Partitioning 2-Edge-Colored Complete Multipartite Graphs into Monochromatic Cycles, Paths and Trees[1]

ZEMIN JIN zeminjin@zjnu.cn
Center for Combinatorics, LPMC, Nankai University, Tianjin, China, and Department of Mathematics, Zhejiang Normal University, Jinhua, China

MIKIO KANO kano@cis.ibaraki.ac.jp
Department of Computer and Information Science Ibaraki University, Hitachi, Japan

XUELIANG LI[2] lxl@nankai.edu.cn
Center for Combinatorics, LPMC, Nankai University, Tianjin, China

BING WEI bwei@olemiss.edu
Department of Mathematics, University of Mississippi, Oxford, USA

[1]Research supported by NSFC.
[2]Corresponding author.

ABSTRACT

In this paper we consider the problem of partitioning complete multipartite graphs with edges colored by 2 colors into the minimum number of vertex disjoint monochromatic cycles, paths and trees, respectively. For general graphs we simply address the decision version of these three problems the 2-PGMC, 2-PGMP and 2-PGMT problems, respectively. We show that both 2-PGMC and 2-PGMP problems are NP-complete for complete multipartite graphs and the 2-PGMT problem is NP-complete for bipartite graphs. This also implies that all these three problems are NP-complete for general graphs, which solves a question proposed by the authors in a previous paper. Nevertheless, we show that the 2-PGMT problem can be solved in polynomial time for complete multipartite graphs.

Keywords: complete multipartite graphs, graph partitioning, monochromatic subgraphs, complexity.

AMS subject classification (2000): 68R10, 05C70, 90C27, 68Q17.

1 Introduction

Many combinatorial problems can be described as finding a partition of the vertex set of a given graph into subsets satisfying certain properties. Many graph partitioning problems and their corresponding computational complexity have been well studied (Brandstadt, 1996; Enomoto, 2001; Feder and Motwani, 1995; Garey and Johnson, 1979; Holyer, 1981), most of which are shown to be NP-complete. More general partitioning problems can be found in MacGillivray and Yu (1999) and Feder et al. (1999). A list of graph partitioning problems can be found in the book (Garey and Johnson, 1979).

Some researchers also focused their considerations on graph partitioning problems for edge-colored graphs (Erdös et al., 1991; Gyárfás, 1983; Haxell, 1997; Haxell and Kohayakawa, 1996; Kaneko et al., 2005). Erdös et al. (1991) showed that if the edges of a complete graph are colored by r colors, then the vertex set of the complete graph can be covered by at most $cr \log r$ vertex disjoint monochromatic cycles, where c is a constant. The authors of (Erdös et al., 1991; Haxell, 1997; Haxell and Kohayakawa, 1996; Kaneko et al., 2005) focused their considerations on the problem of determining the minimum number k such that whenever the edges of G are colored by at most r colors, the vertex set of G can be covered by at most k vertex disjoint monochromatic trees or cycles. Jin and Li (2004a) studied the following optimal problems: Given an edge-colored graph G, find the minimum number of vertex disjoint monochromatic trees, cycles and paths, respectively, which cover the vertex set of G. Note that here a single vertex is also regarded as a monochromatic tree, path or cycle. For convenience, we simply call the decision version of these three problems the PGMT, PGMC and PGMP problem, respectively. We showed that in general all of them are NP-complete and there does not exist a constant factor approximation algorithm for any optimal version of these three problems unless $P = NP$.

Note that the PGMT problem looks like the problem of partitioning a graph into induced forests (Garey and Johnson, 1979). But actually it is not the case. The following facts are easily seen. If G is colored properly, i.e., adjacent edges receive different colors, both the PGMT and PGMP problems are equivalent to the edge cover problem, which can be solved in polynomial time by graph matching algorithm (Garey and Johnson, 1979). If G is colored with one color, i.e., only one color is presented at each vertex, the PGMT problem is equivalent to the spanning tree problem, and can be solved in polynomial time. The PGMC and PGMP problems are equivalent to the Hamiltonian cycle and Hamiltonian path problems, and hence both of them are NP-complete. Jin and Li (2004a) asked the following question: Does the PGMT (PGMC, or PGMP) problem remain to be NP-complete when the edges of G are colored by only 2 colors? For convenience, we simply denote by 2-PGMT, 2-PGMP and 2-PGMC, respectively, the PGMT, PGMP and PGMC problem for a graph G with edge-colored by 2 colors.

Jin and Li (2004b) showed that for any fixed integer $r \geq 5$, if the edges of G are colored by r colors, all the PGMT, PGMC and PGMP problems remain to be NP-complete, where the proofs also imply that for any graph with maximum color degree 2, all these three problems are still NP-complete.

In Sections 2 and 3, we show that both 2-PGMC and 2-PGMP problems are NP-complete for complete and complete bipartite graphs, respectively. Since a complete graph can be viewed as a complete multipartite graph, the former implies that both 2-PGMC and 2-PGMP problems are NP-complete for complete multipartite graphs. In Section 4, we first show that the 2-PGMT problem is NP-complete for general bipartite graphs, nevertheless we then show that the 2-PGMT problem can be solved in polynomial time for complete bipartite and complete multipartite graphs. So, these results imply that for general graphs, all these three problems are NP-complete, which solves a question in (Jin and Li, 2004a).

2 The 2-PGMC and 2-PGMP problems for complete graphs

At first we focus on studying the 2-PGMC problem for complete graphs. For general graphs the 2-PGMC problem (decision version) is defined formally as follows:

THE 2-PGMC PROBLEM

INSTANCE: A graph G with edges colored by 2 colors, and a positive integer k.

QUESTION: Are there k or less vertex disjoint monochromatic cycles, which cover the vertex set of the graph G ?

The corresponding decision versions of the 2-PGMP and 2-PGMT problems can be defined similarly. In the sequel we show that even for $k = 1$, both 2-PGMC and 2-PGMP problems for complete and complete bipartite graphs are NP-complete. Obviously, in general all the three problems are in NP, since a nondeterministic algorithm needs only to guess a set of cycles, paths or trees and check in polynomial time if the cycles, paths or trees in the set are vertex disjoint monochromatic ones and if they cover the vertex set of the given graph.

For the 2-PGMT problem, let the edges of the complete graph K_n be colored by 2 colors, say red and blue. Suppose that the set of all red edges spans a graph G, then the set of all blue edges spans the graph $\overline{G}$. Since for any graph G, at least one of G and its complement graph $\overline{G}$ must be connected, we have a monochromatic spanning tree in K_n. This implies that the 2-PGMT problem for complete graphs can be solved in polynomial time. However, for the other two problems we have the following results.

Theorem 2.1 *The 2-PGMC problem is NP-complete for complete graphs.*

Proof. It is sufficient to show that the problem is NP-complete for $k = 1$. We transform the Hamiltonian path problem into the 2-PGMC problem for complete graph. Let an arbitrary instance of the Hamiltonian path problem be given by a graph G on n vertices. Here we construct a complete graph K_{n+1} with edges colored by 2 colors such that G contains a Hamiltonian path if and only if the constructed complete graph K_{n+1} contains a monochromatic Hamiltonian cycle.

The complete graph K_{n+1} is constructed as follows: Let v be an additional vertex, and let $G^* = G \vee H$, where H consists of the single vertex v, and $G \vee H$, the join graph of G and H, is defined as the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Let $K_{n+1} = G^* \cup \overline{G^*}$, and let every element of $E(G^*)$ be colored by red, while every element of $E(\overline{G^*})$ be colored by blue. We claim that G contains a Hamiltonian path if and only if the constructed complete graph K_{n+1} contains a monochromatic Hamiltonian cycle.

If G contains a Hamiltonian path, denoted by $P = u_1 u_2 \cdots u_n$, then $C = v u_1 u_2 \cdots u_n v$ is a monochromatic Hamiltonian cycle in K_{n+1}. Suppose that K_{n+1} contains a monochromatic Hamiltonian cycle, denoted by $C = v u_1 u_2 \cdots u_n v$. Since every edge incident to v is colored by red, every edge on C must appear in the graph G. This implies that $u_1 u_2 \cdots u_n$ is a Hamiltonian path of G. The proof is complete. ∎

Theorem 2.2 *The 2-PGMP problem is NP-complete for complete graphs.*

Proof. It is sufficient to show that the problem is NP-complete for $k = 1$. Here we also transform the Hamiltonian path problem into the 2-PGMP problem for complete graphs. Let an arbitrary instance of the Hamiltonian path problem be given by a graph G on n vertices. Here we construct a complete graph K_{2n+1} with edges colored by 2 colors such that G contains a Hamiltonian path if and only if the constructed complete graph K_{2n+1} contains a monochromatic Hamiltonian path.

The complete graph K_{2n+1} is constructed as follows: Let v be an additional vertex. Take a disjoint copy G' of G. Let $G^* = (G \cup G') \vee H$, where H consists of the single vertex v. Let $K_{2n+1} = G^* \cup \overline{G^*}$, and let every element of $E(G^*)$ be colored by red, while every element of $E(\overline{G^*})$ be colored by blue. We claim that G contains a Hamiltonian path if and only if the constructed complete graph K_{2n+1} contains a monochromatic Hamiltonian path.

If G contains a Hamiltonian path, denoted by $P = u_1 u_2 \cdots u_n$, then it is easy to see that K_{2n+1} contains a monochromatic Hamiltonian path. Suppose that K_{2n+1} contains a monochromatic Hamiltonian path, denoted by

$Q = x_1x_2 \cdots x_nx_{n+1}x_{n+2} \cdots x_{2n+1}$. Since every edge incident to v is colored by red, and every edge connecting vertices of G and G' is colored by blue, the first or last n vertices of Q must appear in the graph G. This implies that G contains a Hamiltonian path. This completes the proof. ∎

3 The 2-PGMC and 2-PGMP problems for complete bipartite graphs

In this section we consider the 2-PGMC and 2-PGMP problems for complete bipartite graphs. Although a complete graph can be viewed as a multipartite graph, in most cases it is not a bipartite graph. We know from (Golumbic, 1980) that the Hamiltonian path and Hamiltonian cycle problems are NP-complete for bipartite graphs. From this fact, we have the following results.

Theorem 3.1 *The 2-PGMC problem is NP-complete for complete bipartite graphs.*

Proof. It is sufficient to show that the problem is NP-complete for $k = 1$. Here we transform the Hamiltonian path problem for bipartite graph into the 2-PGMP problem for complete bipartite graphs. Let an arbitrary instance of the Hamiltonian path problem for bipartite graphs be given by a bipartite graph $G = G(U, V)$, where $|U| + 1 = |V| = n$. Here we construct a complete bipartite graph $K_{n,n}$ with edges colored by 2 colors such that G contains a Hamiltonian path if and only if the constructed complete bipartite graph $K_{n,n}$ contains a monochromatic Hamiltonian cycle.

The complete bipartite graph $K_{n,n}$ is constructed as follows: Let u be an additional vertex, and let G^* be a graph on the vertex set $V(G) \cup \{u\}$ with edge set $E(G^*) = E(G) \cup \{uv : v \in V\}$. Denote by $\overline{\overline{G^*}}$ the graph on the vertex set $V(G) \cup \{u\}$ with edge set $E(\overline{\overline{G^*}}) = \{xy \notin E(G) : x \in U, y \in V\}$. Let $K_{n,n} = G^* \cup \overline{\overline{G^*}}$, and color every element in $E(G^*)$ by red, while color every element in $E(\overline{\overline{G^*}})$ by blue. We claim that G contains a Hamiltonian path if and only if the constructed complete bipartite graph $K_{n,n}$ contains a monochromatic Hamiltonian cycle.

If G contains a Hamiltonian path, denoted by $P = xu_1 \cdots y$, $x, y \in V$, then $C = uxu_1 \cdots yu$ is a monochromatic Hamiltonian cycle in $K_{n,n}$. Suppose that $K_{n,n}$ contains a monochromatic Hamiltonian cycle, denoted by $C = uxu_1 \cdots yu$. Since every edge incident to u is colored by red, every edge on C must appear in the graph G. This implies that $xu_1 \cdots y$ is a Hamiltonian path of G. The proof is complete. ∎

Theorem 3.2 *The 2-PGMP problem is NP-complete for complete bipartite graphs.*

Proof. It is sufficient to show that the problem is NP-complete for $k = 1$. Here we also transform the Hamiltonian path problem for bipartite graphs into the 2-PGMP problem for complete bipartite graphs. Let an arbitrary instance of the Hamiltonian path problem for bipartite graphs be given by a bipartite graph $G = G(U, V)$, $|U| + 1 = |V| = n$. Here we construct a complete bipartite graph $K_{2n,2n-1}$ with edges colored by 2 colors such that G contains a Hamiltonian path if and only if the constructed complete bipartite graph $K_{2n,2n-1}$ contains a monochromatic Hamiltonian path.

The complete bipartite graph $K_{2n,2n-1}$ is constructed as follows: Let u be an additional vertex. Take a disjoint copy $G^{'} = G^{'}(U^{'}, V^{'})$ of G. Let G^* be the graph on the vertex set $V(G) \cup V(G^{'}) \cup \{u\}$ with edge set $E(G^*) = E(G) \cup E(G^{'}) \cup \{ux : x \in V \cup V^{'}\}$. Denote by $\overline{\overline{G^*}}$ the graph on the vertex set $V(G) \cup V(G^{'}) \cup \{u\}$ with edge set $E(\overline{\overline{G^*}}) = \{xy \notin E(G^{'}) \cup E(G) : x \in U \cup U^{'}, y \in V \cup V^{'}\}$. Let $K_{2n,2n-1} = G^* \cup \overline{\overline{G^*}}$, and color every element in $E(G^*)$ by red, while color every element in $E(\overline{\overline{G^*}})$ by blue. We claim that G contains a Hamiltonian path if and only if the constructed complete bipartite graph $K_{n,n}$ contains a monochromatic Hamiltonian path.

If G contains a Hamiltonian path, denoted by $P = x \cdots y$, then it is easy to see that $K_{2n,2n-1}$contains a monochromatic Hamiltonian path. Suppose that $K_{2n,2n-1}$ contains a monochromatic Hamiltonian path, denoted by $Q = x_1x_2 \cdots x_{2n-1}x_{2n}x_{2n+1} \cdots x_{4n-1}$. Since every edge incident to u is colored by red and every edge connecting vertices of G and $G^{'}$ is colored by blue, the first or last $2n - 1$ vertices of Q must appear in the graph G. This implies that G contains a Hamiltonian path. The proof is complete. ∎

4 The 2-PGMT problem for bipartite and complete multipartite graphs

In this section we first show that the 2-PGMT problem is NP-complete for general bipartite graphs, and then show that the problem can be solved in polynomial time for complete bipartite and complete multipartite graphs. In order to show the first, we need the well-known NP-complete problem the 3SAT problem.

THE 3SAT PROBLEM

INSTANCE: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.

QUESTION: Is there a truth assignment for U such that each $c \in C$ is true ?

Theorem 4.1 *The 2-PGMT problem is NP-complete for bipartite graphs.*

Proof. The 2-PGMT problem is obviously in NP. We complete the proof by showing that for any instance I of the 3SAT problem we can construct a 2-edge colored bipartite graph G_I of polynomial size in terms of the size of the instance I such that I has a satisfying truth assignment if and only if the vertex set of G_I can be partitioned into 2 monochromatic trees, which implies that the 2-PGMT problem is already NP-complete for $k = 2$.

Let I be an instance of the 3SAT problem with clause set $C = \{c_1, c_2, \cdots, c_m\}$ and variable set $U = \{u_1, u_2, \cdots, u_n\}$. We construct a graph G_I as follows: the vertex set of G_I is $C \cup U \cup \{r, b\}$ ($m + n + 2$ vertices). There is an edge between c_i and u_j if and only if u_j or its negation $\overline{u_j}$ belongs to c_i. If u_j belongs to c_i we color the edge $c_i u_j$ by color 1, if $\overline{u_j}$ belongs to c_i we color the edge $c_i u_j$ by color 0. Then, connect r to every u_j and color these edges by color 1, and connect b to every u_j and color these edges by color 0. Obviously, G_I is a 2-edge-colored bipartite graph.

If the vertex set of G_I can be partitioned into 2 monochromatic trees, then we assign a variable u_j with a value "true" or "false" depending on the color 1 or 0 that the monochromatic tree containing the vertex u_j has. It is easily seen that this is a satisfying truth assignment for I. Conversely, if I has a satisfying truth assignment, then the set $\{u_j | u_j$ *is assigned the value "true"*$\} \cup \{c_i | c_i$ *is satisfied by the variables with value "true"*$\} \cup \{r\}$, containing at least one vertex r, induces a subgraph of G_I that has a connected monochromatic spanning subgraph of color 1, and the set of the rest vertices, containing at least one vertex b, induces a subgraph of G_I that has a connected monochromatic spanning subgraph of color 0. So, from the two subgraphs we can get 2 monochromatic trees partitioning the vertex set of G_I. The proof is now complete. ∎

The tree partition number of a graph G with edges colored by r colors is defined to be the minimum number k such that whenever the edges of G are colored with r colors, the vertex set of G can be covered by at most k vertex disjoint monochromatic trees. Kaneko et al. (2005) determined the tree partition number for complete multipartite graphs with edges colored by 2 colors. In the following we show that one can find the minimum number of vertex disjoint monochromatic trees to cover all the vertices of a 2-edge-colored complete multipartite graph in polynomial time. First, we consider complete bipartite graphs.

Theorem 4.2 *The 2-PGMT problem can be solved in polynomial time for complete bipartite graphs.*

Proof. Let $K_{m,n} = K(M, N)$ be a complete bipartite graph with edges colored by red and blue colors. Let R denote the subgraph of $K_{m,n}$ with vertex set the same as $K_{m,n}$ and edge set the set of all red edges and B similarly the subgraph with vertex set the same as $K_{m,n}$ and edge set the set of all blue edges. We distinguish the following cases.

Case 1 One of R and B is connected. Then we have a monochromatic spanning tree, and so we are done.

Case 2 Both R and B are disconnected. Then we need at least two vertex disjoint monochromatic trees to cover the vertex set of $K_{m,n}$. We distinguish the following subcases.

Subcase 2.1 One of R and B contains at least two components such that each of them contains at least one edge. Then, since both R and B have at least two components, we can conclude that both R and B have exactly two such components, say R_1 and R_2, B_1 and B_2, respectively; for otherwise, if one of R and B has more than two such components, the other must be connected, a contradiction. Moreover, we can deduce that $R_1 = K(M_1, N_1)$ and $R_2 = K(M_2, N_2)$ for some partition of $M = M_1 \cup M_2$ and $N = N_1 \cup N_2$, respectively, and $B_1 = K(M_1, N_2)$, $B_2 = K(M_2, N_1)$. So, the vertex set of $K_{m,n}$ can be covered by two vertex disjoint monochromatic red or blue trees.

Subcase 2.2 Both R and B contain at most one component that contains at least one edge. Then, both R and B must contain exact one component that contains at least one edge, denoted by R_0 and B_0, respectively, and so each of the other components contains a single vertex. Since both R and B are disconnected, R_0 must totally cover one and only one of the sets M and N, and the same is true for B_0. We claim that R_0 and B_0 must cover the same set M or N. Otherwise, without loss of generality, let R_0 cover M but not N whereas B_0 cover N but not M, say, $N_1 (\subset N)$ is not covered by R_0 whereas $M_1 (\subset M)$ is not covered by B_0. This implies that all the edges between N_1 and M_1 cannot receive any of the 2 colors, a contradiction and thus the claim follows. So, we assume that both R_0 and B_0 totally cover one and only one common set of M and N, say M but not N. This implies that for any vertex $x \in M$, there are both red and blue edges between x and N. Denote by N_1 and N_2 the sets of vertices not covered by R_0 and B_0, respectively. Then $N \supset N_1 \neq \emptyset$, $N \supset N_2 \neq \emptyset$ and $N_1 \cap N_2 = \emptyset$, and the set $N - N_1 - N_2$, denoted by N_0, is covered by both R_0 and B_0. It is easy to see that every vertex of N_1 is incident only to blue edges, whereas every vertex of N_2 is incident only to red edges.

Let $\mathcal{P}$ be a set of vertex disjoint monochromatic trees with minimum size, which cover the vertex set of $K(M, N)$. If there is a monochromatic tree in $\mathcal{P}$ containing all the vertices of M, then because of the minimality of $\mathcal{P}$ this tree must cover and only covers one of the sets N_1 or N_2. Otherwise, there are two monochromatic trees in $\mathcal{P}$, one red and the other blue, then because of the minimality of $\mathcal{P}$, N_2 must be covered by the red tree, and N_1 must be covered by the blue tree. So, in any case we know that $\mathcal{P}$ must satisfy exactly one of the following properties:

- **(a)** Only one of the sets N_1 and N_2 is covered by a monochromatic tree, and each vertex of the other set forms a tree in $\mathcal{P}$. In this case, M is totally covered by the monochromatic tree.
- **(b)** Both N_1 and N_2 are respectively covered by a monochromatic tree in

$\mathcal{P}$.

In order to find a set of vertex disjoint monochromatic trees with minimum size to cover the vertex set of $K(M,N)$, we first try to find two sets of vertex disjoint monochromatic trees with minimum size, which respectively satisfies property (a) and (b). Clearly, among these two sets, the one with minimum size is, what we want, the set of vertex disjoint monochromatic trees with minimum size which cover the vertex set of $K(M,N)$. Since the set with minimum size satisfying property (a) is trivial, in the following we focus on finding the set with minimum size satisfying property (b).

First, we introduce some notations. For every vertex $x \in N_0$, let $\Gamma_r(x) = \{y \in M : xy \text{ is red}\}$, called the red neighborhood of x. We define an equivalent relation among the vertices of the set N_0 as follows: Two vertices x_1 and x_2 of N_0 are equivalent if and only if $\Gamma_r(x_1) = \Gamma_r(x_2)$. So, according to the red neighborhoods we can partition N_0 into a number of equivalent classes N_{0i}, $i = 1, 2, \cdots, t$. Then, we set up a one to one correspondence between the set of classes N_{0i}, $i = 1, 2, \cdots, t$, and the set of nonempty subsets $\Gamma_r(x_{0i})$ of M, where x_{0i} is a representative of the class N_{0i}. Clearly, $t \leq \min\{n - |N_1| - |N_2|, 2^m - 2\}$. We distinguish the following subsubcases:

Subsubcase 2.2.1 If there is a proper nonempty subset M' of M such that $M' \neq N_{0i}$ for any $i = 1, 2, \cdots, t$, then take a blue tree T_b such that $V(T_b) \cap M = M_b = M'$, and take a red tree T_r such that $V(T_r) \cap M = M_r = M - M' \neq \emptyset$. We then assign that T_b covers N_1 and T_r covers N_2. Next, for any vertex $x \in N_0$ if there is a blue edge between x and M_b, then assign x to the blue tree T_b. Otherwise, all the edges between x and M_b are red. Since $M_b = M'$ is not a red neighborhood for any of the vertices in N_0, there must be a red edge between x and M_r, and then assign x to the red tree T_r. In this way, every vertex of N_0 is either connected to the blue tree T_b or to the red tree T_r, and so the two vertex disjoint monochromatic trees T_b and T_r totally cover the vertex set of $K(M,N)$.

Subsubcase 2.2.2 Otherwise, for every proper nonempty subset M' of M, there is a class N_{0i} such that every vertex of N_{0i} has a red neighborhood equal to M'. Choose a proper nonempty subset M' of M such that M' corresponds to a class N_{0k} that has minimum size. Then, take a blue tree T_b such that $V(T_b) \cap M = M_b = M'$, and take a red tree T_r such that $V(T_r) \cap M = M_r = M - M' \neq \emptyset$. So, T_b covers N_1 and T_r covers N_2. For any vertex $x \in N_0 - N_{0k}$, if there is a blue edge between x and M_b, then assign x to the blue tree T_b. Otherwise, all the edges between x and M_b are red. Since $x \notin N_{0k}$, i.e., $\Gamma_r(x) \neq M_b(= M')$, there must be a red edge between x and M_r, and then assign x to the red tree T_r. In this way, every vertex of $N_0 - N_{0k}$ is either connected to the blue tree T_b or the red tree T_r, and so the two monochromatic trees T_b and T_r together with the vertices of N_{0k} form a vertex disjoint cover of the vertex set of $K(M,N)$. We claim that this is a monochromatic partition with minimum size. In fact, since we want to find a set $\mathcal{P}$ of vertex disjoint monochromatic trees with minimum size satisfying property (b), which cover the vertex set of $K(M,N)$, there must be two monochromatic red and blue trees T_r and T_b in $\mathcal{P}$ which totally cover the sets N_2 and N_1, respectively. Since every vertex of M is incident to both red and blue edges, every vertex of N_1 is incident to only blue edges and every vertex of N_2 is incident to only red edges, because of the minimality of $\mathcal{P}$ we have that T_r union T_b totally covers the set M. Let $M_r \subset V(T_r)$ and $M_b \subset V(T_b)$ such that $M = M_r \cup M_b$, $M_r \neq \emptyset$, $M_b \neq \emptyset$ and $M_r \cap M_b = \emptyset$. Then any tree of $\mathcal{P}$ other than T_r and T_b, if there exists, must be a single vertex of N_0, moreover, every edge between the single vertex and M_b is red and every edge between the single vertex and M_r is blue, that is, the red neighborhood of the single vertex is M_b. Any vertex in N_0 other than this kind of single vertices does not have this property. Denote these single vertices, if there exist, by a set S. Then S has the property that any two vertices in S have the same red neighborhood M_b, and any vertex of N_0 that has this property must belong to S, i.e., there is a class N_{0i} such that $N_{0i} = S$. Because of the minimality of N_{0k}, we have that $|N_{0i}| \geq |N_{0k}|$, and the claim is thus proved. Finally, we claim that under the assumption of this subsubcase, S cannot be empty. Otherwise, the two monochromatic trees T_r and T_b cover the vertex set of $K(M,N)$. Then, consider the proper nonempty subset $M' = V(T_b) \cap M$ of M. M' is not a red neighborhood for any of the vertices in N_0. Otherwise, say that x has the red neighborhood equal to M'. Then, x cannot be assigned to any of the two trees T_r and T_b, which contradicts to that the two trees cover the vertex set of $K(M,N)$.

We claim that both Subsubcases 2.2.1 and 2.2.2 can be done in polynomial time. In fact, to find the equivalent classes can be done in polynomial time, since this only involves checking whether or not the red neighborhoods of two vertices in N_0 are the same. Next, choose t proper nonempty subsets of M randomly or in the following way: generating the k-subsets of M one by one for $k = 1, 2, \cdots$. As soon as a k-subset is generated, we compare it with every $\Gamma_r(x_{0i})(\subset M)$ corresponding to the equivalent class N_{0i}, $i = 1, 2, \cdots, t$, to check whether or not they are equal. By at most t^2 such comparisons, we can decide whether or not there is a proper nonempty subset M' of M such that M' is not a red neighborhood for any of the vertices in N_0. If yes, we have Subsubcase 2.2.1, i.e., there are two monochromatic trees T_b and T_r partition the vertex set of $K(M,N)$, and this partition can be obtained from the subset M' by the way in the proof of Subsubcase 2.2.1. If not, we have Subsubcase 2.2.2, and it can be done in polynomial time to find a class N_{0k} among the classes N_{0i}, $i = 1, 2, \cdots, t$, such that N_{0k} has the minimum size. To check whether or not such a subset M' exists can also be done in polynomial time, because this only involves t^2 comparisons of two subsets of M. This is upper bounded by t^2m^2, which is a very rough estimation. Since $t < n$, we have that $t^2m^2 < (mn)^2$. Obviously, Case 1 and the other subcases of Case 2 can also be done in polynomial time. Therefore, there is an algorithm of polynomial time to solve the 2-PGMT problem for complete bipartite graphs. ∎

Kaneko et al. (2005) proved that, if the edges of a complete k-partite graph, $k \geq 3$, are colored by 2 colors red and blue in such a way that at least one red and one blue edge are incident with every vertex, then it contains a monochromatic spanning tree. For a complete bipartite graph, from the above proof we can see that if every vertex is incident with both some red and blue edges, then its vertex set can be partitioned into at most two vertex disjoint monochromatic trees. In general, for complete multipartite graphs, we can employ a similar proof to get the following result. For convenience of reading, we give part of its proof in detail. One may think of unifying the complete bipartite case and the complete multipartite case together to uniform the results and the proofs. But, we think that that would make the proof unclear. Actually, one can see that the beginnings of the two proofs are quite different.

Theorem 4.3 *The 2-PGMT problem can be solved in polynomial time for complete multipartite graphs.*

Proof. Let $G = K(V_1, V_2, \cdots, V_k)$, $k \geq 3$, be a complete k-partite graph with edge-colored by red and blue colors. We distinguish the following cases.

Case 1 Every vertex is incident to both red and blue edges. Then, from the result of (Kaneko et al. (2005)) we know that G contains a monochromatic spanning tree, and so one monochromatic tree can cover the vertex set of G.

Case 2 Otherwise, there are some vertices that are incident with only red or blue edges. Denote $N_r = \{x \in V(G) : \text{every edge incident to } x \text{ is red}\}$ and $N_b = \{x \in V(G) : \text{every edge incident to } x \text{ is blue}\}$. From the assumption, without loss of generality we can assume that $N_r \neq \emptyset$. We distinguish the following subcases.

Subcase 2.1 If $N_b = \emptyset$, then every vertex is incident to at least one red edge. Whether or not N_r is contained in the same part of G, it is easy to see that G contains a red spanning tree.

Subcase 2.2 Otherwise, $N_b \neq \emptyset$. Clearly, both N_r and N_b are contained in the same part of G, without loss of generality, say in V_1. Let $N_0 = V_1 - N_r - N_b$. Then every vertex of N_0 is incident to both some red and blue edges. This implies that $V(G) - N_b$ and $V(G) - N_r$ can be spanned by a red and blue tree, respectively. Let $\mathcal{P}$ be a set of vertex disjoint monochromatic trees with minimum size, which cover the vertex set of G. If there is a monochromatic tree in $\mathcal{P}$ containing all the vertices of $V(G) - V_1$, then because of the minimality of $\mathcal{P}$ this tree must cover and only covers one of the sets V_r and V_b. Otherwise, there must be two monochromatic trees in $\mathcal{P}$, one red and the other blue, and then because of the minimality of $\mathcal{P}$, N_r must be covered by the red tree, and N_b must be covered by the blue tree.

So, in any case we know that $\mathcal{P}$ must satisfy exactly one of the following properties:

(a) Only one of the sets N_r and N_b is covered by a monochromatic tree, and every vertex of the other set forms a tree in $\mathcal{P}$. In this case, $V(G) - V_1$ is totally covered by the monochromatic tree.

(b) Both N_r and N_b are respectively covered by a monochromatic tree in $\mathcal{P}$.

In order to find a set of vertex disjoint monochromatic trees with minimum size to cover the vertex set of G, we first try to find two sets of vertex disjoint monochromatic trees with minimum size, which respectively satisfies property (a) and (b). Clearly, among these two sets, the one with minimum size is, what we want, the set of vertex disjoint monochromatic trees with minimum size which cover the vertex set of G. Clearly, the set with minimum size satisfying property (a) is trivial. By employing a similar proof to that of Theorem 4.2, we can find the set of vertex disjoint monochromatic trees with minimum size satisfying property (b) in polynomial time. The rest of the proof is omitted. ∎

Remark 1 From the above proofs we can design an efficient algorithm for the 2-PGMT problem for complete bipartite and multipartite graphs.

Remark 2 Using our proof technique we can give another (more natural) proof for the result of Kaneko et al. (2005).

Acknowledgement The authors would like to thank the referees for helpful comments.

References

A. Brandstadt, "Partitions of graphs into one or two independent sets and cliques," *Discrete Math.* vol. 152, pp. 47-54, 1996.

H. Enomoto, "Graph partition problems into cycles and paths," *Discrete Math.* vol. 233, pp. 93-101, 2001.

P. Erdös, A. Gyárfás and L. Pyber, "Vertex coverings by monochromatic cycles and trees," *J. Combin. Theory, Ser. B,* vol. 51, pp. 90-95, 1991.

T. Feder, P. Hell, S. Klein and R. Motwani, "Complexity of graph partition problems," *Proceedings of Thirty-First Annual ACM Symposium on Theory of Computing,* 1999, 464-472.

T. Feder and R. Motwani, "Clique partitions, graph compression and Speeding-Up algorithms," *J. Computer and System Sciences,* vol. 51, pp. 261-272, 1995.

M.R. Garey and D.S. Johnson, *Computers and Intractability,* W.H. Freeman, San Francisco, 1979.

M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs,* Academic Press, New York, 1980.

A. Gyárfás, "Vertex coverings by monochromatic paths and cycles," *J. Graph Theory*, vol. 7, pp. 131-135, 1983.

P.E. Haxell, "Partitioning complete bipartite graphs by monochromatic cycles," *J. Combin. Theory, Ser. B,* vol. 69, pp. 210-218, 1997.

P.E. Haxell and Y. Kohayakawa, "Partitioning by monochromatic trees," *J. Combin. Theory, Ser. B,* vol. 68, pp. 218-222, 1996.

L. Holyer, "The NP-completeness of some edge-partition problems," *SIAM J. Comput.* vol. 10, pp. 713-717, 1981.

Z. Jin and X. Li, "The complexity for partitioning graphs by monochromatic trees, cycles and paths," *International J. Computer Math.* vol. 81, no. 11, pp. 1357-1362, 2004a.

Z. Jin and X. Li, "Vertex partitions of r-edge-colored graphs," *Submitted,* 2004b.

A. Kaneko, M. Kano and K. Suzuki, "Partitioning complete multipartite graphs by monochromatic trees," *J. Graph Theory,* vol. 48, no. 2, pp. 133-141, 2005.

G. MacGillivray and M.L. Yu, "Generalized partitions of graphs," *Disc. Appl. Math.* vol. 91, pp. 143-153, 1999.