Generalization of matching extensions in graphs (II) *

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Abstract

Proposed as a general framework, Liu and Yu [4] (*Discrete Math.* 231 (2001) 311-320) introduced (n, k, d)-graphs to unify the concepts of deficiency of matchings, n-factor-criticality and k-extendability. Let G be a graph and let n, k and d be non-negative integers such that $n + 2k + d \leq |V(G)| - 2$ and $|V(G)| - n - d$ is even. If when deleting any n vertices from G, the remaining subgraph H of G contains a k-matching and each such k-matching can be extended to a defect-d matching in H, then G is called an *(n, k, d)-graph.* In [4], the recursive relations for distinct parameters n, k and d were presented and the impact of adding or deleting an edge also was discussed for the case $d = 0$. In this paper, we continue the study begun in [4] and obtain new recursive results for (n, k, d)-graphs in the general case $d \geq 0$.

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1. Introduction

In this paper we consider only finite, undirected and simple graphs. Denote by $N_G(x)$ set of neighbors of a vertex x in G. If no confusion occurs, we write $N(x)$ for $N_G(x)$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A matching of k edges is called a *k-matching*. Let d be a non-negative integer. A matching is called a *defect-d* matching of G if it covers exactly $|V(G)| - d$ vertices of G. Clearly, a defect-0 matching is a perfect matching. A necessary and sufficient condition for a graph to have a defect-d matching was given by Berge [1].

Theorem 1.1 *(Berge [1]) Let G be a graph and let d be an integer such that $0 \le d \le |V(G)|$ and $|V(G)| \equiv d \pmod 2$. Then G has a defect-d matching if and only if for any $S \subseteq V(G)$*

$$o(G - S) \le |S| + d.$$

For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S and we write $G - S$ for $G[V(G) \setminus S]$. The number of odd components of G is denoted by $o(G)$. Let M be a matching of G. If there is a matching M' of G such that $M \subseteq M'$, then we say that M can be extended to M' or M' is an *extension* of M. If each k-matching can be extended to a perfect matching in G, then G is called *k-extendable*. To avoid triviality, we require that $|V(G)| \ge 2k+2$ for k-extendable graphs. This family of graphs was instroduced by Plummer [6] and studied extensively by Lovász and Plummer [5].

A graph G is called *n-factor-critical* if after deleting any n vertices the remaining subgraph of G has a perfect matching. This concept is introduced by Favaron [2] and Yu [8], independently, which is a generalization of the notions of the well-known factor-critical graphs and bicritical graphs (the cases of $n = 1$ and $n = 2$). Characterizations of n-factor-critical graphs, properties of n-factor-critical graphs and its relationships with other graphic parameters (e.g., degree sum, toughness, binding number, connectivity, etc.) have been discussed in [2], [3] and [8].

Let G be a graph and let n, k and d be non-negative integers such that $|V(G)| \ge n + 2k + d + 2$ and $|V(G)| - n - d$ is even. If when deleting any n vertices from G, the remaining subgraph of G contains a k-matching and each of such k-matchings can be extended to a defect-d matching in the subgraph, then G is called an *(n, k, d)-graph*. This term was introduced by Liu and Yu [4] as a general framework to unify the concepts of defect-d matchings, n-factor-criticality and k-extendability. In particular, $(n, 0, 0)$-graphs are exactly n-factor-critical graphs and $(0, k, 0)$-graphs are just the same as k-extendable graphs. This framework enables the authors to prove a series of general results which include many earlier results of matchig theory as special cases. In [4], Liu and Yu provided the following necessary and sufficient conditions for a graph to be an (n, k, d)-graph.

Theorem 1.2 *A graph G is an (n,k,d)-graph if and only if the following conditions are satisfied.*

(i) For any $S \subseteq V(G)$ and $|S| \geq n$, then

$$o(G-S) \leq |S| - n + d.$$

(ii) For any $S \subseteq V(G)$ such that $|S| \geq n+2k$ and $G[S]$ contains a k-matching,

$$o(G-S) \leq |S| - n - 2k + d.$$

Besides necessary and sufficient conditions, one interesting problem is to find recursive relations for different parameters n, k and d. Here, we list some of the relevant results (i.e., Theorems 1.3-1.6) presented in [4] for the convenience of the reader.

Theorem 1.3 *Every (n,k,d)-graph G is also an (n', k', d)-graph where $0 \leq n' \leq n$, $0 \leq k' \leq k$ and $n' \equiv n \pmod 2$.*

In particular, for $d = 0$, the following result was proved.

Theorem 1.4 *If G is an $(n,k,0)$-graph and $n \geq 1$, $k \geq 2$, then G is a $(n+2, k-2, 0)$-graph.*

The authors in [4] also considered other recursive properties of (n,k,d)-graphs, for instance, determining the parameters $n^{'}, k^{'}$ and $d^{'}$ such that, when adding or deleting an edge from an (n,k,d)-graph, the resulting graph is a $(n^{'}, k^{'}, d^{'})$-graph. The focus in [4] is mostly on the case of $d = 0$ and obtained several interesting results. For graphs obtained by adding an edge to an (n,k,d)-graph, the following result was shown.

Theorem 1.5 *Let G be an $(n,k,0)$-graph with $n, k \geq 1$. Then for any edge $e \notin E(G)$, $G \cup e$ is an $(n, k-1, 0)$-graph.*

Moreover, for graphs obtained by deleting an edge from an (n,k,d)-graph, there is the following result.

Theorem 1.6 *Let G be an $(n,k,0)$-graph, $n \geq 2$ and $k \geq 1$. Then for any edge e of G,*

(i) $G - e$ is an $(n-2, k, 0)$-graph.

(ii) $G - e$ is an $(n, k-1, 0)$-graph.

Note that the recursive results for $d > 0$ are not investigated in [4]. In this paper, our main focus is to extend Theorems 1.4 - 1.6 to the case of $d \geq 0$. The results are natural extensions of those in the case of $d = 0$, but the proofs are somewhat more involved. Section 2 is devoted to recursive relations for graphs obtained by adding an edge to an (n, k, d)-graph. Section 3 presents a recursive relation for graphs obtained by adding a vertex. Similar recursive results for graphs obtained by deleting an edge from an (n, k, d)-graph are presented in Section 4.

2. Recursive relations for adding an edge

In this section, we consider recursive relations for graphs obtained by adding an edge to an (n, k, d)-graph. First we have the following result.

Theorem 2.1 *For any $n > d \geq 0$ and $k \geq 1$, if G is an (n, k, d)-graph, then $G \cup e$ is an $(n, k-1, d)$-graph for any $e \notin E(G)$.*

Proof. For $k = 1$, since G is an $(n, 1, d)$-graph, by Theorem 1.3, it is also an $(n, 0, d)$-graph. Hence $G \cup e$ is an $(n, 0, d)$-graph.

So assume that $k \geq 2$. If $G \cup e$ is not an $(n, k-1, d)$-graph for some edge $e \notin E(G)$, then there exists an n-subset $S' \subseteq V(G)$ and a $(k-2)$-matching $M' = \{x_1y_1, x_2y_2, \ldots, x_{k-2}y_{k-2}\}$ such that the $(k-1)$-matching $e \cup M'$ can not be extended to a defect-d matching of $G - S'$. Let $e = xy$ and $S'' = V(M')$. By Theorem 1.1, there exists a vertex set $S_1 \subseteq G - S' - S'' - x - y$ such that $o(G - S' - S'' - x - y - S_1) \geq |S_1| + d + 1$. Since G is an (n, k, d)-graph, according to Theorem 1.3, it is also an $(n, k-2, d)$-graph. From Theorem 1.2 (ii), $o(G - S' - S'' - x - y - S_1) \leq o(G - S' - S'' - S_1) + 2 \leq |S_1| + d + 2$. By a simple parity argument, we have $o(G - S' - S'' - x - y - S_1) = |S_1| + d + 2$. Let $S_2 = S_1 \cup \{x, y\}$. Then, $o(G - S' - S'' - S_2) = |S_2| + d$.

Claim 1. $S' \cup S_2$ is an independent set in G.

Suppose $e_1 = uv$ is an edge in $G[S' \cup S_2]$. Then $uv \cup M'$ is a $(k-1)$-matching. Let $S = (S' \cup S_2 - u - v) \cup (S'' \cup \{u, v\})$ which is of order $|S_2| + n + 2(k-1) - 2$ and contains a $(k-1)$-matching. Since G is an (n, k, d)-graph, according to Theorem 1.3, G is also an $(n, k-1, d)$-graph. Then from Theorem 1.2 (ii) and recall the fact that $|S_2| \geq 2$, we have

$$o(G - S' - S'' - S_2) = o(G - S) \leq |S| - n - 2(k-1) + d = |S_2| + d - 2,$$

a contradiction.

Let $H = G - S' - S'' - S_2$.

Claim 2. No even component of H is connected to $S' \cup S_2$.

Assume that there is an edge, say $e_2 = uv$, joining an even component C of H to $S_2 \cup S'$, where $u \in S' \cup S_2$ and $v \in V(C)$. Then $e_2 \cup M'$ is a $(k-1)$-matching. Let $S = (S' \cup S_2 - u) \cup (S'' \cup \{u, v\})$ which is of order $n - 1 + |S_2| + 2(k-1)$ and contains a $(k-1)$-matching. Since G is an (n, k, d)-graph, it is also an $(n, k-1, d)$-graph. Hence Theorem 1.2 (ii) implies that $o(G-S) \leq |S| - n - 2(k-1) + d = |S_2| - 1 + d$. However, since the total number of odd components increases by at least one upon deleting v from the even component C, we have that $o(G-S) \geq o(G - S' - S'' - S_2) + 1 = |S_2| + d + 1$, a contradiction.

Claim 3. For every odd component O of H, there do not exist two independent edges $e_3 = u_1v_1$ and $e_4 = u_2v_2$ joining O to $S' \cup S_2$, where $u_1, u_2 \in S' \cup S_2$ and $v_1, v_2 \in V(O)$.

Suppose, to the contrary, that e_3 and e_4 are two such edges. Then $e_3 \cup e_4 \cup M'$ is a k-matching. Let $S = (S' \cup S'' - u_1 - u_2) \cup (S'' \cup \{u_1, u_2, v_1, v_2\})$ which is of order $|S_2| + n - 2 + 2k$ and contains a k-matching. Since G is an (n, k, d)-graph, then according to Theorem 1.2 (ii), we have

$$o(G-S) \leq |S| - n - 2k + d = |S_2| + n - 2 + 2k - n - 2k + d = |S_2| - 2 + d.$$

However, since the total number of odd components does not decrease by deleting v_1 and v_2 from the odd component O, we have $o(G-S) \geq o(G - S' - S'' - S_2) = |S_2| + d$, a contradiction.

According to Claim 3, we conclude that for any odd component O of H, if it is connected to S_2 or S' in graph $G - S''$, then either $|N(V(O)) \cap (S' \cup S_2)| = 1$ or $|N(S' \cup S_2) \cap V(O)| = 1$.

Since G is an (n, k, d)-graph, $G - S''$ is an $(n, 2, d)$-graph by Theorem 1.6 (ii). Suppose that there are h odd components connected to neither S' nor S_2, and t odd components $C_1, C_2, \ldots, C_t$ with $|N(S' \cup S_2) \cap V(C_i)| = 1$, $1 \leq i \leq t$, and $p = |S_2| + d - h - t$ odd components $D_1, D_2, \ldots, D_p$ with $|N(V(D_i)) \cap (S' \cup S_2)| = 1$, $1 \leq i \leq p$. Then $h + t + p = |S_2| + d$. Let $U = \bigcup_{i=1}^{p} N(V(D_i)) \cap (S' \cup S_2) = \{u_1, u_2, \ldots, u_q\}$. We consider the following three cases:

Case 1. $n \leq t$. Let $S_3 = \bigcup_{i=1}^{n} V(C_i) \cap N(S' \cup S_2)$. Then $|S_3| = n$. Now we consider the n-set S_3 and $(k-2)$-matching $M^{'}$. From Claim 1, $S' \cup S_2$ is an independent set in $G - S''$. In $G - S'' - S_3$, $S' \cup S_2$ must be matched by vertices of $|S_2| + d - h - n$ odd components from $C_{n+1}, C_{n+2}, \ldots, C_t, D_1, D_2, \ldots, D_p$ and any maximum matching of $G - S'' - S_3$ must miss at least one vertex from each of h odd components which is connected to neither S' nor S''. Altogether, a maximum matching of $G - S'' - S_3$ will miss at least

$$h + |S_2| + n - (|S_2| + d - h - n) = 2n + 2h - d \geq d + 2$$

vertices (recall that $n > d \geq 0$), which contradicts to the fact that $G - S''$ is an $(n, 2, d)$-graph.

Case 2. $t < n \leq q+t$. Let $S_3 = (\bigcup_{i=1}^{t} V(C_i) \cap N(S' \cup S_2)) \bigcup \{u_1, u_2, \ldots, u_{n-t}\}$. Now we consider the n-set S_3 and $(k-2)$-matching $M^{'}$. Suppose that there are f odd components $D_{i_1}, D_{i_2}, \ldots, D_{i_f}$ among $D_1, D_2, \ldots, D_p$ which are connected to $\{u_1, u_2, \ldots, u_{n-t}\}$ in $G - S''$. It is obvious that $f \geq n-t$. Note that each vertex of $(S' \cup S_2) - S_3$ can only be matched by vertices from $|S_2| + d - h - t - f$ odd components $\{D_1, D_2, \ldots, D_p\} \setminus \{D_{i_1}, D_{i_2}, \ldots, D_{i_f}\}$ in $G - S'' - S_3$. Furthermore, any maximum matching of $G - S'' - S_3$ must miss at least one vertex from D_{i_j}, $1 \leq j \leq f$, and at least one vertex from each of h odd components which is connected to neither S' nor S''. Thus any maximum matching of $G - S'' - S_3$ must miss at least

$$\begin{aligned} f + h + |S_2| + n - (n-t) - (|S_2| + d - h - f - t) &= 2h + 2t + 2f - d \\ &\geq 2h + 2t + 2n - 2t - d \\ &\geq d + 2 \end{aligned}$$

vertices, which implies that $G - S''$ is not an $(n, 2, d)$-graph, a contradiction again.

Case 3. $n > q+t$. Let $S_3 = (\bigcup_{i=1}^{t} V(C_i) \cap N(S' \cup S_2)) \bigcup U \bigcup S_4$, where $S_4 \subseteq S' \cup S_2 - U$ and $|S_4| = n - q - t$. Now we consider the n-set S_3 and $(k-2)$-matching $M^{'}$. Note that any maximum matching of $G - S'' - S_3$ must miss at least one vertex from each of the h odd components connected to neither S' nor S_2 and at least one vertex from $|S_2| + d - h - t$ odd components $D_1, D_2, \ldots, D_p$. Furthermore, $|S_2| + n - (n-t)$ vertices of $S' \cup S_2 - S_3$ must be missed by any maximum matching of $G - S'' - S_3$. Thus any maximum matching of $G - S'' - S_3$ must miss at least

$$h + |S_2| + d - h - t + |S_2| + n - (n-t) = 2|S_2| + d \geq d + 4$$

vertices ($|S_2| \geq 2$), which implies that $G - S''$ is not an $(n, 2, d)$-graph, a contradiction again.

This completes the proof. ∎

Suppose $n, k \geq 1$. Clearly Theorem 1.5 is a special case of Theorem 2.1. Note that the additional condition $n > d$ in Theorem 2.1 is necessary. For example, consider a complete bipartite graph $K_{3,d+2}$ with bipartition $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_{d+2}\}$. Let H be a graph obtained by replacing each w_i by a complete graph K_{2m+1}, $1 \leq i \leq d+2$. Obviously, H is a $(1, 2, d)$-graph, but $H \cup u_1u_2$ is not a $(1, 1, d)$-graph for $d > 0$. An interesting property of the graph H is that H is a $(1, 2, d)$-graph, but not a $(3, 0, d)$-graph for $d > 0$. So the conclusion of Theorem 1.4 does not always hold for $n > d > 0$.

Similarly, under the additional condition $n > d$, we have the following result which extends Theorem 1.4 to the case of $d > 0$.

Theorem 2.2 *For any $n > d \geq 0$ and $k \geq 2$, if G is an (n, k, d)-graph, then G is also an $(n+2, k-2, d)$-graph.*

Proof. Suppose that G is not an $(n+2, k-2, d)$-graph. Then there exist a vertex set S' of order $n+2$ and $(k-2)$-matching M' such that M' can not be extended to a defect-d matching of $G - S'$, i.e., $G - S' - S''$ has no defect-d matchings.

Claim. S' is an independent set in G.

If $e = uv$ is an edge in $G[S']$, then $e \cup M'$ can be extended to a defect-d matching of $G - (S' - u - v)$ since G is an $(n, k-1, d)$-graph, i.e., $G - S' - V(M')$ has a defect-d matching, a contradiction.

Let u, v be two vertices in S' and $G' = G \cup uv$. By Theorem 2.1, G' is an $(n, k-1, d)$-graph. That is, $uv \cup M'$ can be extended to a defect-d matching M of $G - (S' - \{u, v\})$. Then M is also a defect-d matching of $G - S'$ which contains M', a contradiction.

This completes the proof. ∎

3. Recursive relation for adding a vertex

Let G be a graph and $x \notin V(G)$. Denote by $G + x$ the graph obtained by joining each vertex of G to x. Here we consider the recursive result of adding a vertex to an (n, k, d)-graph.

Theorem 3.1 *Let G be an (n, k, d)-graph with $k > 0$ and $n > d$. Then $G + x$ is an $(n+1, k-1, d)$-graph for any vertex $x \notin V(G)$.*

Proof. Denote $G' = G + x$. Let S be an $(n+1)$-set of $V(G')$ and M' a $(k-1)$-matching of $G' - S$. We consider the following cases:

Case 1. $x \in S$. Since G is an (n, k, d)-graph, it is also an $(n, k-1, d)$-graph. Let $S' = S - \{x\}$. Then M' can be extended to a defect-d matching M of $G - S'$. Then M is also a defect-d matching of $G' - S$ which contains the $(k-1)$-matching M'.

Case 2. $x \in V(M')$. Let xy be an edge of the $(k-1)$-matching M'. If $N(y) \cap S \neq \emptyset$, then let $z \in N(y) \cap S$. Then $M'' = (M' - xy) \cup yz$ is a $(k-1)$-matching and $S'' = S - \{z\}$ is an n-set. Hence M'' can be extended to a defect-d matching M of $G - S''$. Then $(M - \{yz\}) \cup \{xy\}$ is also a defect-d matching of $G' - S$ which contains M'. If $N(y) \cap S = \emptyset$, then let z be any vertex of S. According to Theorem 2.1, $G \cup yz$ is an $(n, k-1, d)$-graph. Then $M'' = (M' - xy) \cup yz$ be a $(k-1)$-matching and $S'' = S - \{z\}$ is an n-set. Hence M'' can be extended to a defect-d matching M of $(G \cup yz) - S''$. Then $(M - \{yz\}) \cup \{xy\}$ is also a defect-d matching of $G' - S$ which contains M'.

Case 3. $x \in V(G) - S - V(M')$. Since G is an (n, k, d)-graph, G is an $(n, k-1, d)$-graph. Let y be any vertex of S and set $S' = S - y$. Then M' can be extended to a defect-d matching M of $G - S'$. Then $d_M(y) = 0$ or $d_M(y) = 1$. If $d_M(y) = 0$, then it is obvious that M is also a defect-d matching of $G' - S$ which contains M'. If $d_M(y) = 1$, let $N_M(y) = z$. Then $(M - yz) \cup xz$ is also a defect-d matching of $G' - S$. ∎

4. Recursive relations for deleting an edge

By presenting an example $H \cong dK_{2m+1} \cup K_2$, $m \geq 1$, Liu and Yu [4] observed that Theorem 1.6 (i) does not hold for $d > 0$ in general. Clearly H is a $(2, 1, d)$-graph. But $H - e$ is not a $(0, 1, d)$-graph, where e is the edge in the component K_2 of H. Furthermore, the graph H implies that Theorem 1.6 (ii) does not hold for $d > 0$ as well. Note that the graph H constructed above is not connected. We present a connected example by modifying H as follows. Let $H' = H + u$. It is obvious that H' is a $(3, 1, d)$-graph, but $H' - e$ is not a $(1, 1, d)$-graph. Moreover, H' is a connected counterexample to Theorem 1.6 (ii) for $d > 0$.

In this section, we provide structural theorems for $G - e$ to be an $(n - 2, k, d)$-graph and an $(n, k - 1, d)$-graph, respectively. Also, we discuss the impact of deleting an edge from bipartite (n, k, d)-graphs.

Theorem 4.1 *Let G be an (n, k, d)-graph with $n \geq 2$. Then, for an edge $uv \in E(G)$, $G - uv$ is not an $(n - 2, k, d)$-graph if and only if there exists a vertex subset $S \subseteq V(G)$ with $|S| = n - 2 + 2k$ such that $G[S]$ contains a k-matching and $G - S$ is the union of d odd components, each of which is factor-critical, and the single edge uv.*

Proof. ($\Leftarrow$) The sufficient condition is obvious.

($\Rightarrow$) Let $G' = G - uv$. If G' is not an $(n - 2, k, d)$-graph, then there exists a $(n - 2)$-set $S' \subseteq V(G')$ and a k-matching M' which can not be extended to a defect-d matching of $G' - S'$. Let $S'' = V(M')$. Then, by Theorem 1.1, there exists a vertex set $S_1 \subseteq V(G') - S' - S''$ such that $o(G' - S' - S'' - S_1) \geq |S_1| + d + 1$. Then we have $\{u, v\} \cap (S' \cup S'' \cup S_1) = \emptyset$, for otherwise, since G is an (n, k, d)-graph, from (ii) of Theorem 1.2, we have $o(G' - S' - S'' - S_1) = o(G - S' - S'' - S_1) \leq |S_1| + d$, a contradiction. Since G is an (n, k, d)-graph, we have $o(G' - S' - S'' - S_1) \leq o(G - S' - S'' - S_1) + 2 \leq |S_1| + d + 2$. By a simple parity argument, we have $o(G' - S' - S'' - S_1) = |S_1| + d + 2$. Furthermore, since $|S_1| + d + 2 = o(G' - S' - S'' - S_1) \leq o(G - S' - S'' - S_1) + 2$, we have $o(G - S' - S'' - S_1) = |S_1| + d$. Thus uv must be a bridge of an even component of $G - S' - S'' - S_1$, which implies that $G - S' - S'' - S_1$ contains at least one even component.

Let $H = G - S' - S'' - S_1$.

Claim 1. H has exactly one even component.

Suppose that H has more than one even component. Let C_1 and C_2 be two such even components of H and $x_1 \in V(C_1)$, $x_2 \in V(C_2)$. Since $o(H) = |S_1| + d$ and, by deleting x_1 and x_2 from C_1 and C_2, the total number of the odd components increases by at least two, we have $o(H - x_1 - x_2) \geq |S_1| + d + 2$. However, G is an (n, k, d)-graph, from (ii) of Theorem 1.2, so $o(G - (S' \cup \{x_1, x_2\}) - S'' - S_1) = o(H - x_1 - x_2) \leq |S_1| + d$, a contradiction.

Claim 2. $|S_1| = 0$.

Suppose $|S_1| \geq 1$. Let C be the even component of H, $x \in S_1$, and $y \in V(C)$. Since G is an (n,k,d)-graph, from (ii) of Theorem 1.2, we have $o(H-y) = o(G-(S' \cup \{x,y\}) - S'' - (S_1 - x)) \leq |S_1| + d - 1$. However, the total number of the odd components increases when deleting the vertex y from the even component C. Since $o(H) = |S_1| + d$, we have $o(H-y) \geq |S_1| + d + 1$, a contradiction. Thus $|S_1| = 0$.

Let $S = S' \cup S''$. Then $G - S$ is the union of one even component C which contains edge uv and d odd components $O_1, O_2, \ldots, O_d$. Since $o(G'-S'-S''-S_1) = |S_1|+d+2$ and uv is a bridge of C, without loss of generality, we may assume that $C - uv = O_{d+1} \cup O_{d+2}$. Then $G' - S$ is the union of $d+2$ odd components $O_1, O_2, \ldots, O_{d+2}$. Without loss of generality, suppose $u \in O_{d+1}$ and $v \in O_{d+2}$.

Claim 3. $C \cong K_2$ and each odd component O_i, $1 \leq i \leq d$, is factor-critical.

Suppose that $|V(C)| \geq 4$. Without loss of generality, assume that x is a vertex different from u in O_{d+1}. Since G is an (n,k,d)-graph, from Theorem 1.2 (ii), we have $o(G-(S' \cup \{u,x\}) - S'') \leq d$. However, the total number of the odd components does not decrease by deleting u and x from O_{d+1}, which implies that $o(G-(S' \cup \{u,x\}) - S'') = o(G'-(S' \cup \{u,x\}) - S'') = d+2$, a contradiction. So $|V(C)| = 2$ and $E(C) = \{uv\}$.

If $|O_j| = 1$, for all j, we are done. So suppose that for some j $(1 \leq j \leq d)$, $|O_j| \geq 3$ and there exists a vertex $x \in V(O_j)$ such that $O_j - x$ has no perfect matching. Then any maximum matching of $G-(S' \cup \{u,x\}) - S''$ will miss at least $d+2$ vertices. However, since G is an (n,k,d)-graph, $G-(S' \cup \{u,x\}) - S''$ has a defect-d matching, a contradiction. ∎

From the definition of (n,k,d)-graphs, there exists no such vertex set S mentioned in Theorem 4.1 for $d = 0$. So Theorem 1.6 follows from Theorem 4.1.

Though Theorem 1.6 (i) may not hold for $d > 0$ in general, but there are classes of graphs for which Theorem 1.6 (i) holds for $d > 0$ without the additional condition $n > d$. We will see that bipartite graphs are one of such classes.

Theorem 4.2 *Let G be a bipartite (n,k,d)-graph with $n \geq 2$. Then, for each edge e of G, $G - e$ is an $(n-2,k,d)$-graph.*

Proof. Let $e = uv \in E(G)$. Suppose that $G - uv$ is not an $(n-2,k,d)$-graph. Then, by Theorem 4.1, there exists a vertex set $S \subseteq V(G)$, $|S| = n - 2 + 2k$, such that $G[S]$ contains a k-matching and $G - S$ is the union of d factor-critical components and the single edge $e = uv$ since a bipartite graph of order more than 1 is not factor-critical, each odd component is a singleton, i.e. $|V(G)| = |S| + d + 2 = n + 2k + d$. However, from the definition of the (n,k,d)-graph, we have $n + 2k + d \leq |V(G)| - 2$, a contradiction. ∎

Theorem 1.6 (ii) does not directly extend to the case $d > 0$ in general. However, sometimes we can characterize the edges which cause the statement in Theorem 1.6 (ii) to fail.

Theorem 4.3 *Let G be an (n,k,d)-graph with $k \geq 1$, and $uv \in E(G)$ such that*

$$max\{d_G(u), d_G(v)\} \geq 2k.$$

Then $G - uv$ is not an $(n, k-1, d)$-graph if and only if there exists a vertex subset $S \subseteq V(G)$ with $|S| = n - 2 + 2k$ such that $G[S]$ contains a $(k-1)$-matching and $G - S$ is the union of d factor-critical odd components and the single edge uv.

Proof. ($\Leftarrow$) The sufficient condition is obvious.

($\Rightarrow$) Let $G' = G - uv$. Suppose that G' is not a $(n, k-1, d)$-graph. Then there exist a n-set $S' \subseteq V(G)$ and a $(k-1)$-matching M' which can not be extended to a defect-d matching of $G' - S'$. Denote $V(M')$ by S''. By Theorem 1.1, there exists a vertex set $S_1 \subseteq V(G' - S' - S'')$ such that $o(G' - S' - S'' - S_1) \geq |S_1| + d + 1$. Then we have $\{u, v\} \cap (S' \cup S'' \cup S_1) = \emptyset$, for otherwise, since G is an (n,k,d)-graph, from (ii) of Theorem 1.2, we have $o(G' - S' - S'' - S_1) = o(G - S' - S'' - S_1) \leq |S_1| + d$, a contradiction. Moreover, that G is an (n,k,d)-graph implies $o(G' - S' - S'' - S_1) \leq o(G - S' - S'' - S_1) + 2 \leq |S_1| + d + 2$. By a simple parity argument, we conclude $o(G' - S' - S'' - S_1) = |S_1| + d + 2$ and $o(G - S' - S'' - S_1) = |S_1| + d$. Thus uv must be a bridge of an even component C of $G - S' - S'' - S_1$, which implies that $G - S' - S'' - S_1$ contains at least one even component.

Claim 1. $((N_G(u) \cup N_G(v)) \cap (V(G) - S' - S'')) - \{u, v\} = \emptyset$.

Suppose that ux is an edge in $G - S' - S'' - v$. Since G is an (n,k,d)-graph, $ux \cup M'$ is a k-matching of $G - S'$ which can be extended to a defect-d matching M of $G - S'$. Then M is a defect-d matching which contains M' but not uv, a contradiction.

Claim 1 implies that C is a complete graph consisting of the single edge uv.

Claim 2. $S_1 = \emptyset$.

Without loss of generality, assume that $d_G(u) \geq 2k$ (i.e., $d_G(u) > |S''| + |\{v\}|$). Thus $N(u) \cap S' \neq \emptyset$ or $N(u) \cap S_1 \neq \emptyset$. Consider the case of $N(u) \cap S' \neq \emptyset$. Let $x \in N(u) \cap S'$ and $y \in S_1 \neq \emptyset$. Since G is an (n,k,d)-graph, the k-matching $M' \cup ux$ can be extended to a defect-d matching of $G - (S' \cup y - x)$. Thus $o(G - (S' \cup y - x) - (S'' \cup ux) - (S_1 - y)) \leq |S_1| - 1 + d$. On the other hand, since $o(G - S' - S'' - S_1) = |S_1| + d$ and C is a single edge, $G - (S' \cup y - x) - (S'' \cup ux) - (S_1 - y)$ has $|S_1| + d + 1$ odd components, a contradiction. For the case of $N(u) \cap S_1 \neq \emptyset$, we get a similar contradiction.

Claim 3. C is the only even component of $G - S' - S''$.

The arguments are similar to that of Claim 2. Suppose that there is another even component C' in $G - S' - S''$. Let $y \in V(C')$. Then there exists an edge $ux \in E(C, S')$ so that the k-matching $M' \cup ux$ can be extended to a defect-d matching of $G - (S' \cup y - x)$ which implies that $o(G - (S' \cup y - x) - (S'' \cup ux) - S_1) \leq |S_1| + d$. However, since $o(G - S' - S'' - S_1) = |S_1| + d$ and the number of odd components increases upon deleting y from C', $G-(S'\cup y-x)-(S''\cup ux)-S_1$ has at least $|S_1|+d+2$ odd components, a contradiction.

Claim 4. Each odd component of $G-S'-S''$ is factor-critical.

Suppose that O is an odd component of $G-S'-S''$ which is not factor-critical. Hence there exists a vertex $y\in V(O)$ such that $O-y$ has no perfect matching. Since G is an (n,k,d)-graph, $G-S''$ is an $(n,1,d)$-graph. Thus, for any $x\in N_G(u)\cap S'$, ux can be extended to a defect-d matching of $G-(S'\cup y-x)-S''$, which is impossible since such a matching will miss at least $d+2$ vertices.

Let $S=S'\cup S''$. From the claims above, $G-S$ is the union of d factor-critical odd components and a single edge uv. ∎

Finally, we present an example to show that the restriction $max\{d_G(u),d_G(v)\}\geq 2k$ in Theorem 4.3 is necessary. Let G be the graph with vertices x_1,x_2,x_3,x_4,x_5 and the edges $x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_1,x_2x_4,x_3x_5$. Taking n disjoint copies of G and an edge $e=uv$, join the vertices u and v to x_3 and x_4 in each copy of G. Denote the resulting graph by H. Then $max\{d_H(u),d_H(v)\}=2n+1<2(n+1)$. One can verify that H is an $(1,n+1,n+1)$-graph and $H-uv$ is not an $(1,n,n+1)$-graph. However, for any vertex subset $S\subseteq V(H)$ with $|S|=2n+1$ such that $H[S]$ contains a n-matching, $H-S$ is not the union of $n+1$ factor-critical odd components and a single edge uv.

This article is merely the first of series of investigations of a general framework to unify the various extendabilities and factor-criticalities. So far we have discussed the characterization of (n,k,d)-graphs and the recursive relations only. The important aspects of (n,k,d)-graphs, such as decomposition procedure, Gallai-type structural theorems and algorithms for finding (n,k,d)-graphs, have not been explored yet. More research on this subject will follow.

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