Degree powers in graphs with a forbidden forest

Yongxin Lan[1], Henry Liu[2], Zhongmei Qin[3], Yongtang Shi[1*]

[1]Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
yxlan0@126.com, shi@nankai.edu.cn

[2]School of Mathematics
Sun Yat-sen University, Guangzhou 510275, China
liaozhx5@mail.sysu.edu.cn

[3]College of Science
Chang'an University, Xi'an 710064, China
qinzhongmei90@163.com

5 October 2018

Abstract

Given a positive integer p and a graph G with degree sequence $d_1, \ldots, d_n$, we define $e_p(G) = \sum_{i=1}^{n} d_i^p$. Caro and Yuster introduced a Turán-type problem for $e_p(G)$: Given a positive integer p and a graph H, determine the function $\mathrm{ex}_p(n, H)$, which is the maximum value of $e_p(G)$ taken over all graphs G on n vertices that do not contain H as a subgraph. Clearly, $\mathrm{ex}_1(n, H) = 2\mathrm{ex}(n, H)$, where $\mathrm{ex}(n, H)$ denotes the classical Turán number. Caro and Yuster determined the function $\mathrm{ex}_p(n, P_\ell)$ for sufficiently large n, where $p \geq 2$ and P_ℓ denotes the path on ℓ vertices. In this paper, we generalise this result and determine $\mathrm{ex}_p(n, F)$ for sufficiently large n, where $p \geq 2$ and F is a linear forest. We also determine $\mathrm{ex}_p(n, S)$, where S is a star forest; and $\mathrm{ex}_p(n, B)$, where B is a broom graph with diameter at most six.

Keywords: degree power; Turán-type problem; H-free; forest

AMS Subject Classification (2010): 05C07, 05C35

1 Introduction

For standard graph-theoretic notation and terminology, the reader is referred to [1]. All graphs considered here are finite, undirected, and have no loops or multiple edges. Let G and H be two graphs. The degree of a vertex $v \in V(G)$ and the maximum degree of G are

---

*Corresponding author

1

denoted by $d_G(v)$ and $\Delta(G)$. We use $G \cup H$ to denote the disjoint union of G and H, and $G + H$ for the *join* of G and H, i.e., the graph obtained from $G \cup H$ by adding all edges between G and H. Let kG denote k vertex-disjoint copies of G. For $U \subset V(G)$, let $G[U]$ denote the subgraph of G induced by U. Let K_t, E_t and P_t denote the complete graph, the empty graph, and the path on t vertices, respectively. Let S_r denote the star with maximum degree r. Let M_t be the graph on t vertices with a maximum matching (i.e., $\lfloor \frac{t}{2} \rfloor$ independent edges).

Given a graph H, we say that a graph G is H-*free* if G does not contain H as a subgraph. The classical *Turán number*, denoted by $\text{ex}(n, H)$, is the maximum number of edges in a H-free graph on n vertices. Turán's classical result [15] states that $\text{ex}(n, K_{r+1}) = e(T_r(n))$ for $n \geq r \geq 2$, where $T_r(n)$ denotes the r-*partite Turán graph* on n vertices. Given a graph G whose degree sequence is $d_1, \ldots, d_n$, and a positive integer p, let $e_p(G) = \sum_{i=1}^{n} d_i^p$. Caro and Yuster [5] introduced a Turán-type problem for $e_p(G)$: Determine the function $\text{ex}_p(n, H)$, which is the maximum value of $e_p(G)$ taken over all H-free graphs G on n vertices. Moreover, characterise the *extremal graphs*, i.e., the H-free graphs G on n vertices with $e_p(G) = \text{ex}_p(n, H)$. Clearly, we have $\text{ex}_1(n, H) = 2\text{ex}(n, H)$.

This Turán-type problem has attracted significant interest from many researchers. Caro and Yuster [5] proved that $\text{ex}_p(n, K_{r+1}) = e_p(T_r(n))$ for $p = 1, 2, 3$. The same result does not hold if r is fixed, and p and n are sufficiently large. For example, if G is the complete bipartite graph with class sizes $\lfloor \frac{n}{2} \rfloor - 1$ and $\lceil \frac{n}{2} \rceil + 1$, then we have $e_4(G) > e_4(T_2(n))$. Hence, we see that the parameter p does play a role in the value of $\text{ex}_p(n, K_{r+1})$ and the extremal graphs. Bollobás and Nikiforov further studied the function $\text{ex}_p(n, K_{r+1})$, where they allowed $p > 0$ to be real. In [2], they proved that for n sufficiently large, $\text{ex}_p(n, K_{r+1}) = e_p(T_r(n))$ for $0 < p < r$, and $\text{ex}_p(n, K_{r+1}) > (1 + \varepsilon)e_p(T_r(n))$ for $p \geq r + \lceil \sqrt{2r} \rceil$ and some $\varepsilon = \varepsilon(r) > 0$. In [3], they proved a result which gives an extension of the Erdős-Stone Theorem [8] by using $e_p(G)$ instead of the number of edges.

When considering cycles as the forbidden subgraphs, Caro and Yuster [5] proved that $\text{ex}_2(n, \mathcal{C}) = e_2(F_n)$ for sufficiently large n, where $\mathcal{C}$ denotes the family of cycles with even length (notice the natural extension of the definition of ex_p to families of graphs), and F_n is the *friendship graph* on n vertices, i.e., F_n is obtained by taking a star on n vertices and adding a maximum matching on the set of leaves. They also showed that F_n is the unique extremal graph, and remarked that the same result also holds for $p > 2$. Nikiforov [13] proved that $\text{ex}_p(n, C_{2k+2}) = (1 + o(1))kn^p$, where C_t denotes the cycle of order t, and this settled a conjecture of Caro and Yuster. Gu et al. [11] proved that for $p \geq 1$, there exists a constant $c = c(p)$ such that the following holds: If $\text{ex}_p(n, C_5) = e_p(G)$ for some C_5-free graph G of order n, then G is a complete bipartite graph with class sizes $cn + o(n)$ and $(1 - c)n + o(n)$.

A *linear forest* (resp. *star forest*) is a forest whose connected components are paths (resp. stars). There are many known results about the function $\text{ex}_p(n, F)$ where F is a linear forest. For the case of the classical Turán number $\text{ex}(n, F)$, one of the earliest results is the case when $F = P_\ell$ is a path. Erdős and Gallai [7] proved in 1959 that $\text{ex}(n, P_\ell) \leq (\frac{\ell}{2} - 1)n$ for $\ell \geq 2$, and if $\ell - 1$ divides n, then equality holds only for the graph with vertex-disjoint copies of $K_{\ell-1}$. Motivated by this result, Erdős and Sós [6] in 1963 made the conjecture that the same result holds for any tree, i.e., if T is a tree on $t \geq 2$ vertices, then we have $\text{ex}(n, T) \leq (\frac{t}{2} - 1)n$. This long-standing conjecture remains open, and many partial results are

known. The result of Erdős and Gallai was also sharpened by Faudree and Schelp [9], when they determined the function $\mathrm{ex}(n, P_\ell)$ exactly as well as the extremal graphs. When F has more components, Erdős and Gallai [7] also proved that $\mathrm{ex}(n, kP_2) = \binom{k-1}{2} + (k-1)(n-k+1)$ for $k \geq 2$ and sufficiently large n, where the unique extremal graph is $K_{k-1} + E_{n-k+1}$. Very recently, this result was extended by Bushaw and Kettle [4], who determined the function $\mathrm{ex}(n, kP_\ell)$ for $k \geq 2$, $\ell \geq 3$ and sufficiently large n. Their result was further generalised by Lidický et al. [12], who determined the function $\mathrm{ex}(n, F)$ for an arbitrary linear forest F and sufficiently large n. In these two results, the extremal graph is unique. Lidický et al. [12] also determined the function $\mathrm{ex}(n, S)$ for an arbitrary star forest S and sufficiently large n, and characterised the extremal graphs.

On the other hand, Caro and Yuster [5] determined the function $\mathrm{ex}_p(n, P_\ell)$ for $p \geq 2$, $\ell \geq 3$ and sufficiently large n. The extremal graph is again unique, and is significantly different to the extremal graphs of $\mathrm{ex}(n, P_\ell)$ obtained by Faudree and Schelp [9]. They also determined the functions $\mathrm{ex}_p(n, S_r)$ and $\mathrm{ex}_p(n, S_r^*)$, and their extremal graphs, where S_r^* is the graph obtained by attaching a pendent edge at a leaf of S_r.

This paper will be organised as follows. In Section 2, we will state precisely the previously known results about the function $\mathrm{ex}_p(n, F)$, for various forests F. In Sections 3 and 4, we will determine the function $\mathrm{ex}_p(n, F)$ when F is a linear forest, a star forest, and a broom with diameter at most 6 (A *broom* is a path with a star attached at one end). Our results can be regarded as extensions to many of these previously known results from [4, 5, 7, 12]. Unless otherwise stated, we assume that n is always sufficiently large, and we will make no serious attempt to minimise the lower bound on n. Without going into details, we remark that every large lower bound on n depends only on the forest F, and not the parameter p.

2   Known results

In this section, we will review many of the known results about the function $\mathrm{ex}_p(n, F)$, for various forests F. Some of these results will also be helpful for us to present our results in Sections 3 and 4. First, we collect the results where F is a single component. When F is a path, Caro and Yuster [5] observed that for $p \geq 1$, we have

$$\mathrm{ex}_p(n, P_2) = 0, \quad \text{and} \quad \mathrm{ex}_p(n, P_3) = \begin{cases} n-1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}. \end{cases} \tag{1}$$

Moreover, the unique extremal graph for $\mathrm{ex}_p(n, P_3)$ is M_n, the graph on n vertices with a maximum matching. For $F = P_\ell$, Erdős and Gallai [7] proved the following result, as we have mentioned in the introduction.

Theorem 1. [7] *For $\ell \geq 2$, we have $\mathrm{ex}(n, P_\ell) \leq (\frac{\ell}{2} - 1)n$. Moreover, if $\ell - 1$ divides n, then equality holds only for the graph with vertex-disjoint copies of $K_{\ell-1}$.*

Inspired by Theorem 1, Erdős and Sós [6] made the conjecture that the same result holds for any tree: If T is a tree on $t \geq 2$ vertices, then $\mathrm{ex}(n, T) \leq (\frac{t}{2} - 1)n$. This long-standing conjecture remains open, and many partial results are known. Theorem 1 was subsequently sharpened by Faudree and Schelp [9], when they managed to determine $\mathrm{ex}(n, P_\ell)$ exactly, as well as all the extremal graphs.

3

Theorem 2. [9] *Let $\ell \geq 2$ and $n = a(\ell - 1) + b$, where $a \geq 0$ and $0 \leq b < \ell - 1$. We have*

$$\text{ex}(n, P_\ell) = a\binom{\ell - 1}{2} + \binom{b}{2}.$$

Moreover, the extremal graphs are:

- $aK_{\ell-1} \cup K_b$,

- $a'K_{\ell-1} \cup \left(K_{\ell/2-1} + E_{\ell/2+(a-a'-1)(\ell-1)+b}\right)$, *where ℓ is even, $a > 0$, $b = \frac{\ell}{2}$ or $b = \frac{\ell}{2} - 1$ and $0 \leq a' < a$.*

Caro and Yuster [5] determined the function $\text{ex}_p(n, P_\ell)$ for $p \geq 2$, $\ell \geq 4$, and sufficiently large n, and they showed that the extremal graph is unique. To state their result, we define the graph $H(n, \ell)$ as follows. Let $b = \lfloor \frac{\ell}{2} \rfloor - 1$. Then $H(n, \ell) = K_b + E_{n-b}$ if ℓ is even, and $H(n, \ell)$ is $K_b + E_{n-b}$ with an edge added to E_{n-b} if ℓ is odd.

Theorem 3. [5] *Let $p \geq 2$, $\ell \geq 4$, and $n \geq n_0(\ell)$ be sufficiently large. Then*

$$\text{ex}_p(n, P_\ell) = e_p(H(n, \ell))$$
$$= \begin{cases} b(n-1)^p + (n-b-2)b^p + 2(b+1)^p & \text{if } \ell \text{ is odd,} \\ b(n-1)^p + (n-b)b^p & \text{if } \ell \text{ is even,} \end{cases}$$

where $b = \lfloor \frac{\ell}{2} \rfloor - 1$. Moreover, $H(n, \ell)$ is the unique extremal graph.

They remarked that the extremal graph $H(n, \ell)$ for $\text{ex}_p(n, P_\ell)$, with $p \geq 2$, is very different from the extremal graphs for $\text{ex}(n, P_\ell)$ in Theorem 2. This is because $H(n, \ell)$ has large maximum degree, which plays a role in making the value of $e_p(H(n, \ell))$ large, when $p \geq 2$.

When $F = S_r$ is a star, Caro and Yuster [5] made the observation that $\text{ex}_p(n, S_r)$ is attained by a graph L on n vertices which is an extremal graph for $\text{ex}(n, S_r)$. Clearly if $n \leq r - 1$, we have $L = K_n$. For $n \geq r$, we have L is an $(r-1)$-regular graph if $(r-1)n$ is even, and L has $n-1$ vertices of degree $r-1$ and one vertex of degree $r-2$ if $(r-1)n$ is odd. We call such a graph L a *near $(r-1)$-regular graph*, since L is as close to being $(r-1)$-regular as possible. It is well-known and easy to show that such graphs L exist. Note that we have $e(L) = \lfloor \frac{(r-1)n}{2} \rfloor$. Thus, the observation of Caro and Yuster is the following.

Proposition 4. [5] *Let $p \geq 1$, and let S_r be the star with maximum degree $r \geq 1$.*

(a) *If $n \leq r - 1$, then $\text{ex}_p(n, S_r) = n(n-1)^p$. Moreover, the unique extremal graph is K_n.*

(b) *If $n \geq r$, then*

$$\text{ex}_p(n, S_r) = \begin{cases} (n-1)(r-1)^p + (r-2)^p & \text{if } (r-1)n \text{ is odd,} \\ n(r-1)^p & \text{if } (r-1)n \text{ is even.} \end{cases}$$

Moreover, the extremal graphs are the near $(r-1)$-regular graphs on n vertices.

For $\ell \geq 4$ and $s \geq 0$, let $B_{\ell,s}$ be the graph on $\ell + s$ vertices, obtained by adding s pendent edges to a penultimate vertex v of P_ℓ. Such a graph $B_{\ell,s}$ is a *broom*, and v is the *centre* of the broom.
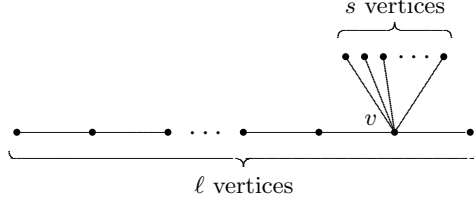

Figure 1. The broom graph $B_{\ell,s}$

It is interesting to study Turán-type problems for brooms, because a broom may be considered as a generalisation of both a path and a star. Sun and Wang [14] determined the function $\mathrm{ex}(n, B_{4,s})$ for $s \geq 1$, as follows.

Theorem 5. [14] *Let $s \geq 1$ and $n \geq s+4$. Let $n = a(s+3)+b$, where $a \geq 1$ and $0 \leq b < s+3$. We have*

$$\mathrm{ex}(n, B_{4,s}) = \begin{cases} (a-1)\dbinom{s+3}{2} + \left\lfloor \dfrac{(s+1)(s+3+b)}{2} \right\rfloor & \text{if } s \geq 3 \text{ and } 2 \leq b \leq s, \\ a\dbinom{s+3}{2} + \dbinom{b}{2} & \text{otherwise.} \end{cases}$$

Roughly speaking, in Theorem 5, the value of $\mathrm{ex}(n, B_{4,s})$ is attained as follows. If b is close to either 0 or $s+3$, then we would take the graph $aK_{s+3} \cup K_b$. Otherwise, we would take a graph $(a-1)K_{s+3} \cup L$, where L is a near $(s+1)$-regular graph on $s+3+b$ vertices.

Sun and Wang also determined the function $\mathrm{ex}(n, B_{5,s})$ for $s \geq 1$ and $n \geq s+5$. However, their result is complicated to state in full. A key result that they proved is the following.

Theorem 6. [14] *Let $s \geq 1$ and $n \geq s+5$. Let $n = a(s+4)+b$, where $a \geq 1$ and $0 \leq b < s+4$. We have*

$$\mathrm{ex}(n, B_{5,s}) = \begin{cases} (a-1)\dbinom{s+4}{2} + \mathrm{ex}(s+4+b, B_{5,s}) & \text{if } 1 \leq b \leq s, \\ a\dbinom{s+4}{2} + \dbinom{b}{2} & \text{if } b \in \{0, s+1, s+2, s+3\}. \end{cases}$$

Similarly, in Theorem 6, the value of $\mathrm{ex}(n, B_{5,s})$ is attained by $aK_{s+4} \cup K_b$ if b is either 0 or close to $s+4$. Otherwise, we would take a graph $(a-1)K_{s+4} \cup L$, where L is an extremal graph for $B_{5,s}$ on $s+4+b$ vertices.

Caro and Yuster [5] determined the function $\mathrm{ex}_p(n, B_{4,s})$, for $p \geq 2$ and sufficiently large n. They remarked that the result is very different to Proposition 4, even though $B_{4,s}$ is very close to being a star.

Proposition 7. [5] *Let $p \geq 2$, $s \geq 1$, and $n > 2(s+4)$. Then $\mathrm{ex}_p(n, B_{4,s}) = e_p(S_{n-1}) = (n-1)^p + (n-1)$. Moreover, S_{n-1} is the unique extremal graph.*

5

Now we consider the case when the forest F has more than one component. When $F = kP_2$, the classical Turán number $\mathrm{ex}(n, kP_2)$ was determined by Erdős and Gallai [7].

Theorem 8. [7] *Let $k \geq 2$ and $n > \frac{5k}{2} - 1$. We have $\mathrm{ex}(n, kP_2) = \binom{k-1}{2} + (k-1)(n-k+1)$. Moreover, $K_{k-1} + E_{n-k+1}$ is the unique extremal graph.*

For $n \leq \frac{5k}{2} - 1$, Erdős and Gallai also determined $\mathrm{ex}(n, kP_2)$ and the extremal graphs, which are different from those in Theorem 8. For the function $\mathrm{ex}(n, kP_3)$, Yuan and Zhang [16] obtained the following result.

Theorem 9. [16] *Let $k \geq 2$ and $n > 5k - 1$. We have $\mathrm{ex}(n, kP_3) = \binom{k-1}{2} + (k-1)(n-k+1) + \lfloor \frac{n-k+1}{2} \rfloor$. Moreover, $K_{k-1} + M_{n-k+1}$ is the unique extremal graph.*

In fact, Yuan and Zhang completely determined $\mathrm{ex}(n, kP_3)$ and the extremal graphs for all n, which solved a conjecture of Gorgol [10]. Bushaw and Kettle [4] had previously proved the case of Theorem 9 for $n \geq 7k$.

Next, there are results for the case when $F = \bigcup_{i=1}^{k} P_{\ell_i}$ is a linear forest, where $k \geq 2$, and we may assume that $\ell_1 \geq \cdots \geq \ell_k \geq 2$. To describe the results, we define the graph $H(n, F)$ as follows. Let $b = \sum_{i=1}^{k} \lfloor \frac{\ell_i}{2} \rfloor - 1$. Then, $H(n, F)$ is $K_b + E_{n-b}$ with a single edge added to E_{n-b} if all ℓ_i are odd, and $H(n, F) = K_b + E_{n-b}$ otherwise. Note that $H(n, F)$ is F-free. Indeed, if $H(n, F)$ contains F, then the path in F of order ℓ_i must use at least $\lfloor \frac{\ell_i}{2} \rfloor$ vertices of the K_b. But this cannot happen for every path in F, by the definition of b.

In the case when $F = kP_\ell$, we write $H(n, k, \ell)$ for $H(n, F)$. We have already seen the results for $\mathrm{ex}(n, kP_\ell)$ when $\ell = 2, 3$ (Theorems 8 and 9). For $\ell \geq 4$, Bushaw and Kettle [4] proved the following result.

Theorem 10. [4] *Let $k \geq 2$, $\ell \geq 4$, and $n \geq 2\ell + 2k\ell(\lceil \frac{\ell}{2} \rceil + 1)\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$. We have*

$$\mathrm{ex}(n, kP_\ell) = e(H(n, k, \ell)) = \binom{k\lfloor \frac{\ell}{2} \rfloor - 1}{2} + \left(k\left\lfloor \frac{\ell}{2} \right\rfloor - 1\right)\left(n - k\left\lfloor \frac{\ell}{2} \right\rfloor + 1\right) + c,$$

where $c = 1$ if ℓ is odd, and $c = 0$ if ℓ is even. Moreover, $H(n, k, \ell)$ is the unique extremal graph.

This result was extended by Lidický et al. [12], who determined $\mathrm{ex}(n, F)$ for an arbitrary linear forest $F \neq kP_3$.

Theorem 11. [12] *Let $k \geq 2$, and $F = \bigcup_{i=1}^{k} P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\mathrm{ex}(n, F) = e(H(n, F)) = \binom{\sum_{i=1}^{k} \lfloor \frac{\ell_i}{2} \rfloor - 1}{2} + \left(\sum_{i=1}^{k} \left\lfloor \frac{\ell_i}{2} \right\rfloor - 1\right)\left(n - \sum_{i=1}^{k} \left\lfloor \frac{\ell_i}{2} \right\rfloor + 1\right) + c,$$

where $c = 1$ if all ℓ_i are odd, and $c = 0$ otherwise. Moreover, $H(n, F)$ is the unique extremal graph.

6

Finally, Lidický et al. [12] determined the function $\mathrm{ex}(n, F)$, when F is a star forest and n is sufficiently large. Let $F = \bigcup_{i=1}^{k} S_{r_i}$, where $r_1 \geq \cdots \geq r_k \geq 1$. To describe their result, we define a graph $G(n, F)$ as follows. Let $i, r \geq 1$, and L be a graph on $n - i + 1$ vertices which is an extremal graph for S_r. Thus L is a near $(r-1)$-regular graph, and $e(L) = \lfloor \frac{r-1}{2}(n - i + 1) \rfloor$. Let $G(n, i, r) = K_{i-1} + L$. Now, let $G(n, F)$ be any graph $G(n, i, r_i)$ where $e(G(n, i, r_i))$ is maximised over $1 \leq i \leq k$. Note that each of $G(n, i, r_i)$ and $G(n, F)$ can be one of many possible graphs.

Observe that $G(n, i, r_i)$ is F-free for all $1 \leq i \leq k$. Indeed, if $G(n, i, r_i) = K_{i-1} + L$ as defined and contains a copy of F, then each star $S_{r_1}, \ldots, S_{r_{i-1}}$ must have at least one vertex from the K_{i-1}, and S_{r_i} is not a subgraph of L.

Lidický et al. [12] proved that the graphs $G(n, F)$ are extremal for F.

Theorem 12. [12] *Let $k \geq 2$, and $F = \bigcup_{i=1}^{k} S_{r_i}$ be a star forest, where $r_1 \geq \cdots \geq r_k \geq 1$ are the maximum degrees of the components. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\mathrm{ex}(n, F) = e(G(n, F))$$
$$= \max_{1 \leq i \leq k} \left\{ (i-1)(n - i + 1) + \binom{i-1}{2} + \left\lfloor \frac{r_i - 1}{2}(n - i + 1) \right\rfloor \right\}.$$

Moreover, the extremal graphs are the graphs $G(n, F)$.

3   Linear and star forests

We now study the function $\mathrm{ex}_p(n, F)$, where F is a linear forest or a star forest, $p \geq 2$, and n is sufficiently large. We assume throughout this section that F has at least two components, since the single component case is covered by (1), Theorem 3, and Proposition 4.

We first consider the case when F is a star forest. Recall that S_r is the star with maximum degree r. Let $F = \bigcup_{i=1}^{k} S_{r_i}$, where $r_1 \geq \cdots \geq r_k \geq 1$. Our first result is the following. It turns out that $\mathrm{ex}_p(n, F)$ is attained by the graphs $G(n, k, r_k)$.

Theorem 13. *Let $k, p \geq 2$, and $F = \bigcup_{i=1}^{k} S_{r_i}$ be a star forest, where $r_1 \geq \cdots \geq r_k \geq 1$ are the maximum degrees of the components. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\mathrm{ex}_p(n, F) = e_p(G(n, k, r_k)).$$

Moreover, the extremal graphs are the graphs $G(n, k, r_k)$.

Proof. Since $G(n, k, r_k)$ does not contain a copy of F, we have $\mathrm{ex}_p(n, F) \geq e_p(G(n, k, r_k))$. To prove the theorem, it suffices to show that any F-free graph G on n vertices with $G \neq G(n, k, r_k)$ satisfies $e_p(G) < e_p(G(n, k, r_k))$.

It is easy to calculate that

$$e_p(G(n, k, r_k)) = \begin{cases} (k-1)(n-1)^p + (n-k+1)(r_k + k - 2)^p \\ \qquad \text{if one of } r_k - 1 \text{ and } n - k + 1 \text{ is even} \\ (k-1)(n-1)^p + (n-k)(r_k + k - 2)^p + (r_k + k - 3)^p \\ \qquad \text{if } r_k - 1 \text{ and } n - k + 1 \text{ are odd} \end{cases}$$

$$= (k-1)n^p + o(n^p). \tag{2}$$

We may assume that there exists a subset $U \subset V(G)$ of order $k - 1$ such that every vertex $v \in U$ has degree $d_G(v) \geq \sum_{i=1}^{k} r_i + k$. Otherwise, if G has at most $k - 2$ such vertices, then using $p \geq 2$ and (2), we have

$$e_p(G) < (k-2)n^p + (n-k+2)\left(\sum_{i=1}^{k} r_i + k \right)^p < (k-1)n^p + o(n^p)$$

$$= e_p(G(n,k,r_k)).$$

Now, we prove that $G \subset G(n, k, r_k)$, which implies that $e_p(G) < e_p(G(n, k, r_k))$. Recall that $G(n, k, r_k) = K_{k-1} + L$, where L is a graph on $n - k + 1$ vertices which is an extremal graph for S_{r_k}. Thus by identifying U with K_{k-1} and $G - U$ with L, we have $G \subset G(n, k, r_k)$ if we can show that $G - U$ is S_{r_k}-free. Suppose that there is a copy of S_{r_k} in $G - U$. Then, using the fact that $d_G(v) \geq \sum_{i=1}^{k} r_i + k$ for every $v \in U$, we can find vertex-disjoint copies of $S_{r_1}, \ldots, S_{r_{k-1}}$, using vertices in U as their centres, and with their neighbours in $G - U$, not using the vertices of the S_{r_k}, as leaves. This gives a copy of F in G, a contradiction. □

Now, we consider the case when F is an arbitrary linear forest. Let $F = \bigcup_{i=1}^{k} P_{\ell_i}$, where $k \geq 2$ and $\ell_1 \geq \cdots \geq \ell_k \geq 2$. For the case when $F = kP_3$, we can set $r_1 = \cdots = r_k = 2$ in Theorem 13 to obtain the following result, which can be considered as an extension to Theorem 9.

Corollary 14. *Let $k, p \geq 2$ and $n \geq n_0(k)$ be sufficiently large. We have $\mathrm{ex}_p(n, kP_3) = e_p(K_{k-1} + M_{n-k+1})$. Moreover, $K_{k-1} + M_{n-k+1}$ is the unique extremal graph.*

Now, let $F \neq kP_3$. We shall prove the following result, which can be considered as an extension to Theorem 11.

Theorem 15. *Let $k, p \geq 2$, and $F = \bigcup_{i=1}^{k} P_{\ell_i}$ be a linear forest, where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k \geq 2$ and $\ell_i \neq 3$ for some i. Let $n \geq n_0(F)$ be sufficiently large. We have*

$$\mathrm{ex}_p(n, F) = e_p(H(n, F)).$$

Moreover, $H(n, F)$ is the unique extremal graph.
In particular, if $F = kP_\ell$ and $\ell \neq 3$, then

$$\mathrm{ex}_p(n, kP_\ell) = e_p(H(n, k, \ell)).$$

Moreover, $H(n, k, \ell)$ is the unique extremal graph.

Before we prove Theorem 15, we first recall a lemma of Caro and Yuster [5].

Lemma 16. *[5] Let $b \geq 1$ and $p \geq 2$ be integers. Let G be a graph on n vertices such that $e(G) \leq (b + \frac{1}{2})n$. Let $d_1 \geq \cdots \geq d_n$ be the degree sequence of G. Then, if $d_b \leq 0.65n$, we have $e_p(G) \leq cn^p + O(n^{p-1})$ for some constant c with $0 < c < b$.*

Although Lemma 16 is not stated explicitly in [5], it can be seen easily in the proof of Lemma 3.5 in [5].

Proof of Theorem 15. Since $H(n, F)$ is F-free, we have $\mathrm{ex}_p(n, F) \geq e_p(H(n, F))$. Hence, it suffices to show that any F-free graph G on n vertices with $G \neq H(n, F)$ has $e_p(G) < e_p(H(n, F))$. Assume the contrary, and let G be an F-free graph on n vertices, that is maximal in the sense that $e_p(G) = \mathrm{ex}_p(n, F) \geq e_p(H(n, F))$ and $G \neq H(n, F)$.

Let $b = \sum_{i=1}^{k} \lfloor \frac{\ell_i}{2} \rfloor - 1 \geq 1$. By the definition of $H(n, F)$, it is easy to calculate that

$$e_p(H(n, F)) = \begin{cases} b(n-1)^p + (n-b-2)b^p + 2(b+1)^p & \text{if all } \ell_i \text{ are odd} \\ b(n-1)^p + (n-b)b^p & \text{otherwise} \end{cases}$$

$$= bn^p + o(n^p). \tag{3}$$

According to Theorem 11, we have

$$e(G) \leq \binom{b}{2} + b(n-b) + 1 \leq bn. \tag{4}$$

Let $d_1 \geq \cdots \geq d_n$ be the degree sequence of G. Let $X \subset V(G)$ be the set of vertices with degrees $d_1, \ldots, d_b$, and $Y = V(G) \setminus X$. By Lemma 16 and using (3) and (4), we may assume that $d_b > 0.65n$. Let $A \subset Y$ be the set of vertices that have a neighbour in X, and $B = Y \setminus A$. Note that any two vertices $u, v \in X$ have at least $2(0.65n - 1) - (n-2) = 0.3n$ common neighbours in G, and hence at least $0.3n - (b-2) > 0.29n$ common neighbours in A. This means that for any set $Y' \subset Y$ with $|Y'|$ depending only on $\ell_1, \ldots, \ell_k$ (and hence $|Y'|$ is much smaller than n), u and v have a common neighbour in $A \setminus Y'$. Likewise, any vertex of X has a neighbour in $A \setminus Y'$. We now prove a series of claims.

Claim 17. *If ℓ_i is odd for all i, then $G[Y]$ does not contain a copy of P_3 with an end-vertex in A. If ℓ_i is even for some i, then $G[Y]$ does not contain an edge with an end-vertex in A.*

Proof. Suppose first that ℓ_i is odd for all i, and that $G[Y]$ contains a path $c_1c_2a_1$ with $a_1 \in A$. Let $y_1 \in X$ be a neighbour of a_1, and $y_2 \in X \setminus \{y_1\}$. Then, y_1 and y_2 have a common neighbour $a_2 \in A \setminus \{c_1, c_2, a_1\}$. Repeating this procedure, we can obtain a path $c_1c_2a_1y_1a_2y_2 \cdots y_{b-1}a_by_ba_{b+1}$, where $X = \{y_1, \ldots, y_b\}$ and $a_2, a_3, \ldots, a_{b+1} \in A$. This path has $2b+3 = \ell_1 + \sum_{i=2}^{k}(\ell_i - 1)$ vertices, and so it contains vertex-disjoint paths $P_{\ell_1}, P_{\ell_2-1}, \ldots, P_{\ell_k-1}$ with c_1 in the P_{ℓ_1}. Note that each of the paths $P_{\ell_2-1}, \ldots, P_{\ell_k-1}$ has an end-vertex in X, and so we can extend each P_{ℓ_i-1} to P_{ℓ_i} by taking a neighbour of the end-vertex in X. By choosing the $k-1$ neighbours to be distinct vertices in $A \setminus \{c_1, c_2, a_1, \ldots, a_{b+1}\}$, we obtain a copy of F in G, a contradiction.

Now, let $Q = \{1 \leq i \leq k : \ell_i \text{ is even}\} \neq \emptyset$, and suppose that $G[Y]$ contains an edge ca_1 with $a_1 \in A$. As before, we can obtain a path $ca_1y_1a_2y_2 \cdots y_{b-1}a_by_ba_{b+1}$, where $X = \{y_1, \ldots, y_b\}$ and $a_2, a_3, \ldots, a_{b+1} \in A \setminus \{c, a_1\}$. This path has $2b + 2 = \sum_{i \notin Q}(\ell_i - 1) + \sum_{i \in Q} \ell_i$ vertices. We obtain vertex-disjoint paths P_{ℓ_i-1} for $i \notin Q$, and P_{ℓ_i} for $i \in Q$, such that the path using c is P_{ℓ_j}, for some $j \in Q$. Extending each P_{ℓ_i-1} to P_{ℓ_i} for $i \notin Q$, we again have a copy of F in G, a contradiction. $\square$

Claim 18. *$G[B]$ does not contain a copy of P_{ℓ_k}.*

Proof. Suppose that $G[B]$ contains a copy of P_{ℓ_k}. Let $Q_0 = \{1 \leq i \leq k-1 : \ell_i \text{ is even}\}$ and $Q_1 = \{1 \leq i \leq k-1 : \ell_i \text{ is odd}\}$. We can find a path $a_1 y_1 a_2 y_2 \cdots a_b y_b$, where $X = \{y_1, \ldots, y_b\}$ and $a_1, \ldots, a_b \in A$. This path has $2b \geq \sum_{i \in Q_0} \ell_i + \sum_{i \in Q_1} (\ell_i - 1)$ vertices, and hence contains vertex-disjoint paths P_{ℓ_i} for $i \in Q_0$, and $P_{\ell_i - 1}$ for $i \in Q_1$. Extending each $P_{\ell_i - 1}$ to P_{ℓ_i} for $i \in Q_1$, we have a copy of F in G, a contradiction. $\qquad\square$

Claim 19. $B = \emptyset$.

Proof. Suppose that $B \neq \emptyset$. If $\ell_k = 2$, then note that Claims 17 and 18 imply that $G[Y]$ does not contain any edges. This means that G is a subgraph of $H(n, F)$, and we have $e_p(G) < e_p(H(n, F))$, a contradiction. Now, let $\ell_k \geq 3$. We will derive a contradiction by constructing a new F-free graph G' such that $e_p(G') > e_p(G)$.

Note that by Claim 17, every vertex of B has at most one neighbour in G lying in A. Since Claim 18 implies that $G[B]$ is P_{ℓ_k}-free, by Theorem 1, $G[B]$ contains at most $(\frac{\ell_k}{2} - 1)|B|$ edges. Hence, there exists a vertex $v \in B$ with at most $\ell_k - 2$ neighbours in $G[B]$. Now in G, in view of Claim 17, one of the following holds.

(i) $d_G(v) = 1$, with the only neighbour of v, say u, lying in A.

(ii) $0 \leq d_G(v) \leq \ell_k - 2$, with all neighbours of v lying in B.

Delete all edges adjacent to v in G, connect v to all vertices of X, and denote the new graph by G'. We claim that G' is also F-free. Indeed, if G' contains a copy of F, then exactly one path of F, say P_{ℓ_j}, must use an edge vy_1, for some $y_1 \in X$. If v is not an end-vertex of such a P_{ℓ_j}, then the P_{ℓ_j} also contains another neighbour $y_2 \in X$ of v. We can find a common neighbour $v' \in A$ of y_1 and y_2 in G which is not used in the copy of F. If v is an end-vertex of the P_{ℓ_j}, then we take $v' \in A$ to be any neighbour of y_1 not in the copy of F. Replacing v with v' on the P_{ℓ_j}, we obtain a copy of F in G, a contradiction.

We now show that $e_p(G') > e_p(G)$. Consider the effect of the transformation from G to G' on the degree sequence. The degrees of the vertices of X have increased by one. The degree of v has not decreased, since $d_{G'}(v) - d_G(v) \geq b - (\ell_k - 2) \geq (2\lfloor \frac{\ell_k}{2} \rfloor - 1) - (\ell_k - 2) \geq 0$. The degrees of the neighbours of v in G have decreased by 1. Since every vertex of X has degree at least $0.65n$, the total increase in $e_p(G') - e_p(G)$ contributed by the vertices of X is at least

$$b((0.65n + 1)^p - (0.65n)^p) = bp(0.65n)^{p-1} + o(n^{p-1}).$$

Also, $d_b > 0.65n$ implies that $|B| < 0.35n$. If (i) holds, then Claim 17 implies that in G, u has no neighbours in A, and hence $d_G(u) < 0.35n + b < 0.36n$. The decrease in $e_p(G') - e_p(G)$ contributed by u is at most

$$(0.36n)^p - (0.36n - 1)^p = p(0.36n)^{p-1} + o(n^{p-1}).$$

Suppose that (ii) holds. Then in G, every neighbour of v (lying in B) cannot have a neighbour in A, in view of Claim 17. Hence, every neighbour of v has degree at most $0.35n$ in G. The total decrease in $e_p(G') - e_p(G)$ contributed by the neighbours of v is at most

$$(\ell_k - 2)((0.35n)^p - (0.35n - 1)^p) \leq (\ell_k - 2)p(0.36n)^{p-1} + o(n^{p-1}).$$

Since $b \geq 2\lfloor\frac{\ell_k}{2}\rfloor - 1 \geq \ell_k - 2$, we have

$$
\begin{aligned}
e_p(G') - e_p(G) &\geq p\Big(b\Big(\frac{0.65}{0.36}\Big)^{p-1} - (\ell_k - 2)\Big)0.36^{p-1}n^{p-1} + o(n^{p-1}) \\
&\geq 0.72(1.8(\ell_k - 2) - (\ell_k - 2))n^{p-1} + o(n^{p-1}) \\
&\geq 0.576n^{p-1} + o(n^{p-1}) > 0.
\end{aligned}
$$

$\square$

By Claim 19, we may assume that $A = Y$ for the rest of the proof.

Claim 20. *If ℓ_i is odd for all i, then $G[Y]$ contains at most one edge. If ℓ_i is even for some i, then $G[Y]$ does not contain an edge.*

Proof. The latter assertion follows immediately from Claim 17 and the fact that $A = Y$. Now, assume that ℓ_i is odd for all i. Then, since we do not have $\ell_1 = \cdots = \ell_k = 3$, we have $\ell_1 \geq 5$. Assume the contrary. Together with Claim 17, the subgraph $G[Y]$ has at least two edges, and its components are single edges and isolated vertices. We consider three cases.

Case 1.

- *Either $\ell_1 = 5$, and there are two edges c_1c_2, c_3c_4 in $G[Y]$, with c_2, c_3 having a common neighbour in X;*

- *Or $\ell_1 \geq 7$, and there are two edges c_1c_2, c_3c_4 in $G[Y]$, with c_2, c_3 having distinct neighbours in X.*

Let $q = \lfloor\frac{\ell_1}{2}\rfloor - 1$. For $\ell_1 = 5$, let c_2, c_3 have a common neighbour $y_1 \in X$, so that $c_1c_2y_1c_3c_4$ is a copy of P_5. For $\ell_1 \geq 7$, let c_2, c_3 have distinct neighbours $y_1, y_q \in X$. Then, as before, we can find a copy of P_{ℓ_1} in the form $c_1c_2y_1a_2y_2\cdots a_qy_qc_3c_4$, where $y_2, \ldots, y_{q-1} \in X$ and $a_2, \ldots, a_q \in Y$. In both cases, we can again find a path $a_{q+1}y_{q+1}a_{q+2}y_{q+2}\ldots a_by_b$, where $X = \{y_1, \ldots, y_b\}$ and $a_{q+1}, a_{q+2}, \ldots, a_b \in Y \setminus \{c_1, c_2, c_3, c_4, a_2, \ldots, a_q\}$. This path has $2(b - q) = \sum_{i=2}^{k}(\ell_i - 1)$ vertices, and so contains vertex-disjoint copies of $P_{\ell_2-1}, \ldots, P_{\ell_k-1}$. As before, we can extend these to copies of $P_{\ell_2}, \ldots, P_{\ell_k}$ so that we have a copy of F in G, a contradiction.

Case 2. $\ell_1 = 5$, and no two vertices from distinct edges in $G[Y]$ have a common neighbour in X.

We shall prove that $e_p(G) < e_p(H(n, F))$, which will contradict the choice of G. Recall that $H(n, F)$ is $K_b + E_{n-b}$ with an edge uv added to the empty class, and note that $b \geq \lfloor\frac{5}{2}\rfloor + \lfloor\frac{3}{2}\rfloor - 1 = 2$. Let $d_H(z)$ denote the degree of a vertex z in $H(n, F)$. Clearly, for every $z \in X$ and every vertex z' in the K_b, we have $d_G(z) \leq n - 1 = d_H(z')$. Now, let $u_1u_2, \ldots, u_{2s-1}u_{2s}$ be all the independent edges in $G[Y]$, for some $s \geq 2$, and let Γ_i be the set of vertices in X that are adjacent to at least one of u_{2i-1} and u_{2i}, for $i = 1, \ldots, s$. We may assume that $|\Gamma_1| \geq \cdots \geq |\Gamma_s| \geq 1$. Note that the Γ_i are pairwise disjoint subsets of X, so that $|\Gamma_2| \leq \frac{b}{2}$. Also, since u_3 has a neighbour in X, we have $|\Gamma_1| \leq b-1$. Hence, $d_G(u_j) \leq b < d_H(u) = d_H(v)$ for $j = 1, 2$; and for every $z' \neq u, v$ in the E_{n-b}, we have $d_G(u_j) \leq \frac{b}{2} + 1 \leq b = d_H(z')$ for $j = 3, \ldots, 2s$; and $d_G(z) \leq b = d_H(z')$ for $z \in Y \setminus \{u_1, \ldots, u_{2s}\}$. The degree sequence of

11

$H(n, F)$ strictly majorises that of G, and therefore, we have $e_p(G) < e_p(H(n, F))$.

Case 3. $\ell_1 \geq 7$, and all vertices of the edges of $G[Y]$ are connected to a single vertex of X.

Let $y_1 \in X$ be this single vertex, and note that $\ell_1 \geq 7$ implies that $b \geq 3$. We construct an F-free graph G' on n vertices such that $e_p(G') > e_p(G)$, which contradicts the choice of G. Let $y_2 \in X \setminus \{y_1\}$, and let Y^* denote the set of non-isolated vertices in $G[Y]$. Observe that $|Y^*| \geq 4$ and no vertex of Y^* is adjacent to any vertex of $X \setminus \{y_1\}$. We construct G' as follows: delete the $\frac{|Y^*|}{2}$ independent edges of $G[Y]$, and join y_2 to each vertex of Y^*.

Similar to Claim 19, we see that G' is an F-free graph. Indeed, if G' contains a copy of F, then exactly one path, say P_{ℓ_j}, must use an edge vy_2, for some $v \in Y^*$. If v is not an end-vertex of such a P_{ℓ_j}, then the other neighbour of v in the P_{ℓ_j} is y_1. Now in G, we can find a common neighbour $v' \in Y$ of y_1 and y_2 which is not used in the copy of F. If v is an end-vertex of the P_{ℓ_j}, then we can take $v' \in Y$ to be any neighbour of y_2 not in the copy of F. Replacing v with v' on the P_{ℓ_j}, we obtain a copy of F in G, a contradiction.

However, the degree sequence of G' strictly majorises that of G, since the degree of y_2 has strictly increased, and all other degrees have not changed. Hence $e_p(G') > e_p(G)$, which is the required contradiction. □

By Claim 20, G is a spanning subgraph of $H(n, F)$. Hence $e_p(G) < e_p(H(n, F))$, which contradicts the choice of G. The proof of Theorem 15 is complete. □

4 Brooms

In this section, we shall consider the function $\mathrm{ex}_p(n, B_{\ell,s})$, where $B_{\ell,s}$ is a broom graph, $p \geq 2$, $\ell \geq 4$, $s \geq 0$, and n is sufficiently large. As we have already seen, Theorems 5 and 6 appear to suggest that the determination of the Turán function $\mathrm{ex}(n, B_{\ell,s})$ and the corresponding extremal graphs may be a complicated problem, in the sense that the potential results may be difficult to state. Somewhat surprisingly, we shall see here that the same problem for $\mathrm{ex}_p(n, B_{\ell,s})$, where $p \geq 2$, may possibly be more manageable. Since the case $\ell = 4$ is covered in Proposition 7, we consider $\ell \geq 5$. Here, we will provide the answers for the cases $\ell = 5, 6, 7$, and present a conjecture for the case of general ℓ. The case $\ell = 5$ turns out to be a rather special case. Although the case $s = 0$ is covered by Theorem 3, we will include this case here since we will obtain some explicit lower bounds for n.

Theorem 21. *Let $p \geq 2$, $s \geq 0$, and $n > (2s + 10)^2$. We have*

$$\mathrm{ex}_p(n, B_{5,s}) = \begin{cases} e_p(H(n, 5)) & \textit{if } s = 0, \\ e_p(K_1 + M_{n-1}) & \textit{if } s \geq 1. \end{cases}$$

Moreover, the unique extremal graph is $H(n, 5)$ if $s = 0$, and $K_1 + M_{n-1}$ if $s \geq 1$.

Theorem 22. *Let $p \geq 2$, $s \geq 0$, and $n > (2s + 12)^2$. We have*

$$\mathrm{ex}_p(n, B_{6,s}) = e_p(H(n, 6)).$$

Moreover, $H(n, 6)$ is the unique extremal graph.

Theorem 23. *Let $p \geq 2$, $s \geq 0$, and $n > (3s + 31)^2$. We have*

$$\mathrm{ex}_p(n, B_{7,s}) = e_p(H(n, 7)).$$

Moreover, $H(n, 7)$ is the unique extremal graph.

In view of Theorems 22 and 23, we believe that the following assertion may be true.

Conjecture 24. *Let $p \geq 2$, $\ell \geq 6$, $s \geq 0$, and $n \geq n_0(\ell, s)$ be sufficiently large. We have*

$$\mathrm{ex}_p(n, B_{\ell,s}) = e_p(H(n, \ell)).$$

Moreover, $H(n, \ell)$ is the unique extremal graph.

That is, Conjecture 24 claims that if n is sufficiently large, then $\mathrm{ex}_p(n, B_{\ell,s})$ is exactly the same as $\mathrm{ex}_p(n, P_\ell)$, with the same unique extremal graph $H(n, \ell)$. If Conjecture 24 is true, then it can be considered as an extension to Theorem 3.

Furthermore, we may also ask for the situation for a *broom forest*, which is a forest whose components are brooms and stars.

Problem 25. *Let $p \geq 2$, and F be a broom forest. For sufficiently large $n \geq n_0(F)$, determine the function $\mathrm{ex}_p(n, F)$ and the extremal graphs.*

Before we prove Theorems 21 to 23, we first prove some auxiliary lemmas.

Lemma 26. *Let $p \geq 2$, $n_1, n_2 \geq \ell$, and $n = n_1 + n_2$.*

(a) If $\ell = 5$, then $e_p(K_1 + M_{n_1-1}) + e_p(K_1 + M_{n_2-1}) < e_p(K_1 + M_{n-1})$.

(b) If $\ell \geq 5$, then $e_p(H(n_1, \ell)) + e_p(H(n_2, \ell)) < e_p(H(n, \ell))$.

Proof. (a) Let $\ell = 5$. Then

$$
\begin{aligned}
e_p(K_1 + M_{n_1-1}) + e_p(K_1 + M_{n_2-1}) &\leq (n_1 - 1)^p + (n_1 - 1)2^p \\
&\quad + (n_2 - 1)^p + (n_2 - 1)2^p \\
&< (n - 1)^p + (n - 2)2^p < e_p(K_1 + M_{n-1}).
\end{aligned}
$$

(b) Let $\ell \geq 5$, and $b = \lfloor \frac{\ell}{2} \rfloor - 1 \geq 1$. Since

$$
\begin{aligned}
e_p(H(n_1, \ell)) + e_p(H(n_2, \ell)) &\leq b(n_1 - 1)^p + (n_1 - b - 2)b^p + 2(b + 1)^p \\
&\quad + b(n_2 - 1)^p + (n_2 - b - 2)b^p + 2(b + 1)^p, \\
e_p(H(n, \ell)) &\geq b(n - 1)^p + (n - b)b^p,
\end{aligned}
$$

it suffices to prove that

$$b(n - 1)^p > b[(n_1 - 1)^p + (n_2 - 1)^p] + 4(b + 1)^p - (b + 4)b^p. \tag{5}$$

Clearly, $n \geq 2\ell \geq 4b + 4$. We have

$$
\begin{aligned}
b(n - 1)^p &> b(n - 2)^p + bp(n - 2)^{p-1} \\
&> b[(n_1 - 1)^p + (n_2 - 1)^p] + 2b(4b + 2)^{p-1},
\end{aligned}
$$

which implies (5), since it is easy to verify that $2b(4b + 2)^{p-1} > 4(b + 1)^p - (b + 4)b^p$. $\qquad\square$

Lemma 27. *Let $p \geq 2$, $\ell \geq 5$, $s \geq 0$. Let G^* be a graph on $h^* > 0$ vertices with $\Delta(G^*) \leq d = d(\ell, s)$. Let $n = h + h^* > (\ell + s + d)^2$ for some $h \geq \ell$.*

(a) If $\ell = 5$, then $e_p(K_1 + M_{h-1}) + e_p(G^*) < e_p(K_1 + M_{n-1})$.

(b) If $\ell \geq 5$, then $e_p(H(h, \ell)) + e_p(G^*) < e_p(H(n, \ell))$.

Proof. (a) Let $\ell = 5$. We have

$$e_p(K_1 + M_{h-1}) + e_p(G^*) \leq (h-1)^p + (h-1)2^p + h^* d^p,$$
$$e_p(K_1 + M_{n-1}) > (n-1)^p + (n-2)2^p.$$

Since $(n-2)2^p \geq (h-1)2^p$, and

$$(n-1)^p - (h-1)^p > (n-h)(n-1)^{p-1} > h^* d^{2p-2} \geq h^* d^p,$$

it follows that $e_p(K_1 + M_{h-1}) + e_p(G^*) < e_p(K_1 + M_{n-1})$.

(b) Let $\ell \geq 5$, and $b = \lfloor \frac{\ell}{2} \rfloor - 1 \geq 1$. Since

$$e_p(H(h, \ell)) + e_p(G^*) \leq b(h-1)^p + (h - b - 2)b^p + 2(b+1)^p + h^* d^p,$$
$$e_p(H(n, \ell)) \geq b(n-1)^p + (n-b)b^p,$$

and $(n-b)b^p > (h - b - 2)b^p$, it suffices to prove that

$$b[(n-1)^p - (h-1)^p] > 2(b+1)^p + h^* d^p.$$

Clearly, $\ell + s \geq \ell \geq 2b + 2$. We have

$$\begin{aligned}
b[(n-1)^p - (h-1)^p] &> (n-h)(n-1)^{p-1} \geq h^*(\ell + s + d)^{2p-2} \\
&\geq h^*(2b + 2 + d)^p > h^*(2b+2)^p + h^* d^p \\
&> 2(b+1)^p + h^* d^p,
\end{aligned}$$

as required. $\qquad \square$

Before we prove the next lemma, we make some definitions. Let C be a connected graph, and $v, x \in V(C)$.

- For $y \in V(C - \{v, x\})$, the edge $e = xy \in E(C)$ is an *x-pendent edge* if x is the only neighbour of y in C.

- Let $y, y' \in V(C - \{v, x\})$ where $xy, xy', yy' \in E(C)$, and y, y' do not have any other neighbours in C. The subgraph $T = C[\{x, y, y'\}]$ is an *x-pendent triangle*.

- Let $z, y, y' \in V(C - \{v, x\})$ where $xy, xy', zy, zy', yy' \in E(C)$, and z, y, y' do not have any other neighbours in C. The subgraph $D = C[\{x, z, y, y'\}]$ is an *x-pendent diamond*.

- For some $t \geq 2$, let $z, y_1, \ldots, y_t \in V(C - \{v, x\})$ where $xy_k, zy_k \in E(C)$ (resp. xy_k, zy_k, $xz \in E(C)$) for every $1 \leq k \leq t$, and $z, y_1, \ldots, y_t$ do not have any other neighbours in C. The subgraph $S = C[\{x, z, y_1, \ldots, y_t\}]$ (resp. $S^+ = C[\{x, z, y_1, \ldots, y_t\}]$) is an *x-pendent spindle* (resp. *x-pendent spindle$^+$*).

14

Lemma 28. *Let $p \geq 2$, $\ell \geq 5$, $s \geq 0$. Let C be a connected $B_{\ell,s}$-free graph, and $v, x \in V(C)$ where $d_C(v) = \Delta(C) \geq \ell + s - 1$. Let C' be a graph that can be obtained from C with any of the following operations.*

 (i) *Delete an x-pendent edge $e = xy$, and add the edge vy.*

 (ii) *Delete the three edges of an x-pendent triangle $T = C[\{x, y, y'\}]$, and add the edges vy, vy'.*

 (iii) *Delete the five edges of an x-pendent diamond $D = C[\{x, z, y, y'\}]$, and add the edges vz, vy, vy'.*

 (iv) *Delete the $2t$ edges of an x-pendent spindle $S = C[\{x, z, y_1, \ldots, y_t\}]$ (for some $t \geq 2$), and add the edges $vz, vy_1, \ldots .vy_t$.*

 (v) *Delete the $2t + 1$ edges of an x-pendent spindle$^+$ $S^+ = C[\{x, z, y_1, \ldots, y_t\}]$ (for some $t \geq 2$), and add the edges $vz, vy_1, \ldots .vy_t$.*

Then C' is also $B_{\ell,s}$-free, $d_{C'}(v) = \Delta(C') \geq \ell + s - 1$, and $e_p(C) < e_p(C')$.

Proof. Clearly we have $d_{C'}(v) = \Delta(C') \geq \ell + s - 1$, since in the transformation from C to C', the only vertex whose degree has increased is v.

Next, let V_1 be the set of neighbours of v in C. Suppose that C' contains a copy of $B_{\ell,s}$, and we are in case (iv) or (v). Then for some $u_1, \ldots, u_m \in \{z, y_1, \ldots, y_t\}$ where $1 \leq m \leq t+1$, the edges $vu_1, \ldots, vu_m$ must be used by the $B_{\ell,s}$, with $u_1, \ldots, u_m$ being leaves. Note that $|V_1 \cup \{v, y_1, \ldots, y_t, z\}| \geq \ell + s + t + 1$, and this means that there are vertices $w_1, \ldots, w_m \in V_1$ which are not used in the $B_{\ell,s}$. Thus we obtain a copy of $B_{\ell,s}$ in C by replacing $vu_1, \ldots, vu_m$ with $vw_1, \ldots, vw_m$, a contradiction. Similar arguments hold if we are in the other three cases, in view of $|V_1 \cup \{v, y\}| \geq \ell + s + 1$; $|V_1 \cup \{v, y, y'\}| \geq \ell + s + 2$; $|V_1 \cup \{v, z, y, y'\}| \geq \ell + s + 3$ for (i), (ii), (iii), respectively. Therefore, C' is $B_{\ell,s}$-free.

It remains to prove that $e_p(C) < e_p(C')$ for each case.

(i) Going from C to C', we see that the degree of v is increased by 1, and the degree of x is decreased by 1. Since $d_C(v) \geq d_C(x) \geq 2$, we have

$$e_p(C') - e_p(C) = (d_C(v) + 1)^p - d_C(v)^p + (d_C(x) - 1)^p - d_C(x)^p$$

$$\geq \sum_{1 \leq j \leq p, \, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j}) + \binom{p}{2} (d_C(v)^{p-2} + d_C(x)^{p-2})$$

$$> 0.$$

(ii) Going from C to C', we see that the degree of v is increased by 2, the degree of x is decreased by 2, and the degrees of y, y' are each decreased from 2 to 1. Since $d_C(v) \geq$

15

$d_C(x) \geq 3$ and $d_C(v) \geq \ell + s - 1 \geq 4$, we have

$$e_p(C') - e_p(C) = (d_C(v) + 2)^p - d_C(v)^p + (d_C(x) - 2)^p - d_C(x)^p + 2(1^p - 2^p)$$

$$\geq \sum_{1 \leq j \leq p,\ j \text{ odd}} \binom{p}{j}(d_C(v)^{p-j} - d_C(x)^{p-j})2^j$$

$$+ \binom{p}{2}(d_C(v)^{p-2} + d_C(x)^{p-2})2^2 + 2(1 - 2^p)$$

$$> 4(4^{p-2} + 3^{p-2}) - 2^{p+1} \geq 0.$$

(iii) Going from C to C', we see that the degree of v is increased by 3, the degree of x is decreased by 2, the degree of z is decreased from 2 to 1, and the degrees of y, y' are each decreased from 3 to 1. Since $d_C(v) \geq d_C(x) \geq 3$ and $d_C(v) \geq \ell + s - 1 \geq 4$, we have

$$e_p(C') - e_p(C) = (d_C(v) + 3)^p - d_C(v)^p + (d_C(x) - 2)^p - d_C(x)^p$$
$$+ (1^p - 2^p) + 2(1^p - 3^p)$$

$$\geq p(3d_C(v)^{p-1} - 2d_C(x)^{p-1}) + \binom{p}{2}(9d_C(v)^{p-2} + 4d_C(x)^{p-2})$$

$$+ \sum_{3 \leq j \leq p,\ j \text{ odd}} \binom{p}{j}(d_C(v)^{p-j}3^j - d_C(x)^{p-j}2^j) + 3 - 2^p - 2 \cdot 3^p$$

$$\geq 2 \cdot 4^{p-1} + 9 \cdot 4^{p-2} + 4 \cdot 3^{p-2} + 3 - 2^p - 2 \cdot 3^p > 0,$$

since $2 \cdot 4^{p-1} + 9 \cdot 4^{p-2} + 3 > 4^p + 3 > 2 \cdot 3^p$ and $4 \cdot 3^{p-2} \geq 2^p$.

(v) Going from C to C', we see that the degree of v is increased by $t + 1$, the degree of x is decreased by $t + 1$, the degree of z is decreased from $t + 1$ to 1, and the degrees of $y_1, \ldots, y_t$ are each decreased from 2 to 1. Since $d_C(v) \geq d_C(x) \geq t + 1$, we have

$$e_p(C') - e_p(C) = (d_C(v) + t + 1)^p - d_C(v)^p + (d_C(x) - (t+1))^p - d_C(x)^p$$
$$+ 1^p - (t+1)^p + t(1^p - 2^p)$$

$$> \sum_{1 \leq j \leq p,\ j \text{ odd}} \binom{p}{j}(d_C(v)^{p-j} - d_C(x)^{p-j})(t+1)^j$$

$$+ \binom{p}{2}(d_C(v)^{p-2} + d_C(x)^{p-2})(t+1)^2 - (t+1)^p - t \cdot 2^p$$

$$\geq 2(t+1)^p - (t+1)^p - t \cdot 2^p = (t+1)^p - t \cdot 2^p > 0.$$

(iv) This follows from (v), since we can obtain the graph C'' from C by adding the edge xz, so that $e_p(C) < e_p(C'') < e_p(C')$. $\qquad\square$

Finally, the next lemma says that, in order to tackle Theorems 21 to 23 and Conjecture 24, it suffices to consider connected broom-free graphs with large maximum degree.

Lemma 29. *Suppose that the following statement is true: Let $p \geq 2$, $\ell \geq 6$, $s \geq 0$. Then there exists $d = d(\ell, s) \geq \ell + s$ such that, for all connected $B_{\ell,s}$-free graph C with $c \geq d$ vertices and $\Delta(C) \geq d - 1$, we have $e_p(C) \leq e_p(H(c, \ell))$, with equality if and only if $C = H(c, \ell)$.*

Then Conjecture 24 holds for $n > (\ell + s + d)^2$.

Similarly, Theorem 21 holds if the following statement is true: Let $p \geq 2$ and $s \geq 0$. Then for all connected $B_{5,s}$-free graph C with $c \geq s + 5$ vertices and $\Delta(C) \geq s + 4$, we have

$$e_p(C) \leq \begin{cases} e_p(H(c,5)) & \text{if } s = 0, \\ e_p(K_1 + M_{c-1}) & \text{if } s \geq 1, \end{cases} \tag{6}$$

with equality if and only if $C = H(c,5)$ for $s = 0$, and $C = K_1 + M_{c-1}$ for $s \geq 1$.

Proof. Suppose that the first assertion in Lemma 29 holds. Let $n > (\ell + s + d)^2$. Clearly the graph $H(n, \ell)$ is $B_{\ell,s}$-free. Now, let G be a $B_{\ell,s}$-free graph on n vertices and $G \neq H(n, \ell)$. Then the assertion of Conjecture 24 follows if we can prove that $e_p(G) < e_p(H(n, \ell))$.

Suppose first that $\Delta(G) \leq d - 2$. Then since $n - 1 > (d+1)^2$, we have

$$\begin{aligned} (n-1)^p = (n-1)(n-1)^{p-1} &> (n-1)(d+1)^{2p-2} \\ &> (n-1)[d^{2p-2} + (2p-2)d^{2p-3}] \\ &> (n-1)d^{2p-2} + 2d^{2p-1} > n(d-2)^p, \end{aligned}$$

so that

$$e_p(G) \leq n(d-2)^p < (n-1)^p < e_p(H(n, \ell)). \tag{7}$$

Now, let $\Delta(G) \geq d - 1$. Let $G^* \subset G$ be the subgraph consisting of the components with maximum degree at most $d - 2$, so that $\Delta(G^*) \leq d - 2$. We have $G = C_1 \cup \cdots \cup C_t \cup G^*$ for some $t \geq 1$, where $C_1, \ldots, C_t$ are the components of G with maximum degree at least $d - 1$. Let $c_i = |V(C_i)| \geq d$. By the assertion in Lemma 29, for every $1 \leq i \leq t$, we have

$$e_p(C_i) \leq e_p(H(c_i, \ell)), \tag{8}$$

with equality if and only if $C_i = H(c_i, \ell)$. We apply (8) to every C_i, and then apply Lemma 26(b) repeatedly $t - 1$ times, and finally Lemma 27(b), if $|V(G^*)| > 0$. We find that $e_p(G) < e_p(H(n, \ell))$, since $G \neq H(n, \ell)$ by assumption.

By a similar argument, using Lemmas 26(a) and 27(a), and setting $\ell = 5$, $d = s + 5$, we see that the second assertion implies Theorem 21 for $n > (2s + 10)^2$. Note that the analogous inequalities to (7) would be

$$e_p(G) \leq n(s+3)^p < (n-1)^p < e_p(H(n,5)) < e_p(K_1 + M_{n-1}).$$

$\square$

We are now ready to prove Theorems 21, 22 and 23. The arguments in all three proofs are similar. In outline, it suffices to verify the statements in Lemma 29 for $\ell = 5, 6, 7$. Let C be a connected $B_{\ell,s}$-free graph on c vertices as defined in the lemma. We may assume that C does not contain any of the pendent subgraphs, otherwise we may apply Lemma 28 to obtain another connected $B_{\ell,s}$-free graph C' with $e_p(C) < e_p(C')$, so that we could consider the argument for C' instead of C. Under this assumption, we then show that $e_p(C) \leq e_p(K_1 + M_{c-1})$ for $\ell = 5$, $s \geq 1$, and $e_p(C) \leq e_p(H(c, \ell))$ otherwise. In each case, equality occurs if and only if C is the corresponding extremal graph.

17

Proof of Theorem 21. It suffices to verify the second statement in Lemma 29. Let C be a connected $B_{5,s}$-free graph with $c \geq s + 5$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq s + 4$. By Lemma 28, we may assume that C does not contain an x-pendent edge xy where $x, y \in V(C - v)$. Otherwise, we may delete xy and add vy to obtain the $B_{5,s}$-free graph C' with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C.

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v. Note that $|V_1| = d_C(v) \geq s + 4$. Also, we have the following properties.

(i) $V_i = \emptyset$ for $i \geq 3$.

(ii) $C[V_2]$ does not contain an edge.

(iii) Every vertex of V_2 has exactly one neighbour in V_1.

(iv) $C[V_1]$ contains at most one edge if $s = 0$, and $C[V_1]$ does not contain a copy of the path P_3 if $s \geq 1$.

Otherwise, suppose that (i) is false. Then we have a copy of $B_{5,s}$, where the path P_5 in $B_{5,s}$ is $x_3 x_2 x_1 v y_1$ with $x_i \in V_i$ for $i = 1, 2, 3$, $y_1 \in V_1$, and the remaining s vertices of the $B_{5,s}$ are all in $V_1 \setminus \{x_1, y_1\}$. Properties (ii) to (iv) also hold for similar reasons. If $V_2 \neq \emptyset$, then we must have an edge $xy \in E(C)$ with $x \in V_1$ and $y \in V_2$. It follows from (i) to (iii) that xy is an x-pendent edge. Therefore, we may assume that $V_2 = \emptyset$. From (iv), we can now easily see that $C \subset H(c, 5)$ if $s = 0$; and $C \subset K_1 + M_{c-1}$ if $s \geq 1$, since $C[V_1]$ consists of independent edges and isolated vertices. Consequently (6) holds, as well as the cases of equality. $\qquad \square$

Proof of Theorem 22. It suffices to verify the first statement in Lemma 29 for $\ell = 6$, with $d = s + 6$. Let C be a connected $B_{6,s}$-free graph with $c \geq s + 6$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq s + 5$. We may assume that C does not contain an x-pendent edge or an x-pendent triangle, where $x \in V(C - v)$. Otherwise in either case, we may obtain the $B_{6,s}$-free graph C' as described in Lemma 28 with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C.

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v. Note that $|V_1| = d_C(v) \geq s + 5$. Also, we have the following properties.

(i) $V_i = \emptyset$ for $i \geq 4$.

(ii) $C[V_i]$ does not contain a copy of the path P_{5-i}, for $i = 1, 2, 3$.

(iii) Every vertex of V_3 has exactly one neighbour in V_2.

Otherwise if any of (i) to (iii) is false, then we can easily find a copy of $B_{6,s}$ with centre v. By (i) to (iii), we may assume that $V_3 = \emptyset$, otherwise we have an x-pendent edge $xy \in E(C)$ where $x \in V_2$ and $y \in V_3$. Next, suppose that we have an edge yy' in $C[V_2]$. If y and y' have distinct neighbours in V_1, then C again has a copy of $B_{6,s}$. Thus, y and y' must each have exactly one neighbour in V_1, which is a common neighbour $x \in V_1$, and from (ii) with $i = 2$, we have $C[\{x, y, y'\}]$ is an x-pendent triangle. Therefore, we may assume that $C[V_2]$ does not contain an edge. Since no x-pendent edge xy exists where $x \in V_1$, $y \in V_2$, this means that every vertex of V_2 must have at least two neighbours in V_1. Now, if $C[V_1 \cup V_2]$ contains

a copy of P_4, say $y_1x_1x_2y_2$, then we must have $y_1, y_2 \in V_2$, otherwise C has a copy of $B_{6,s}$. Moreover, $x_1, x_2 \in V_1$, since $C[V_2]$ does not contain an edge. But then, y_1 has a neighbour in $V_1 \setminus \{x_1\}$, and we have another copy of P_4 with an end-vertex in V_1, yielding a copy of $B_{6,s}$ in C. Therefore, $C[V_1 \cup V_2]$ does not contain a copy of P_4, and we can easily deduce that $C[V_1 \cup V_2]$ is a subgraph whose components are stars and triangles.

Let C^* be the graph obtained from C by adding all edges from v to V_2. Note that by replacing $C^* - v$ with the star of the same order, we obtain the graph $H(c, 6)$. We shall show that this operation does not decrease the value of e_p. Consider the following operations.

(A) Suppose that $C^* - v$ contains two star components, say with centres x and y, and the leaves at y are $y_1, \ldots, y_m$ for some $m \geq 0$. We delete the edges $yy_1, \ldots, yy_m$ and add the edges $xy, xy_1, \ldots, xy_m$. The increase in the value of e_p is

$$(d_{C^*}(x) + m + 1)^p - d_{C^*}(x)^p + 2^p - (m+1)^p > 2^p > 0.$$

(B) Suppose that $C^* - v$ contains at least two triangle components, say with vertices $x_1, \ldots, x_{3m}$ for some $m \geq 2$. We delete the $3m$ edges of the triangles, and connect x_1 to $x_2, \ldots, x_{3m}$. The increase in the value of e_p is

$$(3m)^p - 3^p + (3m-1)(2^p - 3^p) = (m^p - 3m)3^p + (3m-1)2^p > 0.$$

(C) Suppose that $C^* - v$ contains a star and a triangle component, exactly one of each. Let x be the centre of the star, and note that since $|V(C^* - v)| = c - 1 \geq s + 5 \geq 5$, we have $d_{C^*}(x) \geq 2$. We delete the three edges of the triangle and connect x to its three vertices. The increase in the value of e_p is

$$(d_{C^*}(x) + 3)^p - d_{C^*}(x)^p + 3(2^p - 3^p).$$

If $p = 2$, then the increase is $6d_{C^*}(x) - 6 > 0$. If $p \geq 3$ and $d_{C^*}(x) = 2$, then the increase is $5^p + 2^{p+1} - 3^{p+1} > 0$. Otherwise if $p \geq 3$ and $d_{C^*}(x) \geq 3$, then the increase is at least

$$3p\,d_{C^*}(x)^{p-1} + 3(2^p - 3^p) \geq 3^{p+1} + 3(2^p - 3^p) > 0.$$

Therefore where possible, we apply operation (B), followed by successive applications of operation (A), and finally operation (C). We remark that during the applications of these operations, we may encounter a copy of $B_{6,s}$ in an intermediate graph. We obtain $e_p(C) \leq e_p(C^*) \leq e_p(H(c, 6))$. Equality occurs if and only if $C = C^*$ and $C^* - v$ is itself a star. That is, if and only if $C = H(c, 6)$. $\qquad\square$

Proof of Theorem 23. It suffices to verify the first statement in Lemma 29 for $\ell = 7$, with $d = 2s + 24$. Let C be a connected $B_{7,s}$-free graph with $c \geq 2s + 24$ vertices, and $v \in V(C)$ with $d_C(v) = \Delta(C) \geq 2s + 23$. We may assume that C does not contain an x-pendent edge, triangle, diamond, spindle, or spindle$^+$, where $x \in V(C - v)$. Otherwise, we may obtain the $B_{7,s}$-free graph C' as described in Lemma 28 with $e_p(C) < e_p(C')$ and $d_{C'}(v) = \Delta(C')$, and consider the graph C' instead of C.

For $i \geq 1$, let V_i be the set of vertices of C at distance i from v. Note that $|V_1| = d_C(v) \geq 2s + 23$. Also, we have the following properties.

19

(i) $V_i = \emptyset$ for $i \geq 5$.

(ii) $C[V_i]$ does not contain a copy of the path P_{6-i}, for $i = 1, 2, 3, 4$.

(iii) Every vertex of V_4 has exactly one neighbour in V_3.

Otherwise if any of (i) to (iii) is false, then we can easily find a copy of $B_{7,s}$ with centre v. Proceeding exactly the same way as we did in Theorem 22, by avoiding a copy of $B_{7,s}$, or an x-pendent edge or triangle for some $x \in V(C - v)$, we can obtain the following facts.

- We may assume that $V_4 = \emptyset$.

- We may assume that $C[V_3]$ does not contain an edge, and that every vertex of V_3 has at least two neighbours in V_2.

Suppose that we have $z \in V_3$. Then the only neighbours of z in C are $y_1, \ldots, y_t \in V_2$, for some $t \geq 2$. In order to avoid a copy of $B_{7,s}$, note that each y_k must have exactly one neighbour in V_1, which is a common neighbour $x \in V_1$. Also, no y_k can have a neighbour in $V_3 \setminus \{z\}$. Finally, no edge e in $C[V_2]$ can be incident with any y_k, except when $t = 2$ and $e = y_1 y_2$. Thus, $C[\{x, z, y_1, \ldots, y_t\}]$ is an x-pendent diamond or spindle. Therefore, we may further assume that $V_3 = \emptyset$.

By (ii) with $i = 2$, we see that the components of $C[V_2]$ are stars and triangles. Suppose that we have a star component in $C[V_2]$ with centre z and leaves $y_1, \ldots, y_t$, for some $t \geq 2$. Then as before, the vertices y_k must each have exactly one neighbour in V_1, which is a common neighbour $x \in V_1$. If z has a neighbour $x' \in V_1 \setminus \{x\}$, then we have a copy of $B_{7,s}$ with centre v, where the P_7 is $y_1 x y_2 z x' v x''$ for some $x'' \in V_1 \setminus \{x, x'\}$, and the s leaves are in $V_1 \setminus \{x, x', x''\}$. Therefore, x must be the unique neighbour of z in V_1, and $C[\{x, z, y_1, \ldots, y_t\}]$ is an x-pendent spindle$^+$.

Thus, we may assume that the components of $C[V_2]$ are triangles, single edges, and isolated vertices. We consider the behaviour of the edges that connect these components to V_1, keeping in mind that we should avoid creating a copy of $B_{7,s}$.

- If $y_1, y_2, y_3 \in V_2$ form a triangle in $C[V_2]$, then y_1, y_2, y_3 must have a unique common neighbour in V_1, and they do not have any other neighbours in V_1.

- Let $y_1 y_2$ be a single edge in $C[V_2]$. If y_1, y_2 have exactly one common neighbour $x \in V_1$, then exactly one of y_1, y_2 has at least one neighbour in $V_1 \setminus \{x\}$, otherwise $C[\{x, y_1, y_2\}]$ is an x-pendent triangle or there is a copy of $B_{7,s}$. If y_1, y_2 have exactly two common neighbours $x_1, x_2 \in V_1$, then both y_1, y_2 cannot have a neighbour in $V_1 \setminus \{x_1, x_2\}$. Also, y_1, y_2 cannot have at least three common neighbours in V_1. The remaining possibility is that y_1, y_2 have no common neighbour in V_1.

- If y is an isolated vertex in $C[V_2]$, then y must have at least two neighbours in V_1, otherwise there is an x-pendent edge xy, for some $x \in V_1$.

Let $\tilde{C} = (C - v) - E(C[V_1])$, i.e., $\tilde{C}$ is the subgraph on $V_1 \cup V_2$, with the edges of $C[V_1]$ deleted. Then, when considering the components of $\tilde{C}$, these components are the subgraphs as shown in Figure 2(a). We refer the subgraphs illustrated as *Type 1* to *Type 5*.
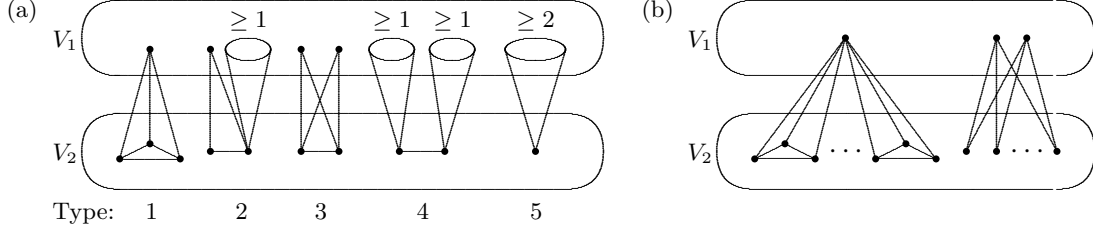
20

Figure 2. (a) Types 1 to 5 subgraphs; (b) How Type 1 and Type 5 subgraphs intersect

For two such subgraphs S, S', we see that, in order to avoid creating a $B_{7,s}$ with centre v, we would need to have $V(S) \cap V(S') \cap V_1 = \emptyset$ in most cases. The only exceptions are when S and S' are of Type 1 and meeting at one vertex in V_1, or they are of Type 5 with order 3 and meeting at exactly two vertices in V_1. Indeed, we may have at least two subgraphs meeting in V_1 in these two exceptional cases, as shown in Figure 2(b). We next eliminate these two possibilities.

Suppose first that we have $x \in V_1$, with exactly $m \geq 2$ Type 1 subgraphs meeting at x. Let $y_1, \ldots, y_{3m} \in V_2$ be the vertices of the triangles in V_2. We delete the $6m$ edges of the Type 1 subgraphs, and add the edges $vy_1, \ldots, vy_{3m}$. Then the degree of v is increased by $3m$, the degree of x is decreased by $3m$, and the degrees of the y_k are decreased from 3 to 1. Since $d_C(v) \geq d_C(x) > 3m$, the increase in the value of e_p is

$$(d_C(v) + 3m)^p - d_C(v)^p + (d_C(x) - 3m)^p - d_C(x)^p + 3m(1^p - 3^p)$$

$$> \sum_{1 \leq j \leq p,\, j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x)^{p-j})(3m)^j$$

$$+ \binom{p}{2}(d_C(v)^{p-2} + d_C(x)^{p-2})(3m)^2 - 3m \cdot 3^p$$

$$> 18m^2(3m)^{p-2} - m \cdot 3^{p+1} = 2m^p \cdot 3^p - m \cdot 3^{p+1} \geq 4m \cdot 3^p - m \cdot 3^{p+1} > 0.$$

Next, suppose that we have exactly $m \geq 2$ Type 5 subgraphs of order 3 meeting at $x_1, x_2 \in V_1$. Let $y_1, \ldots, y_m \in V_2$ be the vertices of these subgraphs in V_2. Note that the neighbours of x_1 (resp. x_2) are precisely v and the y_k, and possibly x_2 (resp. x_1), otherwise there is a copy of $B_{7,s}$. Thus $d_C(x_1), d_C(x_2) \in \{m+1, m+2\}$. Suppose first that $m \leq s+1$. Note that $d_C(v) \geq 2s + 23 > 2(m+2) \geq d_C(x_1) + d_C(x_2)$. We delete the $2m$ edges of the Type 5 subgraphs, and add the edges $vy_1, \ldots, vy_m$. Then the degree of v is increased by m, the degrees of x_1, x_2 are decreased by m, and the degrees of the y_k are decreased from 2 to

21

1. The increase in the value of e_p is

$$(d_C(v) + m)^p - d_C(v)^p + (d_C(x_1) - m)^p - d_C(x_1)^p + (d_C(x_2) - m)^p$$
$$- d_C(x_2)^p + m(1^p - 2^p)$$
$$> \sum_{1 \le j \le p,\ j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x_1)^{p-j} - d_C(x_2)^{p-j}) m^j$$
$$+ \binom{p}{2} (d_C(v)^{p-2} + d_C(x_1)^{p-2} + d_C(x_2)^{p-2}) m^2 - m \cdot 2^p$$
$$> ((2m)^{p-2} + 2m^{p-2}) m^2 - m \cdot 2^p = (2^{p-2} + 2) m^p - m \cdot 2^p > 0.$$

Secondly, let $m \ge s + 2$. Suppose that there is an edge $x'y'$ where either $x' \in V_1 \setminus \{x_1, x_2\}$ and $y' \in (V_1 \cup V_2) \setminus \{x_1, x_2, y_1, \ldots, y_m\}$, or $x' = x_2$ and $y' \in V_1 \setminus \{x_1, x_2\}$. Then there is a copy of $B_{7,s}$ with centre x_1, where the P_7 is $y'x'vx_2y_2x_1y_1$ or $wvy'x_2y_2x_1y_1$ for some $w \in V_1 \setminus \{x_1, x_2, y'\}$, and the s leaves are $y_3, \ldots, y_{s+2}$. Similarly, we cannot have an edge x_1y' for every $y' \in V_1 \setminus \{x_1, x_2\}$. It follows that all the edges of C are those connecting v to V_1, and all edges between $\{x_1, x_2\}$ and $\{y_1, \ldots, y_m\}$, and possibly x_1x_2. Now, let C' be the graph obtained by deleting the edges $x_2y_1, \ldots, x_2y_m$ and adding the edges $vy_1, \ldots, vy_m$. Since $d_C(v) \ge d_C(x_2)$, the increase in the value of e_p is

$$(d_C(v) + m)^p - d_C(v)^p + (d_C(x_2) - m)^p - d_C(x_2)^p$$
$$\ge \sum_{1 \le j \le p,\ j \text{ odd}} \binom{p}{j} (d_C(v)^{p-j} - d_C(x_2)^{p-j}) m^j + \binom{p}{2} (d_C(v)^{p-2} + d_C(x_2)^{p-2}) m^2 > 0.$$

Moreover, we see that the degree sequence of C' is majorised by the degree sequence of $K_2 + E_{c-2}$, by identifying $\{v, x_1\}$ with K_2, and the remaining vertices of C' with E_{c-2}. It follows that $e_p(C) < e_p(C') \le e_p(K_2 + E_{c-2}) < e_p(H(c, 7))$.

Therefore, we may assume that no two of the subgraphs as shown in Figure 2(a) meet in V_1. For such a subgraph S, let $[S]$ denote the component of $C - v$ containing S. We consider the structure of $[S]$, so as to avoid a copy of $B_{7,s}$. Clearly if S is of Type 1, then $[S] = S$. If S is of Type 2, 3 or 4, then either $[S] = S$, or $|V(S) \cap V_1| = 2$, and the edge connecting the two vertices of $V(S) \cap V_1$ is in $[S]$. Finally, let S be of Type 5, with $V(S) \cap V_1 = \{x_1, \ldots, x_t\}$ for some $t \ge 2$. It is easy to check that any additional vertices and edges in $[S]$ are as follows. If $t = 2$, then we may possibly have the edge x_1x_2, and either x_1 and x_2 are connected to a vertex of $V_1 \setminus \{x_1, x_2\}$, or only one of x_1, x_2 is connected to some vertices of $V_1 \setminus \{x_1, x_2\}$. If $t = 3$, then we may possibly have any number of the edges x_1x_2, x_1x_3, x_2x_3, or none of these three edges and only one of x_1, x_2, x_3 is connected to some vertices of $V_1 \setminus \{x_1, x_2, x_3\}$. If $t \ge 4$, then we may either have exactly one edge in $\{x_1, \ldots, x_t\}$, or no edge in $\{x_1, \ldots, x_t\}$ and only one of $x_1, \ldots, x_t$ is connected to some vertices of $V_1 \setminus \{x_1, \ldots, x_t\}$.

We see that all such components $[S]$ can be classified into exactly one of three types:

(I) A subgraph of K_4.

(II) A $H(c', 5)$ for some $5 \le c' \le c - 1$ (i.e., a star on c' vertices with an edge connecting two leaves).

(III) A double star with at least five vertices (i.e., two disjoint stars with an edge connecting their centres). A star itself is a special case of a double star.

Moreover, by (ii) with $i = 1$, we see that if $Y = V_1 \setminus \bigcup V([S])$, where the union is taken over all such subgraphs S in Figure 2(a), then $C[Y]$ is P_5-free. It is easy to show that the components of $C[Y]$ must also be one of the types (I), (II) or (III). Consequently, if we connect v to all vertices of V_2 to obtain the graph C^*, then the components of $C^* - v$ are of the types (I), (II) or (III). Note that by replacing $C^* - v$ with the graph $H(c - 1, 5)$, we obtain the graph $H(c, 7)$. We shall show that this operation does not decrease the value of e_p. Consider the following operations.

(A) Suppose that $C^* - v$ has a double star component with at least five vertices, which is not a star. Let the centres be x, y, and the leaves at y be $y_1, \ldots, y_m$, for some $m \geq 1$. We may assume that $d_{C^*}(x) \geq d_{C^*}(y) = m + 2$. We obtain the star with the same order by deleting the edges $yy_1, \ldots, yy_m$, and adding the edges $xy_1, \ldots, xy_m$. If $m \geq 2$, then the increase in the value of e_p is

$$(d_{C^*}(x) + m)^p - d_{C^*}(x)^p + 2^p - (m + 2)^p > pd_{C^*}(x)^{p-1}m - (m + 2)^p$$
$$\geq 2(m + 2)^{p-1}m - (m + 2)^p \geq 0.$$

If $m = 1$, then we have $d_{C^*}(x) \geq 4$. In this case, the increase in the value of e_p is

$$(d_{C^*}(x) + 1)^p - d_{C^*}(x)^p + 2^p - 3^p > pd_{C^*}(x)^{p-1} + 2^p - 3^p \geq 2 \cdot 4^{p-1} + 2^p - 3^p > 0.$$

(B) Suppose that we have two components $C_1, C_2 \subset C^* - v$ with c_1 and c_2 vertices, where $c_1 \geq c_2 \geq 5$, and C_1 (resp. C_2) is either a star or the graph $H(c_1, 5)$ (resp. $H(c_2, 5)$). If C_1 is a star, we add an edge to create $H(c_1, 5)$, and likewise for C_2, so that we have the graphs $H(c_1, 5)$ and $H(c_2, 5)$. We then delete all edges of the $H(c_2, 5)$, and connect all of its vertices to the universal vertex of the $H(c_1, 5)$, thus obtaining the subgraph $H(c_1 + c_2, 5)$. The increase in the value of e_p is at least

$$(c_1 + c_2)^p - c_1^p + 2^p - c_2^p + 2(2^p - 3^p) > pc_1^{p-1}c_2 - 2 \cdot 3^p > 0.$$

Let R be the subgraph of $C^* - v$ consisting of the components which are subgraphs of K_4. We have $d_{C^*}(y) \leq 4$ for all $y \in V(R)$. Let $|V(R)| = r$.

(C) Suppose $r \geq 16$. We replace R with the star of order r, with centre $x \in V(R)$. The increase in the value of e_p is

$$r^p - d_{C^*}(x)^p + \sum_{y \in V(R-x)} (2^p - d_{C^*}(y)^p) > r^p - r \cdot 4^p \geq r(16^{p-1} - 4^p) \geq 0.$$

(D) Suppose that $1 \leq r \leq 15$, and the subgraph $C^* - (\{v\} \cup V(R))$ is $H(c_1, 5)$. Recall that $|V(C^* - v)| = c - 1 \geq 2s + 23 \geq 23$, and thus $c_1 \geq 8$. We delete all edges of R, and connect all vertices of R to the universal vertex of the $H(c_1, 5)$, to form a copy of $H(c - 1, 5)$. Since $c_1 + r = c - 1$, the increase in the value of e_p is

$$(c - 1)^p - c_1^p + \sum_{y \in V(R)} (2^p - d_{C^*}(y)^p) \geq (c_1 + r)^p - c_1^p + r(2^p - 4^p)$$
$$> pc_1^{p-1}r - r \cdot 4^p \geq r(2 \cdot 8^{p-1} - 4^p) \geq 0.$$

23

Therefore where possible, we apply operation (C), then apply operation (A) to all double stars in V_1, followed by successive applications of operation (B), and finally operation (D). Again, it is possible to encounter a copy $B_{7,s}$ in an intermediate graph during the applications of these operations. We obtain $e_p(C) \leq e_p(C^*) \leq e_p(H(c,7))$. Equality occurs if and only if $C = C^*$ and $C^* - v$ is the graph $H(c-1,5)$. That is, if and only if $C = H(c,7)$.

The proof of Theorem 23 is now complete. $\qquad\qquad\qquad\qquad\square$

Acknowledgements

References

[1] B. Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998, xiv+394 pp.

[2] B. Bollobás, V. Nikiforov, Degree powers in graphs with forbidden subgraphs, *Electron. J. Combin.* 11 (2004), R42.

[3] B. Bollobás, V. Nikiforov, Degree powers in graphs: the Erdős-Stone Theorem, *Combin. Probab. Comput.* 21 (2012), 89–105.

[4] N. Bushaw, N. Kettle, Turán numbers of multiple paths and equibipartite forests, *Combin. Probab. Comput.* (2011), 837–853.

[5] Y. Caro, R. Yuster, A Turán type problem concerning the powers of the degrees of a graph, *Electron. J. Combin.* 7 (2000), R47.

[6] P. Erdős, Extremal problems in graph theory, in: Theory of Graphs and its applications, Proc. Sympos. Smolenice, Prague (1964), 29–36.

[7] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* 10 (1959), 337–356.

[8] P. Erdős, A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946), 1087–1091.

[9] R.J. Faudree, R.H. Schelp, Path Ramsey numbers in multicolorings, *J. Combin. Theory Ser. B* 19 (1975), 150–160.

[10] I. Gorgol, Turán numbers for disjoint copies of graphs, *Graphs Combin.* 27 (2011), 661–667.

[11] R. Gu, X. Li, Y. Shi, Degree powers in C_5-free graphs, *Bull. Malays. Math. Sci. Soc.* 38 (2015), 1627–1635.

[12] B. Lidický, H. Liu, C. Palmer, On the Turán number of forests, *Electron. J. Combin.* 20 (2013), P62.

[13] V. Nikiforov, Degree powers in graphs with a forbidden even cycle, *Electron. J. Combin.* 16 (2009), R107.

[14] Z-H. Sun, L-L. Wang, Turán's problem for trees, *J. Comb. Number Theory* 3 (2011), 51–69.

[15] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok.* 48 (1941), 436–452.

[16] L-T. Yuan, X-D. Zhang, The Turán number of disjoint copies of paths, *Discrete Math.* 340 (2017), 132–139.