Kirillov's unimodality conjecture for the rectangular Narayana polynomials

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Abstract. In the study of Kostka numbers and Catalan numbers, Kirillov posed a unimodality conjecture for the rectangular Narayana polynomials. We prove that the rectangular Narayana polynomials have only real zeros, and thereby confirm Kirillov's unimodality conjecture. By using an equidistribution property between descent numbers and ascent numbers on ballot paths due to Sulanke and a bijection between lattice words and standard Young tableaux, we show that the rectangular Narayana polynomial is equal to the descent generating function on standard Young tableaux of certain rectangular shape, up to a power of the indeterminate. Then we obtain the real-rootedness of the rectangular Narayana polynomial based on Brenti's result that the descent generating function of standard Young tableaux has only real zeros.

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1 Introduction

The main objective of this paper is to prove a unimodality conjecture for the rectangular Narayana polynomials in the study of Kostka numbers and Catalan numbers. This conjecture was first posed by Kirillov [5] in 1999, and it was restated by himself [6] in 2015. In this paper we prove that the rectangular Narayana polynomials have only real zeros, an even stronger result than Kirillov's conjecture.

Let us begin with an overview of Kirillov's conjecture. Throughout this paper, we abbreviate the vector $(m, m, \ldots, m)$ with n occurrences of m as

(m^n) for any positive integer m and n. We say that a word $\mathtt{w} = w_1 w_2 \cdots w_{nm}$ in symbols $1, 2, \ldots, m$ is a lattice word of weight (m^n), if the following conditions hold:

(a) each i between 1 and m occurs exactly n times and

(b) for each $1 \le r \le nm$ and $1 \le i \le m-1$, the number of i's in $w_1 w_2 \cdots w_r$ is not less than the number of $(i+1)$'s.

Given a word $\mathtt{w} = w_1 w_2 \cdots w_p$ of length p, we say that i is an ascent of $\mathtt{w}$ if $w_i < w_{i+1}$, and a descent of $\mathtt{w}$ if $w_i > w_{i+1}$. Denote the number of ascents of $\mathtt{w}$ by $\mathrm{asc}(\mathtt{w})$, and the number of descents $\mathrm{des}(\mathtt{w})$. For any m and n, the rectangular Narayana polynomial $N(n, m; t)$ is defined by

$$N(n, m; t) = \sum_{\mathtt{w} \in \mathcal{N}(n,m)} t^{\mathrm{des}(\mathtt{w})}, \tag{1.1}$$

where $\mathcal{N}(n, m)$ is the set of lattice words of weight (m^n). Note that $N(n, 2; t)$ is the classical Narayana polynomial, and $N(n, 2; 1)$ is the classical Catalan number, see [6]. For this reason, $N(n, m; 1)$ is called the rectangular Catalan number.

Kirillov's conjecture is concerned with the unimodality of the rectangular Narayana polynomial $N(n, m; t)$. Recall that a sequence $\{a_0, a_1, \ldots, a_n\}$ of positive real numbers is said to be unimodal if there exists an integer $i \ge 0$ such that

$$a_0 \le \cdots \le a_{i-1} \le a_i \ge a_{i+1} \ge \cdots \ge a_n,$$

and it is said to be log-concave if, for each $1 \le i \le n-1$, there holds

$$a_i^2 \ge a_{i-1} a_{i+1}.$$

Clearly, for a sequence of positive numbers, its log-concavity implies unimodality. Given a polynomial with real coefficients

$$f(t) = \sum_{k=0}^{n} a_k t^k,$$

we say that it is unimodal (or log-concave) if its coefficient sequence $\{a_0, a_1, \ldots, a_n\}$ is unimodal (resp. log-concave). Kirillov proposed the following conjecture.

Conjecture 1.1 ([6, Conjecture 2.5]). *For any m and n, the rectangular Narayana polynomial $N(n, m; t)$ is unimodal as a polynomial of t.*

In this paper, we give an affirmative answer to the above conjecture. Instead of directly proving its unimodality, we shall show that the rectangular

Narayana polynomial $N(n, m; t)$ has only real zeros. By the well known Newton's inequality, if a polynomial with nonnegative coefficients has only real zeros, then its coefficient sequence must be log-concave and hence unimodal. Thus, from the real-rootedness of $N(n, m; t)$ we deduce its log-concavity and unimodality.

The remainder of this paper is organized as follows. In Section 2, we show that the rectangular Narayana polynomial $N(n, m; t)$ is equal to the descent generating function on standard Young tableaux of shape (n^m), up to a power of t. We use a result of Sulanke [8] that the ascent and descent statistics are equidistributed over the set of ballot paths. In Section 3, we first prove the real-rootedness of the descent generating function on standard Young tableaux, and then obtain the real-rootedness of $N(n, m; t)$.

2 Tableau interpretation

The aim of this section is to interpret the rectangular Narayana polynomials as the descent generating function on standard Young tableaux.

Let us first recall some definitions. Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, its Young diagram is defined to be an array of squares in the plane justified from the top left corner with l rows and λ_i squares in row i. By transposing the diagram of λ, we get the conjugate partition of λ, denoted λ'. A cell (i, j) of λ is in the i-th row from the top and in the j-th column from the left. A semistandard Young tableau (SSYT) of shape λ is a filling of its diagram by positive integers such that it is weakly increasing in every row and strictly increasing down every column. The type of T is defined to be the composition $\alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. If T is of type α with $\alpha_i = 1$ for $1 \leq i \leq |\lambda|$ and $\alpha_i = 0$ for $i > |\lambda|$, then it is called a standard Young tableau (SYT) of shape λ. Let $\mathcal{T}_\lambda$ denote the set of SYTs of shape λ. Given a standard Young tableau, we say that i is a descent of T if $i + 1$ appears in a lower row of T than i. Define the descent set $D(T)$ to be the set of all descents of T, and denote by $\mathrm{des}(T)$ the number of descents of T.

The main result of this section is as follows.

Theorem 2.1. *For any positive integers m and n, we have*

$$N(n, m; t) = t^{1-m} \sum_{T \in \mathcal{T}_{(n^m)}} t^{\mathrm{des}(T)}. \tag{2.1}$$

To prove the above result, we need a bijection between the set of lattice paths and the set of standard Young tableaux. Here we use a very natural bijection ϕ between the lattice word of weight (m^n) and the standard Young tableau of shape (n^m), see [3, p. 92], [4, p. 221] and [7]. To be self-contained, we shall give a description of this bijection in the following.

Given a lattice word $\mathtt{w} = w_1 \cdots w_{nm}$ of weight (m^n), let $T = \phi(\mathtt{w})$ be the tableau of shape (n^m) obtained by filling the square (i, j) with k provided that w_k is the j-th occurrence of i in $\mathtt{w}$ from left to right. Clearly, T is a standard Young tableau. Conversely, given a standard Young tableau T of shape (n^m), define a word $\mathtt{w}$ by letting w_i to be j if i is in the j-th row of T. It is easy to verify that $\mathtt{w} = \phi^{-1}(T)$. Figure 2.1 gives an illustration of this bijection, where T is of shape (4^3) and $\mathtt{w}$ is of weight (3^4).

$$\mathtt{w} = \ 121113223233 \mapsto \ T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 7 & 8 & 10 \\ \hline 6 & 9 & 11 & 12 \\ \hline \end{array}$$

Figure 2.1: Bijection between standard Young tableaux and lattice words

By using the above bijection ϕ, we obtain the following result.

Lemma 2.2. *For any positive integers m and n, we have*

$$\sum_{T \in \mathcal{T}_{(n^m)}} t^{\mathrm{des}(T)} = \sum_{\mathtt{w} \in \mathcal{N}(n,m)} t^{\mathrm{asc}(\mathtt{w})}. \tag{2.2}$$

Proof. Suppose that $T = \phi(\mathtt{w})$. Note that if i is an ascent in $\mathtt{w}$, i.e. $w_i < w_{i+1}$, then $i+1$ is filled in the w_{i+1}-th row, which is lower than the row including i in T. Hence, i is a descent of T. Conversely, given a tableau T, let i be a descent of T and $\mathtt{w} = \phi^{-1}(T)$. Since i is in a row above that of $i+1$, it follow that that $w_i < w_{i+1}$. Hence, i is an ascent of $\mathtt{w}$. Therefore, the bijection ϕ sends the set of ascents in $\mathtt{w}$ to the set of descents of $T = \phi(\mathtt{w})$ and hence $\mathrm{asc}(\mathtt{w}) = \mathrm{des}(T)$. This completes the proof. $\square$

To prove Theorem 2.1, it remains to show that

$$t^{1-m} \sum_{\mathtt{w} \in \mathcal{N}(n,m)} t^{\mathrm{asc}(\mathtt{w})} = \sum_{\mathtt{w} \in \mathcal{N}(n,m)} t^{\mathrm{des}(\mathtt{w})}. \tag{2.3}$$

In fact, this has been established by Sulanke [8], who stated it in terms of ballot paths. In the following, we shall give an overview of Sulanke's result.

Recall that a ballot path for m-candidates is an m-dimensional lattice path running from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$ with the steps:

$$\begin{aligned} X_1 &:= (1,0,\ldots,0),\\ X_2 &:= (0,1,\ldots,0),\\ \vdots \quad & \qquad \vdots\\ X_m &:= (0,0,\ldots,1), \end{aligned}$$

and lying in the region

$$\{(x_1,x_2,\ldots,x_m) : 0 \le x_1 \le x_2 \le \ldots \le x_m\}.$$

Denote by $\mathcal{C}(m,n)$ the set of all such paths.

For any path $P := p_1p_2\ldots p_{mn} \in \mathcal{C}(m,n)$, the number of ascents of P is defined by

$$\mathrm{asc}(P) := |\{i : p_ip_{i+1} = X_jX_l, j < l\}|,$$

and the number of descents of P by

$$\mathrm{des}(P) := |\{i : p_ip_{i+1} = X_jX_l, j > l\}|.$$

Sulanke [8] obtained the following result by a nice bijection.

Lemma 2.3 ([8, Proposition 2])**.** *For any positive integers m and n, we have*

$$\sum_{P\in\mathcal{C}(m,n)} t^{\mathrm{asc}(P)} = \sum_{P\in\mathcal{C}(m,n)} t^{\mathrm{des}(P)-m+1}. \tag{2.4}$$

Note that there is an obvious bijection between $\mathcal{C}(m,n)$ and $\mathcal{N}(n,m)$: given a path $P \in \mathcal{C}(m,n)$, simply replace each step X_i of P by the symbol $m-i+1$, and the resulting word $\mathtt{w}$ is clearly a lattice word of $\mathcal{N}(n,m)$. Moreover, we have $\mathrm{asc}(P) = \mathrm{des}(\mathtt{w})$ and $\mathrm{des}(P) = \mathrm{asc}(\mathtt{w})$. With this bijection, Sulanke's result can be restated as (2.3).

Proof of Theorem 2.1. Combining (1.1), (2.2) and (2.3), we immediately obtain the desired result. □

3 Real zeros

In this section, we aim to prove the real-rootedness of rectangular Narayana polynomials. Our main result of this section is as follows.

Theorem 3.1. *The rectangular Narayana polynomial $N(n,m;t)$ has only real zeros for any m and n.*

By Theorem 2.1, we only need to show that the following polynomial

$$\sum_{T \in \mathcal{T}_{(n^m)}} t^{\mathrm{des}(T)}$$

has only real zeros. In fact, Brenti [2] has already obtained a more general result during the study of the Neggers-Stanley Conjecture, see also Brändén [1].

Theorem 3.2 ([2, p. 60, Proof of Theorem 5.3.2])**.** *For any integer partition λ, the polynomial*

$$\sum_{T \in \mathcal{T}_{\lambda}} t^{\mathrm{des}(T)}$$

has only real zeros.

Now we can give a proof of Theorem 3.1.

Proof of Theorem 3.1. This follows from Theorems 2.1 and 3.2. □

As an immediate corollary of Theorem 3.1, we obtain the following result, which gives an affirmative answer to Kirillov's conjecture.

Corollary 3.3. *The rectangular Narayana polynomial $N(n, m; t)$ is unimodal for any m and n.*

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