The Binding Number of a Digraph*

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Abstract. Caccetta-Häggkvist's Conjecture discusses the relation between the girth $g(D)$ of a digraph D and the minimum outdegree $\delta^+(D)$ of D. The special case when $g(D) = 3$ has lately attracted wide attention. For an undirected graph G, the binding number $bind(G) \geq \frac{3}{2}$ is a sufficient condition for G to have a triangle (cycle with length 3). In this paper we generalize the concept of binding numbers to digraphs and give some corresponding results. In particular, the value range of binding numbers is given, and the existence of digraphs with a given binding number is confirmed. By using the binding number of a digraph we give a condition that guarantees the existence of a directed triangle in the digraph. The relationship between binding number and connectivity is also discussed.

Key words: binding number; girth; directed graph; Caccetta-Häggkvist Conjecture

1 Introduction

Throughout the paper we consider only simple digraphs without loops and parallel arcs. Terminology and notation not defined here can be found in [4].

Let $D = (V(D), A(D))$ be a digraph. For a vertex $v \in V(D)$, by $N_D^+(v)$ and $N_D^-(v)$ we denote the set of outneighbors and the set of inneighbors of v respectively, i.e., $N_D^+(v) = \{u \in V(D) \mid vu \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) \mid uv \in A(D)\}$. For a subset S of $V(D)$, we use $N_D^+(S)$ and $N_D^-(S)$ for $\cup_{v\in S} N_D^+(v)$ and $\cup_{v\in S} N_D^-(v)$, respectively. The *outdegree* and *indegree* of v, denoted by $d_D^+(v)$ and $d_D^-(v)$ respectively, are defined as $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$. We set $\delta^+(D) = \min\left\{d_D^+(v) \mid v \in V(D)\right\}$, $\delta^-(D) = \min\left\{d_D^-(v) \mid v \in V(D)\right\}$ and $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. When no confusion occurs, we use $N^+(v)$, $N^-(v)$, $N^+(S)$, $N^-(S)$, $d^+(v)$, $d^-(v)$, δ^+, δ^- and δ for

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$N_D^+(v)$, $N_D^-(v)$, $N_D^+(S)$, $N_D^-(S)$, $d_D^+(v)$, $d_D^-(v)$, $\delta^+(D)$, $\delta^-(D)$ and $\delta(D)$, respectively. D is called *d-regular* if $d^+(v) = d^-(v) = d$ for every vertex $v \in V(D)$. The girth of D, denoted by $g(D)$, is the length of a shortest directed cycle of D. In the following, we always assume $|V(D)| = n$ and $\delta(D) = \delta$.

In 1970, Behzad et al. [1] proposed the following

Conjecture 1. [1] Let D be a d-regular digraph. Then $g(D) \leq \lceil \frac{n}{d} \rceil$.

Caccetta and Häggkvist made the following more general conjecture in [7].

Conjecture 2. (Caccetta-Häggkvist Conjecture [7]) Let D be a digraph with $\delta^+(D) \geq d$. Then $g(D) \leq \lceil \frac{n}{d} \rceil$.

These conjectures have received much attention in recent years. Conjecture 1 has been proved for $d = 2$ by Behzad [2] and for $d = 3$ by Bermond [3]. Conjecture 2 has been proved for $d = 2$ by Caccetta and Häggkvist [7], for $d = 3$ by Hamidoune [11] and for $d = 4, 5$ by Hoàng and Reed [12]. In [8], Chvátal and Szemerédi established the bound $\frac{n}{d} + 2500$ for arbitrary values of d. Nishimura [14] reduced the additive constant from 2500 to 304. Some further developments on these conjectures can be found in [13, 16, 17].

An interesting special case of Conjecture 2 is: Any directed graph with n vertices and minimum outdegree at least $n/3$ has a directed triangle (a directed cycle of length 3). Several papers have been devoted to this special case, see Bondy [5], Shen [15], Graaf et. al [10], Goodman [9], and Broersma and Li [6]. However, this conjecture has not been resolved to date.

The corresponding problem on the existence of triangles in undirected graphs has been studied by considering a graph invariant called the binding number of a graph. For a simple graph G without loops and multiple edges, its *binding number* is defined as

$$bind(G) = \min_{\substack{\emptyset \neq S \subseteq V(G) \\ N(S) \neq V(G)}} \left\{ \frac{|N(S)|}{|S|} \right\},$$

where $N(S) = \{u \,|\, uv \in E(G), v \in S\}$. This parameter was introduced by Woodall [19] in 1973. He conjectured that: If a graph G has $bind(G) \geq 3/2$, then it contains a triangle. This conjecture was confirmed by Shi in 1984.

Theorem 1. **[18]** *Let G be a simple graph with $bind(G) \geq \frac{3}{2}$. Then G contains a triangle.*

In this paper, motivated by the concept of binding number of graphs, we introduce the binding number of a digraph in Section 2. Some basic results on this parameter are also obtained in this section. In Section 3, we show that a digraph with binding number at least $\frac{\sqrt{5}+1}{2}$ has a directed triangle. Two conjectures on the girth of a digraph in terms of the binding number are also posed.

2 The Binding Number of a Digraph

Definition 1. *The binding number $bind(D)$ of a digraph D is defined as*

$$bind(D) = \min_{\substack{\emptyset \neq S \subseteq V(D) \\ N(S) \neq V(D)}} \left\{ \frac{|N(S)|}{|S|} \right\},$$

$$\text{where } N(S) = \begin{cases} N^+(S), & \text{if } |N^+(S)| < |N^-(S)|; \\ N^+(S) \text{ or } N^-(S), & \text{if } |N^+(S)| = |N^-(S)|; \\ N^-(S), & \text{if } |N^+(S)| > |N^-(S)|. \end{cases}$$

By the definition, for any digraph D, the binding number $bind(D)$ is a rational number, and for any arc $uv \notin A(D)$, we have $bind(D) \leq bind(D + uv)$. Let v be the vertex of D with $\delta = d^+(v)$ or $d^-(v)$. Set $S = \{v\}$. Then we have $bind(D) \leq \frac{|N(S)|}{|S|} = \delta$. The following theorem provides an alternative definition of the binding number.

Theorem 2. *Let D be a digraph. Then $bind(D) = \min_{\substack{\emptyset \neq S \subseteq V(D) \\ N(S) \neq V(D)}} \left\{ \frac{|V(D)|-|S|}{|V(D)|-|N(S)|} \right\}$.*

Proof. Firstly, we prove that $bind(D) \leq \frac{|V(D)|-|S|}{|V(D)|-|N(S)|}$ for any nonempty set $S \subseteq V(D)$ with $N(S) \neq V(D)$.

Given any nonempty set $S \subseteq V(D)$ with $N(S) \neq V(D)$. If $|N(S)| = |N^+(S)| \leq |N^-(S)|$, then let $T = V(D) \setminus N^+(S)$. If $s \in S$, then $s \notin N^-(T)$, so we know $N^-(T) \subseteq V(D) \setminus S$. By the definition of the binding number, we have $bind(D) \leq \frac{|N^-(T)|}{|T|} \leq \frac{|V(D) \setminus S|}{|T|} = \frac{|V(D)|-|S|}{|V(D)|-|N^+(S)|}$.

If $|N(S)| = |N^-(S)| \leq |N^+(S)|$, then we can show similarly that $bind(D) \leq \frac{|V(D)|-|S|}{|V(D)|-|N^-(S)|}$. Hence $bind(D) \leq \frac{|V(D)|-|S|}{|V(D)|-|N(S)|}$.

Next, we show that there exists a nonempty set $S_0 \subseteq V(D)$ such that $bind(D) = \frac{|V(D)|-|S_0|}{|V(D)|-|N(S_0)|}$. By the definition of the binding number, let the nonempty set $T_0 \subseteq V(D)$ be the vertex set satisfying $bind(D) = \frac{|N(T_0)|}{|T_0|}$. Without loss of generality, we may assume that $N(T_0) = N^+(T_0)$. Let $S_0 = V(D) \setminus N^+(T_0)$. Then $T_0 \cap N^-(S_0) = \emptyset$.

(1) If $bind(D) = 0$, then $N^+(T_0) = \emptyset$ and $S_0 = V(D)$. Moreover, $N(S_0) \neq V(D)$ since $T_0 \nsubseteq N^-(S_0)$. Therefore $\frac{|V(D)|-|S_0|}{|V(D)|-|N(S_0)|} = 0 = bind(D)$.

(2) If $bind(D) > 0$, we have $\delta \geq 1$. And if $t \notin T_0$, then it should be that $N^+(t) \nsubseteq N^+(T_0)$ and $N^+(t) \neq \emptyset$. Otherwise, let $T_1 = \{t\} \cup T_0$, we have $bind(D) = \frac{|N(T_0)|}{|T_0|} > \frac{|N(T_1)|}{|T_1|}$, which is a contradiction to the definition of $bind(D)$. Hence $N^+(t) \cap S_0 \neq \emptyset$, that is, $t \in N^-(S_0)$. So we have $T_0 \cup N^-(S_0) = V(D)$. We also have $T_0 = V(D) \setminus N^-(S_0)$ since $T_0 \cap N^-(S_0) = \emptyset$. Therefore $bind(D) = \frac{|N^+(T_0)|}{|T_0|} = \frac{|V(D) \setminus S_0|}{|V(D) \setminus N^-(S_0)|} = \frac{|V(D)|-|S_0|}{|V(D)|-|N(S_0)|}$.

This completes the proof. $\square$

The next two corollaries follow from Theorem 2 immediately.

Corollary 1. *Let D be a digraph. Then $bind(D) \le \frac{n-1}{n-\delta}$.*

Proof. Assume that $d^+(v) = \delta$, and let $S = \{v\}$. Then $|N(S)| = \delta$. By Theorem 2, we have $bind(D) \le \frac{n-|S|}{n-|N(S)|} = \frac{n-1}{n-\delta}$. □

Corollary 2. *Let D be a digraph. Then $0 \le bind(D) < 2$.*

Proof. It is trivial that $bind(D) \ge 0$. Since D has no loops and parallel arcs, it is trivial that $\delta \le \lfloor \frac{n-1}{2} \rfloor$ for the digraph D. By Corollary 1, we have $bind(D) \le \frac{n-1}{n-\delta}$. It follows immediately that $bind(D) < 2$.

Hence $0 \le bind(D) < 2$. □

From Corollary 2, we see that the binding number of a digraph is a rational number in $[0, 2)$. A natural question is: For a given rational number $r \in [0, 2)$, does there exist a digraph with binding number r? The answer is positive.

Theorem 3. *For every rational number $r \in [0, 2)$, there exits a digraph D with $bind(D) = r$.*

Proof. Suppose that $r = \frac{p}{q}$, where p is a nonnegative integer and q is a positive integer. We distinguish two cases.

Case 1. $0 \le r \le 1$.

Let $K(p, q, p)$ be a complete tripartite graph with three parts of vertex set S_1, S_2, S_3, and $|S_1| = |S_3| = p$, $|S_2| = q$. We construct a digraph D by orienting $K(p, q, p)$ such that the arc set $A(D) = (S_1, S_2) \cup (S_2, S_3) \cup (S_3, S_1)$, where (S_i, S_j) denotes the set of all possible arcs from S_i to S_j.

It is easy to verify that the binding number of the digraph $D = \overrightarrow{K(p, q, p)}$ satisfies $bind(D) = \frac{p}{q}$.

Case 2. $1 < r < 2$.

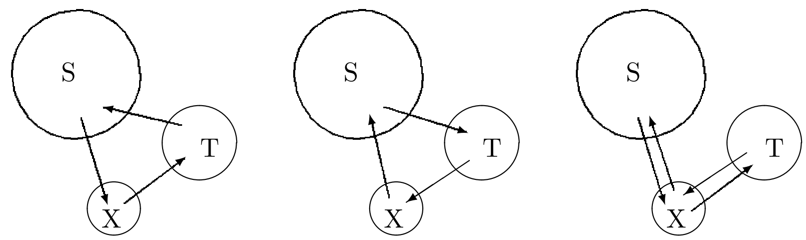
Assume that $d = p-q+1$. We construct a circulant digraph $D = C_S(N)$ such that $n = p+1$, and symbol set $S = \{1, 2, \ldots, d\}$. Here, a *circulant digraph* $C_S(N)$ is defined as the digraph with vertices the elements of $N = \{0, 1, 2, \ldots, n-1\}$ and arcs all pairs of the form $(i, i+s \pmod{n})$ with $i \in N$ and $s \in S$. By the definition of the binding number, for any nonempty vertex set $T \subseteq V(D)$ such that $N(T) \ne V(D)$, it is easy to check that if $T = \{0, 1, 2 \ldots, q-1\}$, then $N(T) = N^+(T) = \{1, 2, \ldots, q, q+1, \ldots, q+d-1\}$ and $bind(D) = \frac{|N(T)|}{|T|} = \frac{p}{q}$. □

Theorem 4. *Let D be a digraph.*

(1) *If D is not strongly connected, then $bind(D) \le 1$;*
(2) *If D is k-connected, but not $(k+1)$-connected, then $bind(D) \le \frac{n+k}{n-k}$.*

Proof. (1) Since D is not strongly connected, there exists a vertex set $S \subset V(D)$ such that $\left(S, \bar{S}\right) = \emptyset$ or $\left(\bar{S}, S\right) = \emptyset$, where $\left(S, \bar{S}\right)$ means the set of arcs from S to $\bar{S}$ in D. Without loss of generality, let $\left(S, \bar{S}\right) = \emptyset$. By the definition of $bind(D)$, we have $bind(D) \leq \frac{|N^+(S)|}{|S|} \leq \frac{|S|}{|S|} = 1$.

(2) Since D is $k-$connected but not $(k+1)$-connected, there exists a vertex set $X \subset V(D)$ with $|X| = k$ such that $D \setminus X$ is not strongly connected. Let T be the smallest strongly connected component of $D \setminus X$. Clearly $|T| \leq \frac{1}{2}(n - \kappa)$. Let $S = V(D) - (X \cup T)$. Then we have $|S| \geq \frac{1}{2}(n - \kappa)$. Considering $N(S)$, we distinguish three cases of D as shown in Fig. 1.



Case 1 Case 2 Case 3

Fig 1. Three cases of D

Note that, in Fig. 1 an arrow from set A to B indicates that three exist arcs from some vertices in A to some vertices in B. Note that X is a minimum vertex cutset to destroy the strongly connectedness of D. So, for any vertex x in X, there are an arc coming to x as well as an arc going out from x. Moreover, there are at least $|X|$ vertices in T belonging to $N^-(S)$ in Case 1 and belonging to $N^+(S)$ in Case 2. So it is easy to see that $|N(S)| \leq |X| + |S|$.

Hence $bind(D) \leq \frac{|N(S)|}{|S|} \leq \frac{|X|+|S|}{|S|} \leq \frac{n+k}{n-k}$. □

The following result gives some properties of the binding numbers for some special digraphs.

Theorem 5. *Let D be a digraph.*

(1) *If $\delta = 0$, then $bind(D) = 0$;*
(2) *If D is a directed cycle, then $bind(D) = 1$;*
(3) *If D is a bipartite digraph with partitions (A, B), then $bind(D) \leq \min\left\{\frac{|A|}{|B|}, \frac{|B|}{|A|}\right\} \leq 1$;*
(4) *If D is a tournament, then $bind(D) = 0$ or $bind(D) \geq \frac{1}{2}$.*

Proof. (1) and (2) are obvious.

(3) It is clear that $N(S) \subseteq B$ if $S = A$. Similarly, $N(S) \subseteq A$ if $S = B$. Then we immediately have $bind(D) \leq \min\left\{\frac{|N(S)|}{|S|}\right\} \leq \min\left\{\frac{|A|}{|B|}, \frac{|B|}{|A|}\right\} \leq 1$.

(4) If $\delta = 0$, we know $bind(D) = 0$ by conclusion (1). Otherwise, since D is a tournament, for any pair of vertices $u, v \in V(D)$, it should be that $uv \in A(D)$

or $vu \in A(D)$. So at most one of $d^+(u) = 0$ and $d^+(v) = 0$ is true, the same to $d^-(u) = 0$ and $d^-(v) = 0$. For any $S \subseteq V(D)$, the deduced subdigraph by S is also tournament, so $|N(S)| = \min\{|N^+(S)|, |N^-(S)|\} \geq |S| - 1$. Let S_0 be the nonempty vertex set such that $bind(D) = \frac{|N(S_0)|}{|S_0|}$. If $|S_0| = 1$, then $bind(D) = 0$ or $bind(D) \geq 1 > \frac{1}{2}$. And if $|S_0| \geq 2$, then $bind(D) = \frac{|N(S_0)|}{|S_0|} \geq \frac{|S_0|-1}{|S_0|} \geq \frac{1}{2}$.

Hence $bind(D) = 0$ or $bind(D) \geq \frac{1}{2}$. □

3 The Binding Number and Girth

As we pointed out earlier, Shi proved that a graph with binding number at least 3/2 has a triangle. For digraphs, we can prove the following result.

Theorem 6. *Let D be a digraph with $bind(D) \geq \frac{\sqrt{5}+1}{2}$. Then $g(D) = 3$.*

Proof. We prove it by contradiction. Assume that there is a digraph D with $bind(D) \geq \frac{1+\sqrt{5}}{2}$ and D does not contain a directed cycle of length 3.

Let $\delta = d^+(v)$. Then $(N^+(v), N^-(v)) = \emptyset$ since D has no directed triangles. Denote $S = N^+(v)$. Then $N^+(S) \cap N^-(v) = \emptyset$, and $|N^-(v)| \geq \delta$. Therefore $bind(D) \leq \frac{|N^+(S)|}{|S|} \leq \frac{|V(D)|-|N^-(v)|}{|S|} \leq \frac{n-\delta}{\delta}$, hence $\delta \leq \frac{n}{bind(D)+1}$.

By Corollary 1, we have $bind(D) \leq \frac{n-1}{n-\delta}$, so $\delta \geq \frac{(bind(D)-1)n+1}{bind(D)}$. Therefore $\frac{(bind(D)-1)n+1}{bind(D)} \leq \delta \leq \frac{n}{bind(D)+1}$, it follows that $n(bind(D))^2-(n-1)bind(D)-n+1 \leq 0$. So we have $bind(D) \leq \frac{n-1+\sqrt{5n^2-6n+1}}{2n} < \frac{1+\sqrt{5}}{2}$, which is a contradiction to the hypothesis. This completes the proof. □

Similar to the undirected graph case, we have the following conjectures.

Conjecture 3. Let D be digraph with $bind(D) \geq \frac{3}{2}$. Then $g(D) = 3$.

The following more general conjecture may also hold.

Conjecture 4. Let D be a digraph with $bind(D) \geq \frac{k}{k-1}$ $(k \geq 3)$. Then $g(D) \leq k$.

In fact, if $bind(D) \geq \frac{k}{k-1}$ $(k \geq 3)$, then it follows from Corollary 1 that $\delta \geq \frac{n+k-1}{k} > \frac{n}{k}$. Therefore, Caccetta-Häggkvist Conjecture is stronger than Conjecture 4. And, Conjecture 3 corresponds to Caccetta-Häggkvist Conjecture in the case of $\delta \geq \frac{n}{3}$.

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