connected and planar and because $\gamma(G) = 0$, it follows from Lemma (2.3) that there is a sequence $(D_1, D_2, \ldots)$ of non-dividing cycles in G such that

(1) for each $i \geq 1$, $I(D_i) \subseteq I(D_{i+1})$,
(2) for each $i \geq 1$, G has no finite $I(D_i)$-bridge,
(3) for each $i \geq 1$, $D_i \cap D_{i+1} \subseteq D_{i+1} \cap D_{i+2}$, and
(4) $\bigcup_{i \geq 1} I(D_i) = G$.

If $D_i \cap D_{i+1} = \emptyset$ for all $i \geq 1$, then by (4), $(D_1, D_2, \ldots)$ is a radial net in G, and hence, G is 2-indivisible, a contradiction. So $D_k \cap D_{k+1} \neq \emptyset$ for some positive integer k. By (3), $D_i \cap D_{i+1} \neq \emptyset$ for all $i \geq k$.

Suppose $D_i \cap D_{i+1}$ consists of a single path for all $i \geq k$. Then by planarity and because G is 4-connected, for any positive integer l, $D_i - V(D_l)$, $i \geq l+1$, are all nonempty and contained in a single component of $G - V(I(D_l))$. By (4), for any finite $X \subseteq V(G)$, $X \subseteq V(I(D_l))$ for some positive integer l. Therefore, $G - X$ has only one infinite component. This shows that G is 2-indivisible, again, a contradiction.

Thus, for some integer $t \geq k$, $D_t \cap D_{t+1}$ consists of at least two vertex disjoint paths, and hence, G has at least two $I(D_t)$-bridges. Because of (2) and since G is 3-indivisible, G has exactly two $I(D_t)$-bridges, both infinite. So $D_t \cap D_{t+1}$ consists of exactly two vertex disjoint paths. Therefore, by (2) and (3), we have

(5) for each $i \geq t$, $D_i \cap D_{i+1}$ consists of exactly two vertex disjoint paths and G has exactly two $I(D_i)$-bridges (both infinite).

By (5) and by (3), we see that

(6) $\partial G = \bigcup_{i \geq 1}(D_i \cap D_{i+1})$ and has exactly two components, each of which is a path, or a 1-way infinite path, or a 2-way infinite path.

Let N_1 and N_2 denote the two components of ∂G. See Figure 4. Note that $I_G(D_t) \cap N_i \neq \emptyset$ for $i = 1, 2$. So there is a separation (G_1, G_2) of G such that $|V(G_1 \cap G_2)|$ is finite, exactly one vertex of $G_1 \cap G_2$ is on N_i for each $1 \leq i \leq 2$, and both G_1 and G_2 are infinite. Among all such separations (G_1, G_2) of G, there is one such that

(7) $|V(G_1 \cap G_2)|$ is minimum.

Let u be the unique vertex of $G_1 \cap G_2 \cap N_1$, and v be the unique vertex of $G_1 \cap G_2 \cap N_2$.

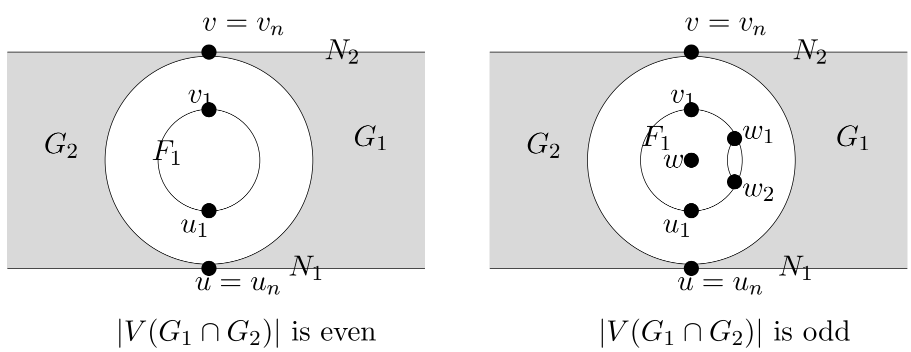


Figure 4: Structure of G

When $|V(G_1 \cap G_2)|$ is even, let $u_n, \ldots, u_1, v_1, \ldots, v_n$ be the vertices in $V(G_1 \cap G_2)$ such that $u_n = u$ and $v_n = v$, and any two consecutive vertices from the sequence are contained in a facial cycle of G. In this case, let F_1 denote the facial cycle of G containing u_1 and v_1 such that $v_1 F_1 u_1 \subseteq G_1$ and $u_1 F_1 v_1 \subseteq G_2$. Since $n \geq 2$ (because G is 4-connected), u_1 and v_1 each have finite degree and the faces of G incident with u_1 or v_1 are bounded by cycles. Hence we may further choose (G_1, G_2), subject to $\{u_n, \ldots, u_1, v_1, \ldots, v_n\} \subseteq V(G_1 \cap G_2)$, so that $v_1 F_1 u_1$ has at least two edges. (Otherwise, we could simply choose F_1 to be the other facial cycle of G containing $u_1 v_1$.) Let $H := G$. See the left part of Figure 4.

When $|V(G_1 \cap G_2)|$ is odd, we let $u_n, \ldots, u_1, w, v_1, \ldots, v_n$ be the vertices in $V(G_1 \cap G_2)$ such that $u_n = u$ and $v_n = v$, and any two consecutive vertices from the sequence are contained in a facial cycle of G. In this case, let D denote the cycle in G such that $I(D) - V(D) = \{w\}$. Because G is 4-connected, D is well defined and $u_1, v_1 \in V(D)$. Without loss of generality, we may assume that $v_1 D u_1 \subseteq G_1$ and $u_1 D v_1 \subseteq G_2$. Notice $n \geq 2$ because G is 4-connected. Hence, $u_1, v_1 \in I(D_i) - V(D_i)$ for all sufficiently large i. This implies that u_1 and v_1 are of finite degree in G and the faces of G incident with

u_1 or v_1 are bounded by cycles. Hence, we may further choose (G_1, G_2) so that, subject to $\{u_n, \dots, u_1, v_1, \dots, v_n\} \subseteq V(G_1 \cap G_2)$, w has at least two neighbors in $v_1Du_1 - \{u_1, v_1\}$. (Otherwise, in $V(G_1 \cap G_2)$ we may replace w with its unique neighbor in $v_1Du_1 - \{v_1, u_1\}$, and continue if necessary. This process must stop because of the above finiteness conditions on u_1 and v_1.) Therefore, let $w_1, w_2 \in V(v_1Du_1) - \{v_1, u_1\}$ be distinct neighbors of w such that v_1, w_1, w_2, u_1 occur on v_1Du_1 in order and w has no neighbor in $V(w_1Dw_2) - \{w_1, w_2\}$. Let $H := (G - w) + w_1w_2$. Let $F_1 = w_2Dw_1 + w_1w_2$, and assume that w_1w_2 is added such that F_1 is a facial cycle of H. See the right part of Figure 4.

Let $(F_1, \dots, F_m)$ be a tight partial net in H such that m is maximum. Note that $m \le n$. By (1) of Lemma (4.1), we may assume that $I_H(F_m)$ is contained in the closed disc bounded by F_m, as illustrated in Figure 4.

Suppose $m < n$. If $F_m \cap (N_1 \cup N_2) \neq \emptyset$, then assume by symmetry that $x \in V(F_m \cap N_1)$. Then $V(F_m \cap N_2) = \emptyset$; for otherwise, let $y \in V(F_m \cap N_2)$, then by (2) of Lemma (4.1), G has a separation (H_1, H_2) with $\{x, y\} \subseteq V(H_1 \cap H_2)$ and $|V(H_1 \cap H_2)| = 2m$, exactly one vertex of $H_1 \cap H_2$ is on N_i for $i = 1, 2$, and both H_1 and H_2 are infinite, contradicting (7). Hence $v_n \notin V(I(F_m))$. Since $v_1 \in V(I(F_m))$ and every pair of consecutive vertices from $v_1, \dots, v_n$ are contained in a facial cycle of G, we must have $v_m \in V(I(F_m))$. Thus there exists some $m \le j \le n$ such that $v_j \in V(I(F_m))$ and $v_{j+1} \notin V(I(F_m))$. This shows that $v_j \in V(F_m)$ (since v_j and v_{j+1} is contained in a facial cycle of G). By (2) of Lemma (4.1), $I_H(F_m)$ has a separation (L_1, L_2) such that $|V(L_1 \cap L_2)| \le 2m$, $\{x, v_j\} \subseteq V(L_1 \cap L_2)$, $v_jF_mx \subseteq L_1$, and $xF_mv_j \subseteq L_2$. Now it is easy to see that G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(L_1 \cap L_2) \cup \{v_{j+1}, \dots, v_n\}$, exactly one vertex of $L_1 \cap L_2$ is on N_i for $i = 1, 2$, and both G_1 and G_2 are infinite, contradicting (7). So $F_m \cap (N_1 \cup N_2) = \emptyset$. Therefore $H - V(I_H(F_m))$ has a unique infinite block B which contains D_i for all sufficiently large i. Let F_{m+1} denote the cycle bounding the face of B containing $I_H(F_m)$. Then we see that $(F_1, \dots, F_{m+1})$ is a tight partial net in H, which contradicts the choice of $(F_1, \dots, F_m)$.

Hence, $m = n \ge 2$. We may assume that the notation is chosen so that for $1 \le i \le n$, $\{u_i, v_i\} \subseteq V(F_i)$, $v_iF_iu_i \subseteq G_1$, and $u_iF_iv_i \subseteq G_2$. Clearly, there is a separation (H_1, H_2) of $I_H(F_n)$ such that $V(H_1 \cap H_2) = \{u_1, \dots, u_n, v_1, \dots, v_n\}$, and for $1 \le i \le n$, $v_iF_iu_i \subseteq H_1$ and $u_iF_iv_i \subseteq H_2$. Then by (7) and by planarity, there is no separation (H_1, H_2) of $I_H(F_n)$ such that $|V(H_1 \cap H_2)| < 2n$, $\{u_n, v_n\} \subseteq V(H_1 \cap H_2)$, $v_nF_nu_n \subseteq H_1$, and $u_nF_nv_n \subseteq H_2$.

When $|V(G_1 \cap G_2)|$ is even, $v_1F_1u_1$ has at least two edges. This, together with (7) and 4-connectivity of G, implies that there exist two disjoint paths in $I_H(F_2)$ from $v_1F_1u_1 - u_1$ or from $v_1F_1u_1 - v_1$ to $v_2F_2u_2$ and internally disjoint from $F_1 \cup F_2$. Hence there is an edge w_1w_2 of $v_1F_1u_1$ such that v_1, w_1, w_2, u_1 occur on $v_1F_1u_1$ in order and there exist two disjoint paths in $I_H(F_2)$ from $v_1F_1w_1$ or from $w_2F_1u_1$ to $v_2F_2u_2$ internally disjoint from $F_1 \cup F_2$. When $|V(G_1 \cap G_2)|$ is odd, then by (7), there exist two disjoint paths in $I_H(F_2)$ from $v_1F_1w_1$ or from $w_2F_1u_1$ to $v_2F_2u_2$ internally disjoint from $F_1 \cup F_2$.

Hence by Lemma (4.3), H has a 2-way infinite F_1-Tutte path P through w_1w_2. Note that G is 4-connected. Therefore, if $|V(G_1 \cap G_2)|$ is even then P is a spanning 2-way infinite path in G, and if $|V(G_1 \cap G_2)|$ is odd then $(P - w_1w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G. □

6 Graphs with $\gamma(G) = \infty$

Let G be a 4-connected 3-indivisible infinite plane graph, and assume $\gamma(G) = \infty$. Let C be a dividing cycle in G, let G' be the subgraph of G contained in the closed disc of the plane bounded by C, and let $G'' = G - (V(G') - V(C))$. Then $G' \cap G'' = C$, both G' and G'' are 2-indivisible, and both G' and G'' are $(4, C)$-connected.

Let S' denote the set of vertices of G' of infinite degree. By Theorem (2.1), there is a set $F' \subseteq E(G')$ incident with vertices in S' such that $G' - F'$ has a net $N' = (C'_1, C'_2, \ldots)$ satisfying the conclusions of Theorem (2.1) (with G', S', F', N' as G, S, F, N, respectively). Similarly, let S'' denote the set of vertices of G'' of infinite degree. Then by Theorem (2.1), there is a set $F'' \subseteq E(G'')$ incident with vertices in S'' such that $G'' - F''$ has a net $N'' = (C''_1, C''_2, \ldots)$ satisfying the conclusions of Theorem (2.1) (with G'', S'', F'', N'' as G, S, F, N, respectively).

Since $\gamma(G) = \infty$, N' or N'' must be a radial net. If both N' and N'' are radial nets, then we say that G admits a *2-way radial net*. If exactly one of N' and N'' is a radial net, then we slightly abuse notation and say that G admits a *mixed net*. We deal with these two types of graphs separately.

(6.1) Theorem. *Suppose that G is a 4-connected 3-indivisible infinite plane graph, and assume that G admits a 2-way radial net. Then G contains a spanning 2-way infinite path.*

Proof. Because G has a 2-way radial net, it follows from Lemma (2.2) that
(1) G is locally finite and every face of G is bounded by a cycle.

Let (K', K'') be a separation of G such that $V(K' \cap K'')$ is finite, and both K' and K'' are infinite. There exists such a separation (K', K'') of G that
(2) $|V(K' \cap K'')|$ is minimum.

When $|V(K' \cap K'')|$ is even (respectively, odd), then let $u_n, \ldots, u_1, v_1, \ldots, v_n$ (respectively, $u_n, \ldots, u_1, w, v_1, \ldots, v_n$) be the vertices in $V(K' \cap K'')$ such that any two consecutive vertices (in cyclic order) from the sequence are contained in a facial cycle of G.

If $|V(K' \cap K'')|$ is even, then let $H := G$ and let F_1 be a facial cycle of H containing $\{u_1, v_1\}$ such that $v_1F_1u_1 \subseteq K''$ and $u_1F_1v_1 \subseteq K'$. Note that $n \geq 2$ (since G is 4-connected). Hence, u_1 and v_1 have finite degrees in G and faces of G incident with u_1

or v_1 are bounded by cycles. Hence, we may choose (K', K'') and F_1 so that $v_1F_1u_1$ has at least two edges.

When $|V(K' \cap K'')|$ is odd, let D be the facial cycle of $G - w$ containing $\{u_1, v_1\}$ such that $v_1Du_1 \subseteq K''$ and $u_1Dv_1 \subseteq K'$. Note that D is uniquely defined because G is 4-connected and every face of G is bounded by a cycle (by (1)). Hence we may further select (K', K'') so that, subject to $\{u_1, \ldots, u_n, v_1, \ldots, v_n\} \subseteq V(K' \cap K'')$, w has at least two neighbors in $v_1Du_1 - \{u_1, v_1\}$. Let w_1, w_2 be distinct neighbors of w in $v_1Du_1 - \{u_1, v_1\}$ such that v_1, w_1, w_2, u_1 occur on v_1Du_1 in order, and w has no neighbor in $w_1Dw_2 - \{w_1, w_2\}$. Let $H := (G - w) + w_1w_2$ and let $F_1 := w_2Dw_1 + w_1w_2$. We may assume that if $w_1w_2 \notin E(D)$ then it is represented by a simple arc in the open disc bounded by D.

We wish to apply Lemma (4.3); therefore we need to show that

(3) there is a tight partial net $(F_1, \ldots, F_n)$ in H.

To prove (3), assume that we have a maximum tight partial net $(F_1, \ldots, F_k)$ in H. For convenience, we may assume that $I_H(F_k)$ is contained in the closed disc bounded by F_k (see (1) of Lemma (4.1)). Suppose $k < n$.

We claim that $H - V(I_H(F_k))$ has a unique infinite block. Otherwise, since H is 3-indivisible (because G is), $H - V(I_H(F_k))$ has a separation (H', H'') such that $|V(H' \cap H'')| \leq 1$ and both H' and H'' are infinite. Since G is 3-indivisible, H' and H'' each have exactly one infinite component. Let L' denote the infinite component of H' and L'' the infinite component of H''. Since G is 4-connected and $|V(H' \cap H'')| \leq 1$, L' has at least three neighbors on F_k. Hence by planarity, there are $\{x, y\} \subseteq V(F_k)$ such that x, y are neighbors of $L' - V(H' \cap H'')$, all neighbors of $L' - V(H' \cap H'')$ in F_k are contained in xF_ky, and all neighbors of $L'' - V(H' \cap H'')$ in F_k are contained in yF_kx. By (2) of Lemma (4.1), $I_H(F_k)$ has a separation (M', M'') such that $|V(M' \cap M'')| \leq 2k$, $\{x, y\} \subseteq V(M' \cap M'')$, $xF_ky \subseteq M'$, and $yF_kx \subseteq M''$. Hence, $H - (V(M' \cap M'') \cup V(H' \cap H''))$ has two infinite components. Therefore, if $|V(K' \cap K'')|$ is even, then $G - (V(M' \cap M'') \cup V(H' \cap H''))$ has two infinite components, and if $|V(K' \cap K'')|$ is odd then $G - (V(M' \cap M'') \cup V(H' \cap H'') \cup \{w\})$ has two infinite components. Since $k < n$ and $|V(H' \cap H'')| \leq 1$, we see that $|V(M' \cap M'') \cup V(H' \cap H'')| \leq 2n - 1$. Thus $|V(M' \cap M'') \cup V(H' \cap H'')| < |V(K' \cap K'')|$ when $|V(K' \cap K'')|$ is even, and $|V(M' \cap M'') \cup V(H' \cap H'') \cup \{w\}| < |V(K' \cap K'')|$ when $|V(K' \cap K'')|$ is odd. This contradicts the minimality of $|V(K' \cap K'')|$ in (2).

Now let B be the unique infinite block of $H - V(I_H(F_k))$. Because of (1), H is locally finite and every face of H is bounded by a cycle, and only finitely many vertices and edges of B are incident with faces of H which are also incident with vertices of F_k. Therefore, since B is 2-connected and locally finite, the face of B containing $I_H(F_k)$ is bounded by a cycle, say F_{k+1}. Because B is the unique infinite block of $H - V(I_H(F_k))$, F_{k+1} is non-dividing, $I_H(F_k) \subseteq I_H(F_{k+1})$, and every $(I_H(F_k) \cup F_{k+1})$-bridge of $I_H(F_{k+1})$ has at most one attachment on F_{k+1}. However, this shows that $(F_1, \ldots, F_{k+1})$ is a tight partial net in H, contradicting the choice of $(F_1, \ldots, F_k)$. Thus, we have (3).

By (1) of Lemma (4.1), we may assume $I_H(F_n)$ is contained in the closed disc bounded by F_n. Because $\{u_1, v_1\} \subseteq V(F_1)$ and any two consecutive vertices from the sequence $u_n, \ldots, u_1, v_1, \ldots, v_n$ are contained in a facial cycle of H, we see that $\{u_i, \ldots, u_1, v_1, \ldots, v_i\} \subseteq V(I_H(F_i))$ for all $1 \leq i \leq n$. Moreover, $\{u_n, v_n\} \subseteq V(F_n)$, as otherwise, $F_n - \{u_n, v_n\}$ is a path or cycle and $G - V(K' \cap K'')$ would have only one infinite component (containing $F_n - \{u_n, v_n\}$), a contradiction. Therefore, because $\{u_1, v_1\} \subseteq V(F_1)$, $v_1 F_1 u_1 \subseteq K''$ (or $v_1 F_1 u_1 \subseteq K'' + w_1 w_2$) and $u_1 F_1 v_1 \subseteq K'$, and any two consecutive vertices from $u_n, \ldots, u_1, v_1, \ldots, v_n$ are contained in a facial cycle of H, it follows from planarity that
(4) for $2 \leq i \leq n$, $\{u_i, v_i\} \subseteq V(F_i)$, $v_i F_i u_i \subseteq K''$, and $u_i F_i v_i \subseteq K'$.

By (2) and (4), $I_H(F_n)$ has no separation (H_1, H_2) such that $|V(H_1 \cap H_2)| < 2n$, $\{u_n, v_n\} \subseteq V(H_1 \cap H_2)$, $v_n F_n u_n \subseteq H_1$, and $u_n F_n v_n \subseteq H_2$.

When $|V(K' \cap K'')|$ is even, $v_1 F_1 u_1$ has at least two edges. This, together with (2) and 4-connectivity of G, implies that there exist two disjoint paths in $I_H(F_2)$ from $v_1 F_1 u_1 - v_1$ or from $v_1 F_1 u_1 - u_1$ to $v_2 F_2 u_2$ internally disjoint from $F_1 \cup F_2$. Hence, there is an edge $w_1 w_2$ of $v_1 F_1 u_1$ such that v_1, w_1, w_2, u_1 occur on $v_1 F_1 u_1$ in order and there are two disjoint paths in $I_H(F_2)$ from $v_1 F_1 w_1$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$ and internally disjoint from $F_1 \cup F_2$. When $|V(K' \cap K'')|$ is odd, then by (2), there are disjoint paths from $v_1 F_2 w_1$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$ and internally disjoint from $F_1 \cup F_2$. Thus by Lemma (4.3),
(5) there is a 2-way infinite F_1-Tutte path P through $w_1 w_2$ in H.

Now it is easy to see that if $|V(K' \cap K'')|$ is even then P is a spanning 2-way infinite path in G, and if $|V(K' \cap K'')|$ is odd then $(P - w_1 w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G. □

Finally, we deal with graphs which admit mixed nets.

(6.2) Theorem. *Let G be a 4-connected 3-indivisible infinite plane graph, and assume that G admits a mixed net. Then G contains a spanning 2-way infinite path.*

Proof. Let C be a dividing cycle in G, let G' denote the subgraph of G contained in the closed disc bounded by C, and let $G'' := G - (V(G') - V(C))$. Because G admits a mixed net, we may assume that G' has a radial net $N' := (C'_1, C'_2, \ldots)$ with $C \subseteq I_{G'}(C'_i)$, and for some (possibly empty) set F'' of edges of G'' incident with vertices of infinite degree in G'', $G'' - F''$ has a ladder net $N'' := (C''_1, C''_2, \ldots)$ with $C \subseteq I_{G''}(C''_1)$. Note that $\partial G''$ is a path, or a 1-way infinite path, or a 2-way infinite path. Also note that each face of G is either a face of G' or a face of G''. Thus, in view of Lemma (2.2),
(1) all but one face of G are bounded by cycles, and $\partial G''$ is precisely the subgraph of G that lies on the boundary of the exceptional face of G.

Because the cycles C''_i are dividing cycles in G and $C''_i \cap \partial G'' \neq \emptyset$ for large i, there is a separation (K', K'') of G such that $C'_j \subseteq K'$ and $C''_j \subseteq K''$ for sufficiently large j,

$V(G_1 \cap G_2) = V(C_i'')$ if $|V(C_i'' \cap \partial G''| \le 2$, and $V(G_1 \cap G_2)$ is the vertex set of the path in C_i'' between two vertices of $\partial G''$ and internally disjoint from $\partial G''$. Hence,
(2) there exists a separation (K', K'') of G such that $V(K' \cap K'')$ is finite, $C_i' \subseteq K'$ for all large i, $C_i'' - V(\partial G'') \subseteq K''$ for all large i, and $1 \le |V(K' \cap K'') \cap V(\partial G'')| \le 2$. Moreover, the vertices in $V(K' \cap K'')$ can be ordered as $x_1, x_2, \dots, x_k$ with $x_1, x_k \in V(\partial G'')$, $x_i \ne x_j$ except possibly $x_1 = x_k$, and for each $1 \le i \le k-1$, $\{x_i, x_{i+1}\}$ is contained in a facial cycle of G.

Note that we include the possibility $1 = |V(K' \cap K'') \cap V(\partial G'')|$, because $\partial G''$ may be a trivial path. We choose (K', K'') such that, subject to conditions in (2),
(3) $|V(K' \cap K'')|$ is minimum.

When $|V(K' \cap K'') \cap V(\partial G'')| = 2$, let $V(K' \cap K'') \cap V(\partial G'') = \{x, y\}$, and otherwise, let $x = y$ be the only vertex in $V(K' \cap K'') \cap V(\partial G'')$. Note that when $x = y$, $G - (V(K' \cap K'') - \{x = y\})$ has two infinite blocks.

If $|V(K' \cap K'') - \{x, y\}|$ is even (respectively, odd), then let $x = u_m, u_{m-1}, \dots, u_1, v_1, \dots v_{m-1}, v_m = y$ (respectively, $x = u_m, u_{m-1}, \dots, u_1, w, v_1, \dots v_{m-1}, v_m = y$) be the vertices in $V(K' \cap K'')$, such that any two consecutive vertices from $u_m, \dots, u_1, v_1, \dots v_m$ (respectively, $u_m, \dots, u_1, w, v_1, \dots v_m$) are contained in a facial cycle of G, and $u_m C_j'' v_m \subseteq C_j'' \cap C_{j+1}''$ for all large j. See Figure 5.

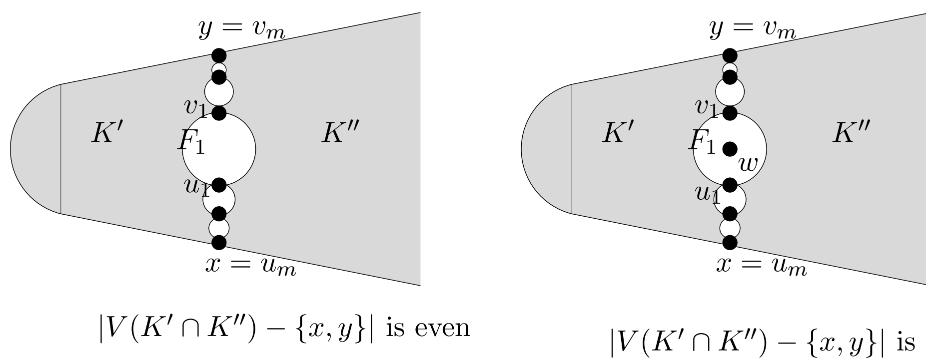


Figure 5: Structure of G

When $|V(K' \cap K'') - \{x, y\}|$ is even, let $H := G$ and let F_1 be the facial cycle of G containing u_1 and v_1 such that $v_1 F_1 u_1 \subseteq K''$ and $u_1 F_1 v_1 \subseteq K'$. Note that $n \ge 2$, and so, $u_1, v_1 \notin \partial G''$. By (1), u_1 and v_1 are of finite degree in G and all faces of G incident with u_1 or v_1 are bounded by cycles. Hence we may choose (G_1, G_2) and F_1 so that $v_1 F_1 u_1$ has at least two edges.

When $|V(K' \cap K'') - \{x, y\}|$ is odd, we have $m \ge 2$ (because G is 4-connected). Let D be the facial cycle of $G - w$ containing u_1 and v_1 such that $v_1 D u_1 \subseteq K''$ and $u_1 D v_1 \subseteq K'$. Because G is 4-connected and $w \notin \partial G''$, D is well defined and $u_1, v_1 \notin$

$V(\partial G'')$. Hence by (1), u_1 and v_1 are of finite degree in G and all faces of G incident with u_1 or v_1 are bounded by cycles. Therefore, we may further choose (K', K'') so that, subject to $\{u_1, \dots, u_m, v_1, \dots, v_m\} \subseteq V(K' \cap K'')$, w has two neighbors w_1 and w_2 in $v_1 D u_1 - \{u_1, v_1\}$ such that v_1, w_1, w_2, u_1 occur on $v_1 D u_1$ in order, and w has no neighbor in $w_1 D w_2 - \{w_1, w_2\}$. Let $F_1 := w_2 D w_1 + w_1 w_2$ and $H := (G - w) + w_1 w_2$. We may assume that if $w_1 w_2 \notin E(D)$ then it is represented by a simple arc in the open disc bounded by D. Clearly H is $(4, F_1)$-connected. Note that $v_1 F_1 u_1 \subseteq K'' + w_1 w_2$ and $u_1 F_1 v_1 \subseteq K'$.

We wish to apply Lemma (4.3). Let $(F_1, \dots, F_n)$ denote a tight partial net in H such that n is maximum. Then $n \leq m$. By (1) of Lemma (4.1), we may assume that $I_H(F_n)$ is contained in the closed disc bounded by F_n.

Since $u_1, v_1 \in V(F_1)$ and because any two consecutive vertices from $u_m, \dots, u_1, v_1, \dots, v_m$ or from $u_m, \dots, u_1, w, v_1, \dots, v_m$ are contained in a facial cycle of G, we see that for each $1 \leq i \leq n$, $\{u_i, \dots, u_1, v_1, \dots, v_i\} \subseteq V(I_H(F_i))$. Therefore, $\{u_n, v_n\} \subseteq V(F_n)$, for otherwise, $F_n - \{u_n, v_n\}$ is a path or a cycle and $H - V(I(F_n))$ would have just one infinite component (containing $F_n - \{u_n, v_n\}$). Hence, again because any two consecutive vertices of $u_m, \dots, u_1, v_1, \dots, v_m$ are contained in a facial cycle of H, we see that $u_i, v_i \in V(F_i)$ for all $1 \leq i \leq n$. By planarity and because $v_1 F_1 u_1 \subseteq K''$ (or $v_1 F_1 u_1 \subseteq K'' + w_1 w_2$) and $u_1 F_1 v_1 \subseteq K'$, we have $v_i F_i u_i \subseteq K''$ and $u_i F_i v_i \subseteq K'$ for $2 \leq i \leq n$.

We consider two cases.

Case 1. $n = m$.

Then $H - V(I_H(F_m))$ has two infinite blocks (which is used when applying Lemma (4.3)). Note that $u_m \neq v_m$, since $H - V(I_H(F_{m-1}))$ has a unique infinite block containing F_m.

Because of the separation (K', K''), $(H - V(I_H(F_m) - V(F_m))) - \{u_m, v_m\}$ has two infinite components. Hence by planarity and by (3), there is no separation (H', H'') of $I_H(F_m)$ such that $|V(H' \cap H'')| < 2m$, $\{u_m, v_m\} \subseteq V(H' \cap H'')$, $v_m F_m u_m \subseteq H''$, and $u_m F_m v_m \subseteq H'$.

When $|V(K' \cap K'')|$ is even, then $m \geq 2$. Since $v_1 F_1 u_1$ has at least two edges and by (3) and 4-connectivity of G, $I_H(F_2)$ has two disjoint paths from $v_1 F_1 u_1 - v_1$ or from $v_1 F_1 u_1 - u_1$ to $v_2 F_2 u_2$ internally disjoint from $F_1 \cup F_2$. Hence there is an edge $w_1 w_2$ of $v_1 F_1 u_1$ such that v_1, w_1, w_2, u_1 occur on $v_1 F_1 u_1$ in order and there are two disjoint paths in $I_H(F_2)$ from $v_1 F_1 w_1$ to $v_2 F_2 u_2$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$, which are internally disjoint from $F_1 \cup F_2$. When $|V(K' \cap K'')|$ is odd, then by (3), there are two disjoint paths in $I_H(F_2)$ from $v_1 F_1 w_1$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$, which are internally disjoint from $F_1 \cup F_2$. Hence, the conditions of Lemma (4.3) are satisfied.

By Lemma (4.3), there is a 2-way infinite F_1-Tutte path P in H through $w_1 w_2$. When $|V(K' \cap K'')|$ is even, we see that P is a spanning 2-way infinite path in G. When $|V(K' \cap K'')|$ is odd, then $(P - w_1 w_2) + \{w, w w_1, w w_2\}$ is a spanning 2-way infinite path in G.

Case 2. $n < m$.

First, we show that $I_H(F_n) \cap \partial G'' = \emptyset$. For otherwise, $F_n \cap \partial G'' \neq \emptyset$. Let z be a vertex contained in $F_n \cap \partial G''$. Then since $(F_1, \ldots, F_n)$ is a tight partial net in H, there are vertices $z_i \in V(F_i)$, $1 \leq i \leq n$ such that $z_n = z$ and any two consecutive vertices from $z_n, \ldots, z_1$ are contained in a facial cycle of H. Thus, by planarity, either $G - \{v_1, \ldots, v_m, z_1, \ldots, z_n\}$ or $G - \{u_1, \ldots, u_m, z_1, \ldots, z_n\}$ has two infinite components. This contradicts (3) because $n < m$.

Next we show that $H - V(I_H(F_n))$ has two infinite blocks. For otherwise, assume that $H - V(I_H(F_n))$ has just one infinite block, say B. Because $F_n \cap \partial G'' = \emptyset$, it follows from (1) that all vertices of F_n have finite degree in H and each face of H incident with a vertex of F_n is bounded by a cycle. Hence the face of B containing $I_H(F_n)$ is incident with only finitely many vertices and edges of H. Since B is 2-connected, the face of B containing $I_H(F_n)$ is bounded by a non-dividing cycle in H, denoted F_{n+1}. Now it is easy to see that $(F_1, \ldots, F_n, F_{n+1})$ is a tight partial net in H, contradicting the choice of $(F_1, \ldots, F_n)$.

Let B', B'' denote the infinite blocks of $H - V(I_H(F_n))$ such that $C''_j \subseteq B''$ and $C'_j \subseteq B'$ for all sufficiently large j. By planarity, we see that the neighbors of B' on F_n are all contained in $u_n F_n v_n$.

When $|V(K' \cap K'')|$ is even, then $m \geq 2$. Since $v_1 F_1 u_1$ has at least two edges and by (3) and 4-connectivity of G, $I_H(F_2)$ has two disjoint paths from $v_1 F_1 u_1 - v_1$ or from $v_1 F_1 u_1 - u_1$ to $v_2 F_2 u_2$ internally disjoint from $F_1 \cup F_2$. Hence there is an edge $w_1 w_2$ of $v_1 F_1 u_1$ such that v_1, w_1, w_2, u_1 occur on $v_1 F_1 u_1$ in order and there are two disjoint paths in $I_H(F_2)$ from $v_1 F_1 w_1$ to $v_2 F_2 u_2$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$ which are internally disjoint from $F_1 \cup F_2$. When $|V(K' \cap K'')|$ is odd, then by (3), there exist two disjoint paths in $I_H(F_2)$ from $v_1 F_1 w_1$ or from $w_2 F_1 u_1$ to $v_2 F_2 u_2$ which are internally disjoint from $F_1 \cup F_2$. Hence, the conditions of Lemma (4.3) are satisfied.

By Lemma (4.3), we see that H contains a 2-way infinite F_1-Tutte path P through $w_1 w_2$. If $|V(K' \cap K'') - \{x, y\}|$ is even, then P is a spanning 2-way infinite path in G. If $|V(K' \cap K'') - \{x, y\}|$ is odd, then $(P - w_1 w_2) + \{w, ww_1, ww_2\}$ is a spanning 2-way infinite path in G. □

It is easy to see that Theorem (1.1) follows from Theorems (5.1), (6.1), and (6.2).

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