Moments about the Mean of the Size of a Self-Conjugate (s,t)-Core Partition

Joseph L.P. Wang[1], Larry X.W. Wang[2], Jane Y.X. Yang[3]

[1,2,3]Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

[1]1120120002@mail.nankai.edu.cn, [2]wsw82@nankai.edu.cn,
[3]yangyingxue@mail.nankai.edu.cn.

Abstract

Johnson proved that if s,t are coprime integers, then the r-th moment of the size of an (s,t)-core is a polynomial of degree $2r$ in t for fixed s. After that, by defining a statistic size on elements of affine Weyl group, which is preserved under the bijection between minimal coset representatives of $\widetilde{\mathfrak{S}}_t/\mathfrak{S}_t$ and t-cores, Thiel and Williams obtained the variance and the third moment about the mean of the size of an (s,t)-core. Later, Ekhad and Zeilberger stated the first six moments about the mean of the size of an (s,t)-core and the first nine moments about the mean of the size of an $(s,s+1)$-core using Maple. To get the moments about the mean of the size of a self-conjugate (s,t)-core, we proceed to follow the approach of Thiel and Williams, however, their approach does not seem to directly apply to the self-conjugate case. In this paper, following Johnson's approach, by Ehrhart theory and Euler-Maclaurin theory, we prove that if s,t are coprime integers, then the r-th moment about the mean of the size of a self-conjugate (s,t)-core is a polynomial of degree $2r$ in t for fixed s. Then, based on a bijection of Ford, Mai and Sze between self-conjugate (s,t)-cores and lattice paths in $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ rectangle and a formula of Chen, Huang and Wang on the size of self-conjugate (s,t)-cores, we obtain the variance, the third moment and the forth moment about the mean of the size of a self-conjugate (s,t)-core.

Keywords: (s,t)-core, self-conjugate partition, lattice path, Ehrhart theory, Euler-Maclaurin theory

AMS Subject Classification: 05A17, 05A15

1 Introduction

The objective of this paper is to give the variance, the third moment and the forth moment about the mean of the size of a self-conjugate (s,t)-core.

A partition is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, and say the length of λ is m. The Young diagram of λ is defined to be a left-justified array of boxes with λ_i boxes in the ith row from the top. For each box B of λ, one can associate its hook length $h(B)$, which is the number of boxes directly below and directly to the right of B (including B itself) in the Young diagram of λ. The conjugation of λ is the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{\lambda_1})$ where

$$\mu_i = \#\{j \colon \lambda_j \geq i, 1 \leq j \leq m\}, \text{ for } 1 \leq i \leq \lambda_1.$$

A partition λ is called a self-conjugate partition if $\lambda = \mu$; we often use λ' to denote the conjugation of λ.

For a positive integer t, a partition λ is a t-core if it has no box with hook length of a multiple of t. Let s be another positive integer, we say that λ is an (s,t)-core if it is simultaneously an s-core and a t-core, if $\lambda = \lambda'$ we say λ is a self-conjugate (s,t)-core.

Let s and t be coprime positive integers. Anderson [3] showed that the number of (s,t)-cores is $\frac{1}{s+t}\binom{s+t}{s}$. Ford, Mai and Sze [8] proved the number of self-conjugate (s,t)-cores is $\binom{\lfloor s/2 \rfloor + \lfloor t/2 \rfloor}{\lfloor s/2 \rfloor}$. Olsson and Stanton [13] proved that there exists a unique (s,t)-core with the maximum size $\frac{(s^2-1)(t^2-1)}{24}$. Armstrong, Hanusa and Jones [4] posed the following conjecture on the average size of an (s,t)-core and the average size of a self-conjugate (s,t)-core.

Conjecture 1.1 *Assume s and t are coprime integers. Then the average size of an (s,t)-core and the average size of a self-conjugate (s,t)-core are both equal to*

$$\frac{(s+t+1)(s-1)(t-1)}{24}. \tag{1.1}$$

Stanley and Zanello [14] showed that the conjecture for the average size of an (s,t)-core holds for $t = s+1$. Based on this work, Aggarwal [1] proved the average size of an $(s, ms+1)$-core is $\frac{(s+ms+2)(s-1)ms}{24}$. Chen, Huang and Wang [6] proved this conjecture for the self-conjugate case. And Johnson [10] proved this conjecture concerning the average size of an (s,t)-core by Ehrhart theory and Euler-Maclaurin theory. He also gave an alternative derivation for the result of Chen, Huang and Wang. After that, Wang [16] gave another proof for Johnson's result by using special cyclic complex-value functions and some special generating functions.

Regarding the size of an (s,t)-core as a random variable, Thiel and Williams [15] extended Johnson's method to compute the variance and the third moment about the mean of the size of an (s,t)-core as follows.

Theorem 1.1 *If s and t are coprime positive integers, then the variance and the third moment about the mean of the size of an (s,t)-core are*

$$\frac{st(s-1)(t-1)(s+t)(s+t+1)}{1440}$$

and
$$\frac{st(s-1)(t-1)(s+t)(s+t+1)(2s^2t+2st^2-3s^2-3t^2-3st-3)}{60480},$$
respectively.

Using Maple, Ekhad and Zeilberger [7] stated new polynomials for the first six moments about the mean of the size of an (s,t)-core, and the first nine moments about the mean of the size of an $(s,s+1)$-core.

Motivated by these works, in this paper we are concerned with the moments about the mean of the size of a self-conjugate (s,t)-core. We found that the approach of Thiel and Williams [15] can not be directly applied to the self-conjugate (s,t)-core. To get the variance of the random size of an (s,t)-core, Thiel and Williams defined a statistic size on elements of affine Weyl group, which is preserved under the bijection of Lascoux [11] between minimal coset representatives of $\widetilde{\mathfrak{S}}_t/\mathfrak{S}_t$ and t-cores. Using this statistic size, they gave another derivation for the maximum size of an (s,t)-core and the average size of an (s,t)-core. Moreover, they obtained the variance and the third moment about the mean of the size of an (s,t)-core. But the statistic size on elements of affine Weyl group in type $\tilde{C}_t$ is not equal to the number of boxes of the corresponding self-conjugate core under the bijection of Hanusa and Jones [9] between the minimal coset representatives of $\widetilde{C}_t/C_t$ and the self-conjugate $2t$-cores.

Let $\xi_{s,t}$ denote the random size of a self-conjugate (s,t)-core in this paper. In order to calculate the moments about the mean of $\xi_{s,t}$, in Section 2, following Johnson's approach, by Ehrhart theory and Euler-Maclaurin theory we first prove that the variance, the third moment and the forth moment about the mean of the random variable $\xi_{s,t}$ are polynomials of degree 4, 6 and 8 in t, respectively. To determine these polynomials, we need some special values. Thus, in Section 3, based on a bijection of Ford, Mai and Sze between self-conjugate (s,t)-cores and lattice paths in $\left\lfloor \frac{s}{2} \right\rfloor \times \left\lfloor \frac{t}{2} \right\rfloor$ rectangle and a formula of Chen, Huang and Wang on the size of self-conjugate (s,t)-cores, we give the formulae for the variance about the mean of $\xi_{s,t}$ for $t=2,3,4$, the third moment about the mean of $\xi_{s,t}$ for $t=2,3,4,5,6$, and the forth moment about the mean of $\xi_{s,t}$ for $t=2,3,4,5,6,7,8$. In Section 4, combining the theorems in Section 2 and the formulae in Section 3, we deduce the variance, the third moment and the forth moment about the mean of $\xi_{s,t}$ by the method of undetermined coefficients. And we state them as the following theorems.

Theorem 1.2 *Let s and t be coprime integers with s odd. Then the variance about the mean of the random variable $\xi_{s,t}$ is*

$$M_2(s,t)=\frac{st(s-1)(s+t)(s+t+1)(2t-3)}{1440}, \quad \text{if } t \text{ is even;}$$

$$M_2(s,t)=\frac{st(s-1)(t-1)(s+t)(2s+2t+3)}{1440}, \quad \text{if } t \text{ is odd.}$$

Theorem 1.3 *Let s and t be coprime integers with s odd. Then the variance about the mean of the random variable $\xi_{s,t}$ is*

$$M_3(s,t) = \frac{1}{120960} st(s-1)(s+t)(s+t+1)(16t^2s^2 - 61ts^2 + 60s^2 - 61t^2s + 16t^3s + 60st + 66t^2 - 27t - 30t^3), \text{ if } t \text{ is even;}$$

$$M_3(s,t) = \frac{1}{120960} st(s-1)(t-1)(s+t)(16ts^3 + 16t^3s + 32t^2s^2 - 30s^3 - 30t^3 - 29ts^2 - 29t^2s - 66s^2 - 66t^2 - 72st - 27s - 27t), \text{ if } t \text{ is odd.}$$

Theorem 1.4 *Let s and t be coprime integers with s odd. Then the variance about the mean of the random variable $\xi_{s,t}$ is*

$$M_4(s,t) = \frac{1}{4838400} st(s-1)(s+t)(s+t+1)(124t^3s^4 - 766t^2s^4 + 1671ts^4 - 1260s^4 + 248t^4s^3 + 3342t^2s^3 - 2520ts^3 - 1532t^3s^3 - 4579t^2s^2 + 621s^2t + 124t^5s^2 + 3975t^3s^2 - 1254t^4s^2 + 1260s^2 + 621st^2 - 3319t^3s + 1260st - 488t^5s + 2304t^4s + 1530t^2 + 528t^5 + 252t^3 - 1512t^4), \text{ if } t \text{ is even;}$$

$$M_4(s,t) = \frac{1}{4838400} st(s-1)(t-1)(s+t)(528s^5 - 1800ts - 488t^5s + 124t^5s^2 + 336t^4s - 1186t^4s^2 + 372t^4s^3 + 2729t^3s + 39t^3s^2 - 1396t^3s^3 + 372t^3s^4 + 3694t^2s^2 + 39t^2s^3 - 1186t^2s^4 + 124t^2s^5 + 2729ts^3 + 336ts^4 - 488ts^5 + 135st^2 + 135s^2t + 1512s^4 + 252s^3 + 1512t^4 + 528t^5 - 1530s^2 - 1530t^2 + 252t^3), \text{ if } t \text{ is odd.}$$

2 The degrees of moments about the mean of $\xi_{s,t}$

In this section, by Ehrhart theory and Euler-Maclaurin theory we will prove that if s, t are coprime positive integers, then the r-th moment about the mean of the random variable $\xi_{s,t}$ is a polynomial of degree $2r$ in t for fixed s.

Let us recall some notions of Ehrhart theory. Given any finite point set $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}^n$, a lattice polytope $P \subset \mathbb{R}^n$ is the smallest convex set containing these points, that is

$$P = \{x_1v_1 + x_2v_2 + \cdots + x_nv_n : \text{ all } x_i \geq 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}.$$

For a positive integer t, define tP to be the polytope obtained by scaling P by t, that is, scaling any point $x \in P$ by $1/t$. For $t \geq 0$, let $L(P,t)$ denote the number of lattice points in tP, that is,

$$L(P,t) = \#\{\mathbb{Z}^n \cap tP\}.$$

Ehrhart showed that $L(P,t)$ is a polynomial of degree n in t. This result is named Ehrhart's theorem. For more detailed introduction to Ehrhart theory, see [5]. Another fact we need is Ehrhart reciprocity [12], which states that

$$L(P,-t) = (-1)^n L(P^\circ, t),$$

where P° denotes the interior of P. The results of Ehrhart theory can be extended to an analogy between integrating a polynomial over a region and summing it over the lattice points in a polytope. Specifically, if f is a polynomial of degree d on $\mathbb{R}^n$, then we have $\int_{tP} f$ is a polynomial of degree $d+n$. Euler-Maclaurin theory says that the discrete analog

$$L(f,P,t) = \sum_{x\in\mathbb{Z}^n\cap tP} f(x)$$

is also a polynomial of degree $d+n$. Ehrhart reciprocity [2] also can be extended as

$$L(f,P,-t) = (-1)^n L(f,P^\circ,t).$$

Recall that a standard simplex $\Delta_n \subset \mathbb{R}^{n+1}$ is a special polytope of dimension n and it can be realized by hyperplane description, namely,

$$\Delta_n = \left\{(x_0,x_1,\ldots,x_n) \in \mathbb{R}^{n+1}\colon x_0+x_1+\cdots+x_n = 1 \text{ and all } x_k \geq 0\right\}.$$

In the case of the standard simplex, the dilate $t\Delta_n$ is given by

$$t\Delta_n = \left\{(x_0,x_1,\ldots,x_n) \in \mathbb{R}^{n+1}\colon x_0+x_1+\cdots+x_n = t \text{ and all } x_k \geq 0\right\}.$$

Note that the points in the interior of $t\Delta_n$ satisfy that $x_k > 0$ for $0 \leq k \leq n$.

In [10], the set of (s,t)-cores is parameterized by the set $TD_s(t)$, where

$$TD_s(t) = \left\{(z_{s,i})_{i\in\mathbb{Z}/s\mathbb{Z}}\colon \sum_{i\in\mathbb{Z}/s\mathbb{Z}} z_{s,i} = t,\ \sum_{i\in\mathbb{Z}/s\mathbb{Z}} iz_{s,i} \equiv 0\ (mod\ s),\ z_{s,i} \equiv 0\ (mod\ 1),\ z_{s,i} \geq 0\right\},$$

and the elements in $TD_s(t)$ are called Johnson's z-coordinates by Wang in [16]. It has been proved [10] that $TD_s(t)$ is a rational simplex and a sublattice of $t\Delta_{s-1}$.

To compute the number of (s,t)-cores and the average size of an (s,t)-core, Johnson [10] established the following relation between (s,t)-cores and lattice points in the simplex $t\Delta_{s-1}$.

Lemma 2.1 *Let s,t be coprime positive integers, then the number of (s,t)-cores is equal to $1/s$ multiplying the number of lattice points in the simplex $t\Delta_{s-1}$, which is a polynomial of degree $s-1$ in t. And the number of self-conjugate (s,t)-cores is equal to the number of lattice points in the simplex $\lfloor t/2\rfloor\Delta_{\lfloor s/2\rfloor}$.*

Moreover, Wang [16] used the z-coordinates to express the size of an (s,t)-core as follows.

Lemma 2.2 *Fix coprime $s, t \geq 1$. The size $|\lambda|$ of an (s,t)-core λ, in the extended z-coordinates, is given by* $f(z) = -\frac{s^2-1}{24} + \frac{s^2-1}{24}\sum_{l\in\mathbb{Z}/s\mathbb{Z}} z_{s,l}^2 + M_2(z)$, *where* $z = (z_{s,0}, \ldots, z_{s,s-1})$ *and* $M_2(z) \in \mathbb{Z}[z_{s,0}, \ldots, z_{s,s-1}]$ *is the "leftover" cyclic homogeneous quadratic with only 'mixed' terms (i.e., no square terms* $z_{s,0}^2, \ldots, z_{s,s-1}^2$*), and with coefficients sum* $-\frac{1}{24}s(s^2-1)$.

By Euler-Maclaurin theory, Johnson gave the following lemma.

Lemma 2.3 *Let s, t be coprime positive integers, then the sum of sizes over all (s,t)-cores is equal to $1/s$ multiplying the sum of $f(z)$ over the lattice points in the simplex $t\Delta_{s-1}$ where $f(z)$ is defined in Lemma 2.2, which is a polynomial of degree $s+1$ in t.*

Combining Lemma 2.1 and Lemma 2.3, Johnson proved the following theorem.

Theorem 2.4 *For fixed s, and t relatively prime to s, then the average size of an (s,t)-core is a polynomial of degree 2 in t.*

In [10], Johnson proved that the set of self-conjugate (s,t)-cores can be parameterized by the subset of points $(z_{s,i})_{i\in\mathbb{Z}/s\mathbb{Z}} \in TD_s(t)$ satisfying the symmetry $z_{s,i} = z_{s,-i}$, which corresponds to the set of lattice points in the simplex $\lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor}$. In other words, assume $s = 2k+1$, a lattice point $u = (u_0, u_1, \ldots, u_k) \in \lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor}$ corresponds to a self-conjugate (s,t)-core, which the corresponding point $z \in TD_s(t)$ is of the form $z = (2u_0+1, u_1, \ldots, u_k, u_k, \ldots, u_1)$, if t is odd; and $z = (2u_0, u_1, \ldots, u_k, u_k, \ldots, u_1)$, if t is even.

Thus, by Lemma 2.2, we can express the size of a self-conjugate (s,t)-core as follows.

Lemma 2.5 *If s and t are coprime positive integers, assume s is odd, say $s = 2k+1$, then the size of a self-conjugate (s,t)-core λ is given by*

$$f_1(u) = \begin{cases} f(2u_0+1, u_1, \ldots, u_k, u_k, \ldots, u_1), & \textit{if } t \textit{ is odd;} \\ f(2u_0, u_1, \ldots, u_k, u_k, \ldots, u_1), & \textit{if } t \textit{ is even.} \end{cases} \tag{2.1}$$

where $u = (u_0, u_1, \ldots, u_k)$ is a lattice point in the simplex $\lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor}$, and f is defined in Lemma 2.2.

To get the moments about the mean of the random variable $\xi_{s,t}$, we first prove that the r-th moment of $\xi_{s,t}$ is a polynomial of degree $2r$ in t.

Theorem 2.6 *For fixed s, and t relatively prime to s. Then $\mathbf{E}[\xi_{s,t}^r]$ is a polynomial of degree $2r$ in t.*

Proof. Let $D_{s,t}$ denote the set of all self-conjugate (s,t)-cores and $|\lambda|$ denote the random variable $\xi_{s,t}$, then

$$\mathbf{E}[\xi_{s,t}^r] = \mathbf{E}[|\lambda|^r] = \frac{\sum_{\lambda \in D_{s,t}} |\lambda|^r}{|D_{s,t}|}.$$

Since s and t are coprime, without loss of generality, we assume that s is odd, say $s = 2k+1$.

To prove $\mathbf{E}[\xi_{s,t}^r]$ is a polynomial of degree $2r$ in t, we first prove that $\sum_{\lambda \in D_{s,t}} |\lambda|^r$ and $|D_{s,t}|$ are both polynomials in t. First, we concern with $|D_{s,t}|$. Denote $|D_{s,t}|$ by $F(s,t)$. We claim that $F(s,t)$ is a polynomial of degree k in t. By Lemma 2.1,

$$\begin{aligned} F(s,t) &= \sum_{u \in \mathbb{Z}^{k+1} \cap \lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor}} 1 \\ &= \binom{\lfloor t/2 \rfloor + \lfloor s/2 \rfloor}{\lfloor s/2 \rfloor} \\ &= \binom{\lfloor t/2 \rfloor + k}{k}. \end{aligned}$$

Obviously, it is a polynomial of degree k in t.

We proceed to consider $\sum_{\lambda \in D_{s,t}} |\lambda|^r$. Let $G_r(s,t)$ denote $\sum_{\lambda \in D_{s,t}} |\lambda|^r$. According to Lemma 2.5, we get

$$G_r(s,t) = \sum_{u \in \mathbb{Z}^{k+1} \cap \lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor}} f_1^r(u). \tag{2.2}$$

By the definition of f in Lemma 2.2, we have that $f_1^r(u)$ is a polynomial of degree $2r$ in u. It follows from Euler-Maclaurin theory that $G_r(s,t)$ is a polynomial of degree $k+2r$ in t.

To prove that the r-th moment of $\xi_{s,t}$ is a polynomial of degree $2r$ in t, we aim to show that each root of $F(s,t)$ is also a root of $G_r(s,t)$.

When t is odd, since $F(s,t) = \binom{\lfloor t/2 \rfloor + k}{k}$, the roots of $F(s,t)$ are $-1, -3, \ldots, -(2k-1)$. On the other hand, by Ehrhart reciprocity, we obtain that

$$G_r(s,-x) = (-1)^k \sum_{u \in \Omega} f_1^r(u) \tag{2.3}$$

where

$$\begin{aligned} \Omega &= \mathbb{Z}^{k+1} \cap (-\lfloor -x/2 \rfloor \Delta_{\lfloor s/2 \rfloor})^\circ \\ &= \left\{ (u_0, u_1, \ldots, u_k) \in \mathbb{N}^{k+1} \colon u_0 + u_1 + \cdots + u_k = -\lfloor -x/2 \rfloor \text{ and all } u_i \geq 1 \right\}. \end{aligned}$$

For $-\lfloor -x/2 \rfloor \leq k$, by the definition of Ω, the interior of $-\lfloor -x/2 \rfloor \Delta_{\lfloor s/2 \rfloor}$ is empty. Thus $G_r(s,t)$ vanishes at $t = -1, -3, \ldots, -(2k-1)$. It follows that $G_r(s,t)/F(s,t)$ is a polynomial of degree $2r$ in t.

Similarly, one can show that $\mathbf{E}[\xi_{s,t}^r]$ is a polynomial of degree $2r$ in t when t is even. Note that in this case, both $F(s,t)$ and $G_r(s,t)$ vanish at $t = -2, -4, \ldots, -2k$. This completes the proof. ∎

From Theorem 2.6, we can deduce the following theorem which determines the degree of the r-th moment about the mean of $\xi_{s,t}$.

Theorem 2.7 *For fixed s, and t relatively prime to s, the r-th moment about the mean $\mathbf{E}[(\xi_{s,t} - \mathbf{E}[\xi_{s,t}])^r]$ of the random variable $\xi_{s,t}$ is a polynomial of degree $2r$ in t, for $r \geq 2$. If we denote $\mathbf{E}[(\xi_{s,t} - \mathbf{E}[\xi_{s,t}])^r]$ by $M_r(s,t)$, then*

$$M_r(s,t) = M_r(s, -s-t),$$

for s is odd.

Proof. Let $M_r(s,t)$ denote the r-th moment about the mean of the random variable $\xi_{s,t}$, then

$$\begin{aligned} M_r(s,t) &= \mathbf{E}[(\xi_{s,t} - \mathbf{E}[\xi_{s,t}])^r] \\ &= \sum_{i=0}^{r} \binom{r}{i} \mathbf{E}[\xi_{s,t}^i](-\mathbf{E}[\xi_{s,t}])^{r-i} \\ &= \sum_{i=0}^{r} \binom{r}{i} \frac{G_i(s,t)}{F(s,t)} \left(-\frac{(s-1)(t-1)(s+t+1)}{24} \right)^{r-i}. \end{aligned} \tag{2.4}$$

By Theorem 2.6, $\frac{G_i(s,t)}{F(s,t)}$ is a polynomial of degree $2i$ in t. On the other hand, it is obvious that $\left(-\frac{(s-1)(t-1)(s+t+1)}{24}\right)^{r-i}$ is a polynomial of degree $2r - 2i$ in t. Thus, by (2.4) we find that $M_r(s,t)$ is a polynomial of degree $2r$ in t.

Now we are in the position to prove $M_r(s,t) = M_r(s,-s-t)$. Replacing t with $-s-t$ in (2.4) leads to

$$M_r(s,-s-t) = \sum_{i=0}^{r} \binom{r}{i} \frac{G_i(s,-s-t)}{F(s,-s-t)} \left(-\frac{(s-1)(-s-t-1)(-t+1)}{24} \right)^{r-i}. \tag{2.5}$$

Thus, to prove $M_r(s,t) = M_r(s,-s-t)$, it suffices to show

$$\frac{G_i(s,t)}{F(s,t)} = \frac{G_i(s,-s-t)}{F(s,-s-t)}. \tag{2.6}$$

By Ehrhart reciprocity, we have

$$G_i(s,-s-t) = (-1)^k \sum_{u \in \Omega} f_1^i(u)$$

where

$$\Omega = \mathbb{Z}^{k+1} \cap \left(- \left\lfloor \frac{-s-t}{2} \right\rfloor \Delta_{\lfloor s/2 \rfloor} \right)^{\circ}. \tag{2.7}$$

Since

$$\begin{aligned}
&\mathbb{Z}^{k+1} \cap \left(- \left\lfloor \frac{-s-t}{2} \right\rfloor \Delta_{\lfloor s/2 \rfloor} \right)^{\circ} \\
&= \Big\{ (u_0, u_1, \ldots, u_k) \in \mathbb{N}^{k+1} \colon u_0 + u_1 + \cdots + u_k = - \left\lfloor \frac{-s-t}{2} \right\rfloor \text{ and all } u_i \geq 1 \Big\} \\
&= \Big\{ (u_0, u_1, \ldots, u_k) \in \mathbb{N}^{k+1} \colon u_0 + u_1 + \cdots + u_k = \lfloor t/2 \rfloor \text{ and all } u_i \geq 0 \Big\} \\
&= \lfloor t/2 \rfloor \Delta_{\lfloor s/2 \rfloor},
\end{aligned}$$

we arrive at that $G_i(s, -s-t) = (-1)^k G_i(s, t)$ by combining (2.2) and (2.7). Similarly, we get $F(s, -s-t) = (-1)^k F(s, t)$. It implies that (2.6) holds. Combining (2.4)–(2.6), we conclude that $M_r(s, -s-t) = M_r(s, t)$. This completes the proof. ∎

3 Moments of the random variable $\xi_{s,t}$ for special t

In Section 2, we have proved that if s, t are coprime positive integers, then $\mathbf{E}[(\xi_{s,t} - \mathbf{E}[\xi_{s,t}])^r]$ is a polynomial of degree $2r$ in t for $r \geq 2$. If we have $2r+1$ values of this polynomial, we can determine this polynomial. Hence, to determine these polynomials, we will give the formulae for the variance about the mean for $t = 2, 3, 4$, and the third moment about the mean for $t = 2, 3, 4, 5, 6$, and the forth moment about the mean of $\xi_{s,t}$ for $t = 2, 3, 4, 5, 6, 7, 8$ in this section.

We begin with a review of the work on the structure of self-conjugate (s, t)-cores. Throughout this section, let λ be a self-conjugate partition, we define

$$MD(\lambda) = \Big\{ h \mid h \text{ is the hook length of a cell on the main diagonal of } \lambda \Big\}.$$

It is clear that MD(λ) uniquely determines λ. Ford, Mai and Sze [8] characterized the main diagonal hook lengths set of a self-conjugate t-core. We restate this characterization as the following theorem.

Theorem 3.1 *Let λ be a self-conjugate partition. Then λ is a t-core if and only if both of the following hold:*

(1)If $h \in MD(\lambda)$ with $h > 2t$, then $h - 2t \in MD(\lambda)$;

(2)If $h, l \in MD(\lambda)$, then $h + l \not\equiv 0$ (mod $2t$).

To describe the main diagonal hook lengths of a self-conjugate (s, t)-core, Ford, Mai and Sze [8] introduced an integer array $A = (A_{i,j})_{1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor}$, where

$$A_{i,j} = st - (2j-1)s - (2i-1)t, \ 1 \leq i \leq \lfloor s/2 \rfloor, \ 1 \leq j \leq \lfloor t/2 \rfloor. \tag{3.1}$$

83	67	51	35	19	3
57	41	25	9	−7	−23
31	15	−1	−17	−33	−49
5	−9	−27	−43	−59	−75

Figure 1: A lattice path in the array $A(8, 13)$

Let $\mathcal{P}(A)$ be the set of lattice paths in A from the lower-left corner to the upper-right corner. See, Figure 1 for an example of $s = 8$, $t = 13$, and the solid lines represent a lattice path in $\mathcal{P}(A)$. For a lattice path in $\mathcal{P}(A)$, let $M_A(P)$ denote the set of positive entries $A_{i,j}$ below P and the absolute values of negative entries above P.

Ford, Mai and Sze [8] stated the following theorem.

Theorem 3.2 *Assume that s and t are coprime. Let A be the array as given in* (3.1). *Then there is a bijection ϕ between the set $\mathcal{P}(A)$ of lattice paths and the set of self-conjugate (s, t)-cores such that for $P \in \mathcal{P}(A)$, the set of main diagonal hook lengths of $\phi(P)$ is given by $M_A(P)$.*

Based on Theorem 3.2, Ford, Mai and Sze [8] deduced that the number of self-conjugate (s, t)-cores is

$$\binom{\lfloor s/2\rfloor + \lfloor t/2\rfloor}{\lfloor s/2\rfloor}. \tag{3.2}$$

Chen, Huang and Wang [6] proved the following theorem due to the bijection ϕ, which was used to calculate the expectation of the random variable $\xi_{s,t}$.

Theorem 3.3 *For any lattice path P in $\mathcal{P}(A)$, we have*

$$\mid \phi(A) \mid = \frac{(s^2-1)(t^2-1)}{24} - \sum_{(i,j)\ \text{is above}\ P} A_{i,j}. \tag{3.3}$$

Notice that each lattice path $P \in \mathcal{P}(A)$ corresponds to a unique sequence $(m_1, m_2, \ldots, m_{\lfloor t/2\rfloor})$ satisfying

$$\lfloor s/2\rfloor \geq m_1 \geq \cdots \geq m_{\lfloor t/2\rfloor -1} \geq m_{\lfloor t/2\rfloor} \geq 0,$$

where m_i is the number of boxes in the i-th column above the lattice path P in array A. Hence, Theorem 3.3 can be restated as follow.

Lemma 3.4 *Let s,t be coprime positive integers. Denoting the self-conjugate (s,t)-core whose corresponding sequence is $(m_1, m_2, \ldots, m_{\lfloor t/2 \rfloor})$ by $\lambda(m_1, m_2, \ldots, m_{\lfloor t/2 \rfloor})$, we have*

$$|\lambda(m_1, m_2, \ldots, m_{\lfloor t/2 \rfloor})| = \frac{(s^2-1)(t^2-1)}{24} - \sum_{j=1}^{\lfloor t/2 \rfloor} \sum_{i=0}^{m_j} A_{i,j}. \tag{3.4}$$

Applying Lemma 3.4, we can deduce the formulae for the variance, the third moment and the forth moment about the mean of the random variable $\xi_{s,t}$ for small t. Recall that the r-th moment about the mean of the random variable $\xi_{s,t}$ is given by

$$\mathbf{E}[(\xi_{s,t} - \mathbf{E}[\xi_{s,t}])^r] = \frac{\sum_{\lambda \in D_{s,t}} (|\lambda| - \mathbf{E}[|\lambda|])^r}{|D_{s,t}|}, \tag{3.5}$$

where $D_{s,t}$ denotes the set of all self-conjugate (s,t)-cores.

First, let us consider the case for $t = 2$.

Theorem 3.5 *Let s be an odd positive integer, then the variance about the mean of the random variable $\xi_{s,2}$ is*

$$M_2(s,2) = \frac{s(s-1)(s+2)(s+3)}{720},$$

the third moment about the mean of the random variable $\xi_{s,2}$ is

$$M_3(s,2) = \frac{s(s-1)(s-3)(s+2)(s+5)(s+3)}{30240},$$

and the forth moment about the mean of the random variable $\xi_{s,2}$ is

$$M_4(s,2) = \frac{s(s-1)(s+3)(s+2)(s^4+4s^3-11s^2-30s+84)}{241920}.$$

Proof. Notice that the set of 2-core partitions consists of those partitions of staircase shape $(i, i-1, \ldots, 1)$ for $i \in \mathbb{N}$, along with the empty partition. Since s is odd, we assume that $s = 2k+1$. By (3.2), we have that the number of $(2k+1, 2)$-cores is $k+1$.

Let λ be a self-conjugate $(s,2)$-core. By (1.1), we obtain $\mathbf{E}[|\lambda|] = \frac{k(k+2)}{6}$, that is, the expectation of the random variable $\xi_{s,2}$ is $\frac{k(k+2)}{6}$. Thus, by the definition (3.5), the variance of the random variable $\xi_{s,2}$ can be expressed as follows

$$\begin{aligned} M_2(s,2) &= \frac{\sum_{\lambda \in D_{s,2}} \left(|\lambda| - \frac{k(k+2)}{6}\right)^2}{k+1} \\ &= \sum_{i=1}^{k} \frac{\left(\binom{i+1}{2} - \frac{k(k+2)}{6}\right)^2}{k+1} \end{aligned}$$

$$= \frac{k(3k^3 + 12k^2 + 13k + 2)}{60}. \tag{3.6}$$

Substituting k with $(s-1)/2$ in (3.6) leads to

$$M_2(s,2) = \frac{s(s-1)(s+2)(s+3)}{720}. \tag{3.7}$$

Using the same method, we get that the third moment and the forth moment about the mean of the random variable $\xi_{s,2}$ are

$$M_3(s,2) = \frac{s(s-1)(s-3)(s+2)(s+5)(s+3)}{30240} \tag{3.8}$$

and

$$M_4(s,2) = \frac{s(s-1)(s+3)(s+2)(s^4 + 4s^3 - 11s^2 - 30s + 84)}{241920}, \tag{3.9}$$

respectively. ∎

Next we proceed to give the variance, the third moment and the forth moment about the mean of the random variable $\xi_{s,3}$.

Theorem 3.6 *Let s be odd and relatively prime to 3, then the variance about the mean of the random variable $\xi_{s,3}$ is*

$$M_2(s,3) = \frac{s(s-1)(2s+9)(s+3)}{240},$$

the third moment about the mean of the random variable $\xi_{s,3}$ is

$$M_3(s,3) = \frac{s(s-1)(s+3)(s-3)(2s+11)(s+5)}{2240}$$

and the forth moment about the mean of the random variable $\xi_{s,3}$ is

$$M_4(s,3) = \frac{s(s-1)(s+3)(6s^5 + 63s^4 + 41s^3 - 1092s^2 - 729s + 8127)}{26880}.$$

Proof. When $t = 3$, we have $\lfloor t/2 \rfloor = 1$. Thus the array $A(s,3)$ consists of one column, and $\lfloor s/2 \rfloor$ rows. For example, see Figure 2 for $A(13,3)$. To get the moments about the mean of $\xi_{s,3}$, we need to compute $|D_{s,3}|$, $\mathbf{E}[\xi_{s,3}]$ and the formula for the size of each self-conjugate $(s,3)$-core.

First, let us consider $|D_{s,3}|$. Since $t = 3$, by (3.2) we have

$$|D_{s,3}| = \lfloor s/2 \rfloor + 1. \tag{3.10}$$

By (1.1), we get

$$\mathbf{E}[\xi_{s,3}] = \frac{(s+4)(s-1)}{12}. \tag{3.11}$$

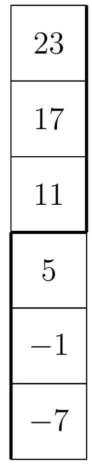

Figure 2: A lattice path in the array $A(13,3)$

Now we are in the position to concern with the size of a self-conjugate $(s,3)$-core. By Lemma 3.4, each $0 \le m_1 \le \lfloor s/2 \rfloor$ corresponds to a unique $(s,3)$-core. Let $\lambda(m_1)$ denote this $(s,3)$-core, then we have

$$D(s,3) = \{\lambda(m_1)\colon\ 0 \le m_1 \le \lfloor s/2 \rfloor\}. \tag{3.12}$$

Setting $j = 1$ in the definition (3.1) of $A_{i,j}$, we have

$$A_{i,1} = 2s - 3(2i-1),\ 1 \le i \le \lfloor s/2 \rfloor. \tag{3.13}$$

Combining (3.4) and (3.13) leads to

$$\begin{aligned} |\lambda(m_1)| &= \frac{s^2-1}{3} - \sum_{i=0}^{m_1} A_{i,1} \\ &= \frac{s^2-1}{3} - 2sm_1 + 3m_1^2. \end{aligned} \tag{3.14}$$

Applying (3.10), (3.11), (3.12) and (3.14) to (3.5), we get

$$M_2(s,3) = \sum_{m_1=0}^{\lfloor s/2 \rfloor} \frac{\left((s^2-1)/3 - 2sm_1 + 3m_1^2 - (s+4)(s-1)/12\right)^2}{\lfloor s/2 \rfloor + 1}. \tag{3.15}$$

Since s is odd and relatively prime to 3, we can assume that $s = 6k+1$ or $s = 6k+5$ for certain nonnegative integer k.

When $s = 6k+1$, the equality (3.15) can be rewritten as

$$M_2(s,3) = \sum_{m_1=0}^{3k} \frac{\left(12k^2 + 4k - 2(6k+1)m_1 + 3m_1^2 - k(6k+5)/2\right)^2}{3k+1}$$

$$= \frac{k(3k+2)(12k+11)(6k+1)}{20}.$$

Substituting k by $\frac{s-1}{6}$ in the above equality, we get

$$M_2(s,3) = \frac{s(s-1)(2s+9)(s+3)}{240}.$$

When $s = 6k+5$, we have

$$M_2(s,3) = \sum_{m_1=0}^{3k+2} \frac{\left(12k^2+20k+8-2(6k+5)m_1+3m_1^2-(2k+3)(3k+2)/2\right)^2}{3k+3}$$
$$= \frac{(6k+5)(3k+2)(3k+4)(12k+19)}{60}.$$

Substituting k by $\frac{s-5}{6}$ in the above equality, we get

$$M_2(s,3) = \frac{s(s-1)(2s+9)(s+3)}{240}.$$

Thus, we assert that when s is odd and relatively prime to 3,

$$M_2(s,3) = \frac{s(s-1)(2s+9)(s+3)}{240}. \tag{3.16}$$

By the same argument, we obtain

$$M_3(s,3) = \frac{s(s-1)(s+3)(s-3)(2s+11)(s+5)}{2240} \tag{3.17}$$

and

$$M_4(s,3) = \frac{s(s-1)(s+3)(6s^5+63s^4+41s^3-1092s^2-729s+8127)}{26880}. \tag{3.18}$$

∎

Now we concern with the case for $t = 4$.

Theorem 3.7 *Let s be an odd positive integer, then the variance about the mean of the random variable $\xi_{s,4}$ is*

$$M_2(s,4) = \frac{s(s-1)(s+4)(s+5)}{72},$$

the third moment and the forth moment about the mean of the random variable $\xi_{s,4}$ are

$$M_3(s,4) = \frac{s(s-1)(s+4)(s+5)(2s^2+8s-27)}{840}$$

and

$$M_4(s,4) = \frac{s(s-1)(s+4)(s+5)(23s^4+184s^3-191s^2-2236s+4046)}{25200},$$

respectively.

35	9
27	1
19	−7
11	−15
3	−23
−5	−31

Figure 3: A lattice path in the array $A(13,4)$

Proof. When $t=4$, we have $\lfloor t/2 \rfloor = 2$. Thus the array $A(s,4)$ consists of two columns and $\lfloor s/2 \rfloor$ rows. For example, see Figure 3 for $A(13,4)$.

To get the moments about the mean of $\xi_{s,4}$, it suffices to obtain $|D_{s,4}|$, $\mathbf{E}[\xi_{s,4}]$ and the formula for the size of each self-conjugate $(s,4)$-core.

First we consider $|D_{s,4}|$ and $\mathbf{E}[\xi_{s,4}]$. Since $t=4$, by (3.2), we have

$$|D_{s,4}| = \binom{\lfloor s/2 \rfloor + 2}{2}. \tag{3.19}$$

By (1.1), we get

$$\mathbf{E}[\xi_{s,4}] = \frac{(s+5)(s-1)}{8}. \tag{3.20}$$

We proceed to consider the size of each self-conjugate $(s,4)$-core. By Lemma 3.4, each sequence (m_1, m_2) satisfying $\lfloor s/2 \rfloor \geq m_1 \geq m_2 \geq 0$ corresponds to a unique $(s,4)$-core, denoted by $\lambda(m_1, m_2)$. Thus

$$D(s,4) = \{\lambda(m_1,m_2)\colon\ \lfloor s/2 \rfloor \geq m_1 \geq m_2 \geq 0\}. \tag{3.21}$$

Setting $j=1,2$ in the definition (3.1) of $A_{i,j}$, we have

$$A_{i,1} = 3s - 4(2i-1),\ 1 \leq i \leq \lfloor s/2 \rfloor \tag{3.22}$$

and

$$A_{i,2} = s - 4(2i-1),\ 1 \leq i \leq \lfloor s/2 \rfloor. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.4), we obtain

$$|\lambda(m_1,m_2)| = \frac{5(s^2-1)}{8} - \sum_{i=0}^{m_1} A_{i,1} - \sum_{j=0}^{m_2} A_{j,2}$$

$$= \frac{5(s^2-1)}{8} - 3sm_1 + 4m_1^2 - sm_2 + 4m_2^2. \tag{3.24}$$

Since s is odd, we can assume $s = 2k+1$ for certain nonnegative integer k. Substituting (3.19), (3.20), (3.21)and (3.24) into (3.5), we get

$$\begin{aligned}
M_2(s,4) &= \frac{\sum_{0\le m_2\le m_1\le \lfloor s/2\rfloor} \left(5(s^2-1)/8 - 3sm_1 + 4m_1^2 - sm_2 + 4m_2^2 - (s+5)(s-1)/8\right)^2}{\binom{\lfloor s/2\rfloor+2}{2}} \\
&= \frac{\sum_{0\le m_2\le m_1\le k} \left(5(k^2+k)/2 - (3m_1+m_2)(2k+1) + 4m_1^2 + 4m_2^2 - k(k+3)/2\right)^2}{\binom{k+2}{2}} \\
&= \frac{k(2k+5)(2k+1)(k+3)}{18}.
\end{aligned}$$

Substituting k by $\frac{s-1}{2}$ in the above equality, we get

$$M_2(s,4) = \frac{s(s-1)(s+4)(s+5)}{72}. \tag{3.25}$$

By the same argument, we have that

$$M_3(s,4) = \frac{s(s-1)(s+4)(s+5)(2s^2+8s-27)}{840} \tag{3.26}$$

and

$$M_4(s,4) = \frac{s(s-1)(s+4)(s+5)(23s^4+184s^3-191s^2-2236s+4046)}{25200}. \tag{3.27}$$

∎

Using the same approach, we can deduce the third moments about the mean of $\xi_{s,t}$ for $t = 5, 6$, and the forth moments about the mean of $\xi_{s,t}$ for $t = 5, 6, 7, 8$. We omit the details of these proofs, and state these results as the following theorems.

Theorem 3.8 *Let s be odd and relatively prime to 5, then the third moment and the forth moment about the mean of the random variable $\xi_{s,5}$ are*

$$M_3(s,5) = \frac{s(s-1)(s+5)(50s^3+589s^2+888s-5535)}{6048} \tag{3.28}$$

and

$$M_4(s,5) = \frac{s(s-1)(s+5)(594s^5+10021s^4+36436s^3-128690s^2-489750s+1294125)}{120960}, \tag{3.29}$$

respectively.

Theorem 3.9 *Let s be odd and relatively prime to 6, then the third moment and the forth moment about the mean of the random variable $\xi_{s,6}$ are*

$$M_3(s,6) = \frac{3s(s-1)(s+6)(s+7)(5s^2+30s-79)}{1120} \tag{3.30}$$

and

$$M_4(s,6) = \frac{s(s-1)(s+7)(s+6)(443s^4+5316s^3+2099s^2-83094s+125316)}{44800}, \tag{3.31}$$

respectively.

Theorem 3.10 *Let s be odd and relatively prime to 7, then the forth moment about the mean of the random variable $\xi_{s,7}$ is*

$$M_4(s,7) = \frac{s(s-1)(s+7)(1594s^5+36673s^4+217805s^3-284860s^2-3232509s+6257937)}{57600}. \tag{3.32}$$

Theorem 3.11 *Let s be an odd positive integer, then the forth moment about the mean of the random variable $\xi_{s,8}$ is*

$$M_4(s,8) = \frac{s(s-1)(s+8)(s+9)(949s^4+15184s^3+24115s^2-292968s+404832)}{21600}. \tag{3.33}$$

4 Moments about the mean of $\xi_{s,t}$

In this section, based on the theorems in Section 2 and 3, we will give the proofs of Theorem 1.2, 1.3 and 1.4.

Before proving the main theorems of this paper, we first concern with the r-th moments $M_r(s,t)$ about the mean of $\xi_{s,t}$ for $t = 0, 1$, which will be used in the following proofs of our main results.

Lemma 4.1 *Let s be an odd integer, then $M_r(s,0) = 0$ and $M_r(s,1) = 0$.*

Proof. Recall that $G_r(s,t)$ is the sum of $|\lambda|^r$ over all self-conjugate (s,t)-cores and $F(s,t)$ is the number of all self-conjugate (s,t)-cores. Since 1-core is the empty partition, we have $G_r(s,1) = 0$ and $F(s,1) = 1$. By (2.4), we get

$$M_r(s,1) = 0. \tag{4.1}$$

For $t=0$, by (2.2) we deduce that

$$G_r(s,0)=f_1^r(0,0,\ldots,0)=(-\frac{s^2-1}{24})^r. \tag{4.2}$$

Substituting the above equality into (2.4), we have

$$\begin{aligned}
M_r(s,0)&=\sum_{i=0}^{r}\binom{r}{i}\frac{G_i(s,0)}{F(s,0)}\left(-\frac{(s-1)(-1)(s+1)}{24}\right)^{r-i}\\
&=\sum_{i=0}^{r}\binom{r}{i}\left(-\frac{s^2-1}{24}\right)^{i}\left(\frac{s^2-1}{24}\right)^{r-i}\\
&=\left(\frac{s^2-1}{24}-\frac{s^2-1}{24}\right)^{r}\\
&=0. \qquad (4.3)
\end{aligned}$$

The proof is completed. ∎

Now we are ready to prove the main results of this paper.

Proof of Theorem 1.2. Corollary 2.7 suggests that $M_2(s,t)$ is a polynomial of degree 4 in t. Thus, if we obtain 5 special values of this polynomial, then we can determine this polynomial by the method of undetermined coefficients. According to the proof of Theorem 2.6, we must treat odd and even values of t separately.

First, let us concern with the case that t is odd. Under the condition that s is odd, we have that $-s,-s-2,-s-4$ are odd. Thus, to determine the formula for $M_2(s,t)$, it suffices to get the values of $M_2(s,t)$ at $t=-s,-s-2,-s-4,1,3$. Corollary 2.7 says that

$$M_2(s,-s-t)=M_2(s,t). \tag{4.4}$$

Thus, from Theorem 3.5, Theorem 3.7 and Lemma 4.1, we get

$$M_2(s,-s)=M_2(s,0)=0, \tag{4.5}$$

$$M_2(s,-s-2)=M_2(s,2)=\frac{s(s-1)(s+2)(s+3)}{720}, \tag{4.6}$$

$$M_2(s,-s-4)=M_2(s,4)=\frac{s(s-1)(s+4)(s+5)}{72}. \tag{4.7}$$

On the other hand, by Theorem 3.6 and Lemma 4.1, we get

$$M_2(s,1)=0, \tag{4.8}$$

$$M_2(s,3)=\frac{s(s-1)(2s+9)(s+3)}{240}. \tag{4.9}$$

The equalities (4.5) and (4.8) suggest that $-s$ and 1 are both roots of $M_2(s,t)$. Notice that $M_2(s,t)$ are polynomials of degree 4 in t. Thus, we can assume that

$$M_2(s,t) = (s+t)(t-1)(t^2x + ty + z), \tag{4.10}$$

where x, y, z are polynomials in s. Substituting (4.6), (4.7) and (4.9) into (4.10) leads to

$$\begin{aligned} 9x + 3y + z &= \frac{s(s-1)(2s+9)}{480}, \\ (s+2)^2x - (s+2)y + z &= \frac{s(s-1)(s+2)}{1440}, \\ (s+4)^2x - (s+4)y + z &= \frac{s(s-1)(s+4)}{288}. \end{aligned}$$

Solving these equations gives

$$x = \frac{1}{720}s(s-1), \tag{4.11}$$

$$y = \frac{1}{720}s^3 + \frac{1}{1440}s^2 - \frac{1}{480}s, \tag{4.12}$$

$$z = 0. \tag{4.13}$$

Substituting (4.11), (4.12) and (4.13) into (4.10), we arrive at

$$M_2(s,t) = \frac{st(s+t)(t-1)(s-1)(2s+2t+3)}{1440}.$$

We turn to the case that t is even. Under the condition that s is odd, we have $-s-1, -s-3$ are even. Thus, to determine the formula for $M_2(s,t)$, we only need the values of $M_2(s,t)$ at $t = 0, -s-1, -s-3, 2, 4$. By Theorem 3.6, Lemma 4.1 and equation (4.4), we have

$$M_2(s,-s-1) = M_2(s,1) = 0, \tag{4.14}$$

$$M_2(s,-s-3) = M_2(s,3) = \frac{s(s-1)(2s+9)(s+3)}{240}. \tag{4.15}$$

On the other hand, from Theorem 3.5, Theorem 3.7 and Lemma 4.1, we deduce that

$$M_2(s,0) = 0, \tag{4.16}$$

$$M_2(s,2) = \frac{s(s-1)(s+2)(s+3)}{720}, \tag{4.17}$$

$$M_2(s,4) = \frac{s(s-1)(s+4)(s+5)}{72}. \tag{4.18}$$

The equalities (4.14) and (4.16) mean that 0 and $-s-1$ are roots of $M_2(s,t)$. Notice that $M_2(s,t)$ are polynomials of degree 4 in t. Thus, we can assume

$$M_2(s,t) = t(s+t+1)(t^2x + ty + z), \tag{4.19}$$

where x, y, z are polynomials in s. Substituting (4.15), (4.17) and (4.18) into (4.19), we obtain

$$\begin{aligned} 4x + 2y + z &= \frac{s(s-1)(s+2)}{1440}, \\ (s+3)^2 x - (s+3)y + z &= \frac{s(s-1)(2s+9)}{480}, \\ 16x + 4y + z &= \frac{s(s-1)(s+4)}{288}. \end{aligned}$$

Solving these equations gives

$$x = \frac{s(s-1)}{720}, \tag{4.20}$$

$$y = \frac{s^3}{720} - \frac{s^2}{288} + \frac{s}{480}, \tag{4.21}$$

$$z = -\frac{s^2(s-1)}{480}. \tag{4.22}$$

Substituting (4.20)–(4.22) into (4.19), we conclude that

$$M_2(s,t) = \frac{st(s+t+1)(2t-3)(s-1)(s+t)}{1440},$$

as desired. ∎

Proof of Theorem 1.3. By Corollary 2.7, $M_3(s,t)$ is a polynomial of degree 6 in t. Thus, if we obtain 7 special values of this polynomial, then this polynomial can be determined. With the same method in the proof of Theorem 1.2, we consider that t is odd and t is even separately.

First, let us concern with the case that t is odd. Under the condition that s is odd, $-s, -s-2, -s-4, -s-6$ are odd. From Corollary 2.7, we get

$$M_3(s, -s-t) = M_3(s,t). \tag{4.23}$$

Combining Theorem 3.5, Theorem 3.7, Theorem 3.9, Lemma 4.1 and equation (4.23), we get

$$M_3(s,-s) = M_3(s,0) = 0, \tag{4.24}$$

$$M_3(s,-s-2) = M_3(s,2) = \frac{s(s-1)(s-3)(s+2)(s+5)(s+3)}{30240}, \tag{4.25}$$

$$M_3(s,-s-4)=M_3(s,4)=\frac{s(s-1)(s+4)(s+5)(2s^2+8s-27)}{840}, \tag{4.26}$$
$$M_3(s,-s-6)=M_3(s,6)=\frac{3s(s-1)(s+6)(s+7)(5s^2+30s-79)}{1120}. \tag{4.27}$$

And from Theorem 3.5, Theorem 3.7, Theorem 3.8 and Lemma 4.1, we have

$$M_3(s,1)=0, \tag{4.28}$$
$$M_3(s,3)=\frac{s(s-1)(s+3)(s-3)(2s+11)(s+5)}{2240}, \tag{4.29}$$
$$M_3(s,5)=\frac{s(s-1)(s+5)(50s^3+589s^2+888s-5535)}{6048}. \tag{4.30}$$

Combining (4.24)–(4.30), we obtain

$$M_3(s,t)=\frac{1}{120960}st(s-1)(t-1)(s+t)(16ts^3+16t^3s+32t^2s^2-30s^3-30t^3-29ts^2 \\ -29t^2s-66s^2-66t^2-72st-27s-27t).$$

Now we turn to the case that t is even. Under the condition that s is odd, then $-s-1,-s-3,-s-5$ are even. Combining Theorem 3.6, Theorem 3.8, Lemma 4.1 and equation (4.23), we get

$$M_3(s,-s-1)=M_3(s,1)=0, \tag{4.31}$$
$$M_3(s,-s-3)=M_3(s,3)=\frac{s(s-1)(s+3)(s-3)(2s+11)(s+5)}{2240}, \tag{4.32}$$
$$M_3(s,-s-5)=M_3(s,5)=\frac{s(s-1)(s+5)(50s^3+589s^2+888s-5535)}{6048}. \tag{4.33}$$

In view of Theorem 3.6, Theorem 3.8 and Lemma 4.1, we have

$$M_3(s,0)=0, \tag{4.34}$$
$$M_3(s,2)=\frac{s(s-1)(s-3)(s+2)(s+5)(s+3)}{30240}, \tag{4.35}$$
$$M_3(s,4)=\frac{s(s-1)(s+4)(s+5)(2s^2+8s-27)}{840}, \tag{4.36}$$
$$M_3(s,6)=\frac{3s(s-1)(s+6)(s+7)(5s^2+30s-79)}{1120}. \tag{4.37}$$

Combining (4.31)–(4.37) immediately induces

$$M_3(s,t)=\frac{1}{120960}st(s-1)(s+t)(s+t+1)(16t^2s^2-61ts^2+60s^2-61t^2s+16t^3s \\ +60st+66t^2-27t-30t^3).$$

∎

Proof of Theorem 1.4. Corollary 2.7 suggests that $M_4(s,t)$ is a polynomial of degree 8 in t. Thus, if 9 special values of this polynomial are known, then we can determine this polynomial by the method of undetermined coefficients. By the same approach to the proof of Theorem 1.2 and Theorem 1.3, we obtain

$$\begin{aligned}M_4(s,t) = {} & \frac{1}{4838400} st(s+t+1)(s-1)(s+t)(124t^3s^4 - 766t^2s^4 + 1671ts^4 - 1260s^4 \\ & + 248t^4s^3 + 3342t^2s^3 - 2520ts^3 - 1532t^3s^3 - 4579t^2s^2 + 621s^2t + 124t^5s^2 \\ & + 3975t^3s^2 - 1254t^4s^2 + 1260s^2 + 621st^2 - 3319t^3s + 1260st - 488t^5s \\ & + 2304t^4s + 1530t^2 + 528t^5 + 252t^3 - 1512t^4), \text{ if } t \text{ is even;}\end{aligned}$$

$$\begin{aligned}M_4(s,t) = {} & \frac{1}{4838400} st(s-1)(t-1)(s+t)(528s^5 - 1800ts - 488t^5s + 124t^5s^2 + 336t^4s \\ & - 1186t^4s^2 + 372t^4s^3 + 2729t^3s + 39t^3s^2 - 1396t^3s^3 + 372t^3s^4 + 3694t^2s^2 \\ & + 39t^2s^3 - 1186t^2s^4 + 124t^2s^5 + 2729ts^3 + 336ts^4 - 488ts^5 + 135st^2 + 135s^2t \\ & + 1512s^4 + 252s^3 + 1512t^4 + 528t^5 - 1530s^2 - 1530t^2 + 252t^3), \text{ if } t \text{ is odd.}\end{aligned}$$

This completes the proof. ∎

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

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