LOCALLY-PRIMITIVE ARC-TRANSITIVE 10-VALENT GRAPHS OF SQUARE-FREE ORDER

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Abstract. In this paper, we present a complete classification for locally-primitive arc-transitive graphs which have square-free order and valency 10. The classification involves nine graphs and three infinite families of graphs.

Keywords: Arc-transitive graph, locally-primitive graph, normal quotient, vertex-stabilizer, (almost) simple group.

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1. introduction

All graphs and groups considered in this paper are assumed to be finite. The notation and terminologies for graphs and permutation groups not defined in this paper are referred to [1] and [6], respectively. For simple groups and their subgroups, we follow the notation used in the Atlas [5] while we sometimes use $\mathbb{Z}_l$ and $\mathbb{Z}_p^k$ to denote respectively the cyclic group of order l and the elementary abelian group of order p^k, where p is a prime.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E, where E is a set of 2-subsets of V. Denote by $\mathsf{Aut}\Gamma$ the automorphism group of Γ. Let $G \leq \mathsf{Aut}\Gamma$, that is, G is a subgroup of $\mathsf{Aut}\Gamma$. Then the graph Γ is called *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V and E, respectively. An *arc* in Γ is an ordered pair of adjacent vertices. The graph Γ is called *G-arc-transitive* if G acts transitively on the set of all arcs in Γ. For $u \in V$, we denote by G_u and $\Gamma(u)$ respectively the vertex-stabilizer of u in G and the set of neighbors of u in Γ, that is,

$$G_u = \{g \in G \mid u^g = u\} \text{ and } \Gamma(u) = \{v \in V \mid \{u, v\} \in E\}.$$

Then the graph Γ is called *G-locally-primitive* if for every $u \in V$ the stabilizer G_u induces a primitive group $G_u^{\Gamma(u)}$ (on $\Gamma(u)$). It is well-known that if Γ is G-locally-primitive then it is G-edge-transitive. Moreover, if Γ is both G-vertex-transitive and G-locally-primitive, then Γ is also G-arc-transitive; in this case, Γ is called *G-locally-primitive arc-transitive.*

In [16], Li et al. give a reduction for locally-primitive arc-transitive of square-free order. It was proved that, for a connected locally-primitive arc-transitive graph Γ with square-free order, if it is not a complete bipartite graph then either $\mathsf{Aut}\Gamma$ is soluble, or Γ is a cover of one of the 'basic' graphs arising from $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$ and a finite number (depending only on the valency of Γ) of other almost simple groups. This result makes it possible to classify such graphs of small valencies. For

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example, the reader may find some classification results on graphs of valency less than 8 in [15, 17, 18]. In this paper we deal with the graphs of valency 10.

Our main result is stated as follow.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected graph of square-free order and valency 10, and let $G \le \mathsf{Aut}\Gamma$. Assume that Γ is G-locally-primitive arc-transitive. Then, up to isomorphism, Γ is one of the following graphs.*

(1) $\mathsf{Aut}\Gamma = \mathrm{PSL}(2, r)$ *for a prime r with $r \equiv \pm 1\,(\mathsf{mod}\ 5)$ and $r \equiv \pm 1\,(\mathsf{mod}\ 12)$, and Γ is constructed in* Example 3.4*; or*
$\mathsf{Aut}\Gamma = \mathbb{Z}_2{\times}\mathrm{PSL}(2, r)$ *for a prime r with $r \equiv \pm 1\,(\mathsf{mod}\ 5)$, $r \equiv \pm 1\,(\mathsf{mod}\ 12)$ and $r^2 \not\equiv 1\,(\mathsf{mod}\ 16)$, and Γ is the standard double cover of the graph in* Example 3.4*.*
(2) $\mathsf{Aut}\Gamma = \mathrm{PGL}(2, r)$ *for a prime r with $r \equiv \pm 1\,(\mathsf{mod}\ 5)$ and $r \not\equiv \pm 1\,(\mathsf{mod}\ 12)$, and Γ is constructed in* Example 3.5*.*
(3) $\mathsf{Aut}\Gamma = \mathrm{P\Sigma L}(2, 25)$*, and Γ is the graph given in* Example 3.7*; or*
$\mathsf{Aut}\Gamma = \mathbb{Z}_2{\times}\mathrm{P\Sigma L}(2, 25)$*, and Γ is the standard double cover of the graph in* Example 3.7*.*
(4) $\mathsf{Aut}\Gamma = \mathrm{S}_7$*, and Γ is the complement graph of $L(\mathsf{K}_7)$, where $L(\mathsf{K}_7)$ is the line graph of the complete graph K_7 of order 7; or*
$\mathsf{Aut}\Gamma = \mathbb{Z}_2{\times}\mathrm{S}_7$*, and Γ is the standard double cover of the complement graph of $L(\mathsf{K}_7)$.*
(5) $\mathsf{Aut}\Gamma = \mathrm{J}_1$*, and Γ is constructed in* Example 3.9*.*
(6) $\mathsf{Aut}\Gamma = \mathrm{S}_{11}$*, and $\Gamma = \mathsf{K}_{11}$; or*
$\mathsf{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{S}_{11}$*, and Γ is the standard double cover of K_{11}.*
(7) $\mathsf{Aut}\Gamma = \mathrm{P\Gamma L}(3, 9).\mathbb{Z}_2$*, and Γ is the point-line incidence graph of the projective plane* $\mathrm{PG}(2, 9)$*.*
(8) $\mathsf{Aut}\Gamma = \mathrm{S}_{19}$*, and $\Gamma = \mathrm{O}_{10}$, the Odd graph of valency* 10*.*

2. Preliminaries

In this section, we collect some elementary results on permutation groups and graphs, which will be used in the following sections.

Let G be a group acting transitively on a finite set V. A nonempty subset B of V is a block of G if either $B^g = B$ or $B^g \cap B =$ for all $g \in G$. A block B of G is nontrivial if $1 < |B| < |V|$. We say the group G is primitive on V if there is no nontrivial block. For $u \in V$ and $B \subseteq V$, let $G_u = \{g \mid u^g = u\}$ and $G_B = \{g \in G \mid B^g = B\}$, called respectively the point-stabilizer of u and the set-wise stabilizer of B in G. If B is a block then B is a G_B-orbit and $G_u \le G_B$ for $u \in B$. Conversely, If $G_u \le H \le G$ for $u \in V$ then it is easily shown that $u^H := \{u^h \mid h \in H\}$ is a block of G. Thus $H \mapsto u^H$ gives a bijection between the subgroups of G containing G_u and the blocks of G containing u. In particular, G is primitive on V if and only if, for $u \in G$, the point-stabilizer G_u is a maximal subgroup of G.

Let G be a group acting transitively on a finite set V, and Let $H \le G$. Then, for $u \in V$, the H-orbit u^H has length $|H : H_u|$, the index of H_u in H. Choose $u \in V$ such that u^H has minimal length among the H-orbits on V. Then the number of H-orbits is no more than $\frac{|V|}{|u^H|} = \frac{|G:G_u|}{|H:H_u|}$. Since $H_u \le G_u$, we have $\frac{|G:G_u|}{|H:H_u|} = \frac{|G||H_u|}{|H||G_u|} \le \frac{|G|}{|H|} = |G : H|$.

If $(|V|, |G:H|) = 1$, that is, $(|G:G_u|, |G:H|) = 1$, then $G = HG_u$ (see [10, I.2.13] for example), and so H is transitive on V. Thus we have the following lemma.

Lemma 2.1. *Let G be a group acting transitively on a finite set V of size n, and Let $H \leq G$. Then H has at most $|G:H|$ orbits on V, and H is transitive on Ω if and only if $G = HG_u$ for $u \in V$. Moreover, if $(|G:H|, n) = 1$ then H is transitive on V.*

Let $\Gamma = (V, E)$ be a graph and $G \leq \mathsf{Aut}\Gamma$. For each $u \in V$, the stabilizer G_u induces a permutation group $G_u^{\Gamma(u)}$. Denote by $G_u^{[1]}$ be the kernel of G_u acting on $\Gamma(u)$. Then $G_u^{\Gamma(u)} \cong G_u/G_u^{[1]}$. Further, the next lemmas are easily shown, refer to [18].

Lemma 2.2. *Assume that $\Gamma = (V, E)$ is a connected G-vertex-transitive graph. Let $N \trianglelefteq G$ such that $N_u^{\Gamma(u)}$ is semiregular for some $u \in V$. Then $N_u^{[1]} = 1$. In particular, $N_u = 1$ if $N_u^{\Gamma(u)} = 1$.*

Lemma 2.3. *Let $\Gamma = (V, E)$ be a connected graph. Let $N \trianglelefteq G \leq \mathsf{Aut}\Gamma$ and $u \in V$. Assume that either N is regular on V, or Γ is a bipartite graph such that N is regular on both the bipartition subsets of Γ. Then $G_u^{[1]} = 1$.*

Lemma 2.4. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph, where $G \leq \mathsf{Aut}\Gamma$. Let N be a normal subgroup of G. If N is not semiregular on V then for $u \in V$ the stabilizer N_u is transitive on $\Gamma(u)$; in particular, N is transitive on E and has at most two orbits on V.*

Now let $\Gamma = (V, E)$ be a connected G-locally-primitive graph, where $G \leq \mathsf{Aut}\Gamma$. Let N be a normal subgroup of G. Suppose that N is intransitive on every G-orbit on V. Let $\mathcal{B}$ be the set of the N-orbits. The *normal quotient* Γ_N is defined as the graph with vertex set $\mathcal{B}$ such that $B_1, B_2 \in \mathcal{B}$ are adjacent if and only if $\{u_1, u_2\} \in E$ for some $u_1 \in B_1$ and $u_2 \in B_2$. The graph Γ is called a (normal) *cover* of Γ_N if, for every edge of $\{B_1, B_2\}$ of Γ_N, the subgraph of Γ induced by $B_1 \cup B_2$ is a matching. By Lemmas 2.3 and 2.4, the following lemma is easily shown, refer to [9, 18].

Lemma 2.5. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive graph, where $G \leq \mathsf{Aut}\Gamma$. Let N be a normal subgroup of G. Assume that N is intransitive on every G-orbit on V. Then one of the following statements holds.*

(i) *Γ is a cover of Γ_N, N is semiregular on V and N itself is the kernel of G acting on $\mathcal{B}$, and Γ_N is (G/N)-locally-primitive.*

(ii) *N has two orbits on V, Γ is bipartite and G-arc-transitive, and either Γ is N-edge-transitive or $G_u^{[1]} = 1$ for every $u \in V$.*

Let $\Gamma = (V, E)$ be a graph. The *standard double cover* of Γ, denoted by $\Gamma^{(2)}$, is the graph defined on $V \times \mathbb{Z}_2$ with edge set $\{\{(u, 0), (v, 1)\} \mid \{u, v\} \in E\}$. It is well-known that $\Gamma^{(2)}$ is connected if and only if Γ is connected and not bipartite. Moreover, each $g \in \mathsf{Aut}\Gamma$ induces an automorphism of $\Gamma^{(2)}$ by

$$(u, i)^g = (u^g, i), \ \ u \in V, \, i \in \mathbb{Z}_2.$$

Further, we have an automorphism interchanging the bipartition subsets of $\Gamma^{(2)}$:

$$\theta : V \times \mathbb{Z}_2 \to V \times \mathbb{Z}_2, \, (u, i) \mapsto (u, i+1), \ \ u \in V, \, i \in \mathbb{Z}_2.$$

Thus, identifying $\mathsf{Aut}\Gamma$ with a subgroup of $\mathsf{Aut}\Gamma^{(2)}$, we have $\mathsf{Aut}\Gamma^{(2)} \geq \mathsf{Aut}\Gamma\times\langle\theta\rangle$. Then the next lemma follows.

Lemma 2.6. *Let Γ be a graph and $G \leq \mathsf{Aut}\Gamma$. If Γ is G-vertex-transitive (resp. arc-transitive or locally-primitive) then $\Gamma^{(2)}$ is G-edge-transitive and $(G\times\langle\theta\rangle)$-vertex-transitive (resp. arc-transitive or locally-primitive).*

Lemma 2.7. *Let $\Gamma = (V, E)$ be a connected $(G\times N)$-arc-transitive graph. Assume $N \cong \mathbb{Z}_2$, Γ is G-edge-transitive and G has two orbits on V. If Γ is a normal cover of Γ_N then $\Gamma \cong (\Gamma_N)^{(2)}$.*

Proof. Take a G-orbit U, and set $N = \langle o\rangle$. Then U^o is the other G-orbit. Define a directed graph Σ on U such that (u, v) is a directed edge of Σ if and only if (u, v^o) is an arc of Γ. Note that o is an automorphism of Γ. Then (u, v^o) is an arc of Γ if and only if (v, u^o) is an arc of Γ. Thus (u, v) is a directed edge of Σ if and only if so does (v, u). Therefore, we may identify Σ with a graph, also denoted by Σ, by viewing two directed edges (u, v) and (v, u) as an edge $\{u, v\}$. Then $\Sigma(u) = \{v \in U \mid v^o \in \Gamma(u)\}$ for $u \in U$. Note that the stabilizer G_u is transitive on $\Gamma(u)$. It follows that G_u is transitive on $\Sigma(u)$, and so Σ is G-arc-transitive as G is transitive on U. Define $\eta : U \cup U^o \to U\times\mathbb{Z}_2$, $u \mapsto (u, 0)$, $u^o \mapsto (u, 1)$. It is easily shown that η is an isomorphism from Γ to $\Sigma^{(2)}$.

Note that N has $|U|$ orbits on V and, for $u \in U$, the N-orbit containing u is $\{u, u^o\}$. Then $\{u, u^o\} \mapsto u$ gives a bijection between the set of N-orbits and U. Since Γ is a bipartite graph, it is easily shown that this bijection is in fact an isomorphism from Γ_N to Σ. Then the lemma follows. □

We end this section by quoting a result on number theory. For positive integers a and n, a prime divisor of $a^n - 1$ is called *primitive* if it does not divide $a^i - 1$ for any positive integer i less than n.

Theorem 2.8 (Zsigmondy)**.** *For integers $a, n \geq 2$, if $a^n - 1$ does not have primitive prime divisors, then either $(a, n) = (2, 6)$, or $n = 2$ and $a + 1$ is a power of 2.*

Corollary 2.9. *Let n be a positive integer, and $r \in \{2, 3\}$. Then $r^n - 1$ has a prime divisor no less than 7 unless $n = 1$ or (r, n) is one of $(2, 2)$, $(2, 4)$, $(3, 2)$ and $(3, 4)$.*

Proof. Suppose that all prime divisors of $r^n - 1$ are less than 7. If $n = 1$ then $r^n - 1$ has no prime divisor no less than 7. Thus let $n \geq 2$. If $r^n - 1$ has no primitive prime divisor, then $(r, n) = (2, 6)$ or $(3, 2)$; in this case, we have $(r, n) = (3, 2)$. Let p be a primitive prime divisor of $r^n - 1$. Then $p \in \{3, 5\}$. For $p = 3$ we have $(r, n) = (2, 2)$, and for $p = 5$ we have $(r, n) = (2, 4)$ or $(3, 4)$. Then the lemma follows. □

3. Coset graphs and examples

Let G be a finite group and H be a core-free subgroup of G, where core-free means that $\cap_{g\in G}H^g = 1$. For $x \in G \setminus H$, the *coset graph* $\mathsf{Cos}(G, H, H\{x, x^{-1}\}H)$ is defined on $[G : H]$, the set of right cosets of H in G, such that Hg_1 and Hg_2 are adjacent whenever $g_2g_1^{-1} \in HxH \cup Hx^{-1}H$. In the case where $HxH = Hx^{-1}H$, we denote this coset graph by $\mathsf{Cos}(G, H, x)$. Note that G may be viewed as a subgroup of $\mathsf{AutCos}(G, H, H\{x, x^{-1}\}H)$, where G acts on $[G : H]$ by right multiplication.

The following statements for coset graphs are well-known.

Lemma 3.1. *Let G be a finite group and H a core-free subgroup of G. Set $\Gamma = \mathsf{Cos}(G, H, H\{x, x^{-1}\}H)$, where $x \in G \setminus H$. Then Γ is both G-vertex-transitive and G-edge-transitive, and*

(i) *Γ is G-arc-transitive if and only if $HxH = HyH$ for some 2-element $y \in \mathbf{N}_G(H \cap H^x) \setminus H$ with $y^2 \in H \cap H^x$; in this case, Γ has valency $|H : (H \cap H^y)|$;*
(ii) *Γ is connected if and only if $\langle H, x\rangle = G$;*
(iii) *for each $\sigma \in \mathsf{Aut}(G)$, there is an isomorphism $Hg \mapsto H^\sigma g^\sigma$, $g \in G$ from Γ to $\mathsf{Cos}(G, H^\sigma, H^\sigma\{x^\sigma, (x^\sigma)^{-1}\}H^\sigma)$.*

Let $\Gamma = (V, E)$ be a graph of valency d, let $\{u, v\} \in E$ and $G \leq \mathsf{Aut}\Gamma$. Set $G_{uv} = G_u \cap G_v$, call the *arc-stabilizer* of (u, v) (and of (v, u)). Assume that Γ is G-arc-transitive. Then G_u is transitive on $\Gamma(u)$, and $d = |\Gamma(u)| = |G_u : G_{uv}|$. Take $x \in G$ with $(u, v)^x = (v, u)$. Then

$$x \in \mathbf{N}_G(G_{uv}) \setminus G_{uv},\ x^2 \in G_{uv}.$$

In particular, the index $|\mathbf{N}_G(G_{uv}) : G_{uv}|$ is even. Note that each element in $\mathbf{N}_G(G_{uv})$ either interchanges u and v, or fixes both of them. Thus this x may be chosen as a 2-element in the normalizer $\mathbf{N}_G(G_{uv})$. Moreover, it is well-know and easily shown that Γ is connected if and only if $\langle x, G_u\rangle = G$. Since G is transitive on V, the map $u^g \mapsto G_u g$ is a bijection between V and $[G : G_u]$. It is easy to show that this map is also an isomorphism from Γ to $\mathsf{Cos}(G, G_u, x)$. Thus the following simple fact holds.

Lemma 3.2. *Let $\Gamma = (V, E)$ ba a graph with $E \neq \emptyset$ and $G \leq \mathsf{Aut}\Gamma$. If Γ is G-arc-transitive then, for $\{u, v\} \in E$ and $x \in G$ with $(u, v)^x = (v, u)$, we have $\Gamma \cong \mathsf{Cos}(G, G_u, x)$; moreover, such an x may be chosen as a 2-element in $\mathbf{N}_G(G_{uv})$, and Γ is connected if and only if $\langle x, G_u\rangle = G$.*

The next result is a direct consequence of Lemmas 3.1 and 3.2.

Lemma 3.3. *Let G be a finite group, and let H be a core-free subgroup of G. Let $K \leq H$ with $|H : K| > 1$. If there is a G-arc-transitive graph $\Gamma = (V, E)$ with $E \neq \emptyset$, $G_u = H$ and $G_{uv} = K$ for $\{u, v\} \in E$, then $|\mathbf{N}_G(K) : K|$ is even and $\mathbf{N}_G(K) \not\leq H$; if further Γ is connected then G is generated by H and some 2-element in $\mathbf{N}_G(K)$.*

Next we shall construct some graphs involved in Theorem 1.1.

Example 3.4. Let $T = \mathrm{PSL}(2, r)$, where r is a prime with $r^2 \equiv 1 \pmod 5$ and $r \equiv \pm 1 \pmod{12}$. Fixes two subgroups K and H of T with $\mathrm{S}_3 \cong K < H \cong \mathrm{A}_5$. Then $\mathbf{N}_T(K) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Let o be the involution in the center of $\mathbf{N}_T(K)$. Then $\langle o, H\rangle = T$, and $\mathsf{Cos}(T, H, o)$ is connected and T-arc-transitive. $\square$

Example 3.5. Let $G = \mathrm{PGL}(2, r)$, where r is a prime with $r^2 \equiv 1 \pmod 5$ and $r \not\equiv \pm 1 \pmod{12}$. Fixes two subgroups K and H of $T = \mathrm{soc}(G)$ with $\mathrm{S}_3 \cong K < H \cong \mathrm{A}_5$. Then $\mathbf{N}_T(K) \cong \mathrm{S}_3$ and $\mathbf{N}_G(K) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Let ι be the involution in the center of $\mathbf{N}_G(K)$. Then $\langle \iota, H\rangle = G$, and $\mathsf{Cos}(G, H, \iota)$ is connected and G-arc-transitive. $\square$

Remark 3.6. We may reconstruct the graphs in Examples 3.4 and 3.5 as follows.

Let $T = \mathrm{PSL}(2, r)$, where r is a prime with $r^2 \equiv 1 \,(\mathsf{mod}\ 5)$. Then T has two conjugacy classes Ω_1 and Ω_2 of subgroups isomorphic to A_5, which are interchanged by each element $\iota \in \mathrm{PGL}(2, r) \setminus \mathrm{PSL}(2, r)$. Moreover, $|\Omega_1| = |\Omega_2| = \frac{r(r^2-1)}{120}$. Let $\epsilon = \pm 1$ with $r + \epsilon$ divisible by 3. Set $\Omega_1 = \{H^g \mid g \in T\}$, where $\mathrm{A}_5 \cong H < T$. Then $\Omega_2 = \{H^{\iota g} \mid g \in T\}$. Let $K < H$ with $|K| = 6$. Then $K \cong \mathrm{S}_3$.

For the dihedral group $\mathrm{D}_{r+\epsilon}$, it is easily shown that its subgroups isomorphic to S_3 form two conjugacy classes if $r+\epsilon$ is divisible by 12, and one conjugacy class otherwise. By [10, II.8.5], two distinct subgroups of T isomorphic to $\mathrm{D}_{r+\epsilon}$ have intersection of order at most 2. Note that each subgroup of T isomorphic to S_3 is contained in some subgroup of T isomorphic to $\mathrm{D}_{r+\epsilon}$. It follows that T has at most two conjugacy classes of subgroups isomorphic to S_3, and T has two conjugacy classes of subgroups isomorphic to S_3 if and only if $r + \epsilon$ is divisible by 12.

(1) Assume that $r + \epsilon$ is divisible by 4. Then T has two conjugacy classes $\Delta_1 := \{K^g \mid g \in T\}$ and Δ_2^ι of subgroups isomorphic to S_3, and $|\Delta_1| = |\Delta_2| = \frac{r(r^2-1)}{24}$.

Enumerating the pairs (X, Y) with $X \in \Omega_i$, $Y \in \Delta_i$ and $Y < X$, we conclude that each $Y \in \Delta_i$ is contained exactly two subgroups in Ω_i, $i = 1, 2$. Since A_5 has 10 subgroups isomorphic to S_3, we know that for each $X \in \Omega_i$, there are exactly 10 other subgroups in Ω_i which intersects X in some subgroup of order 6.

For $i \in \{1, 2\}$, we define a graph Σ_i on Ω_i such that X_1, $X_2 \in \Omega_i$ are adjacent if and only if $|X_1 \cap X_2| = 6$. Then Σ_i is a well-defined graph of valency 10. Let $\theta_i : [G : H] \to \Omega_i$, $Hg \mapsto H^{g\iota^{i-1}}$. Note $H^{g_1\iota^{i-1}} = H^{g_2\iota^{i-1}}$ if and only if $g_2 g_1^{-1}$ normalizes H, yielding $g_2 g_1^{-1} \in H$ as H is a maximal subgroup of T, i.e., $Hg_1 = Hg_2$. It follows that θ_i is a bijection. If Hg_1 and Hg_2 are adjacent in the graph in Example 3.4 then their stabilizers H^{g_1} and H^{g_2} intersect in a subgroup of order 6. Thus both Σ_1 and Σ_2 are isomorphic to the graph given in Example 3.4.

(2) Assume that $r + \epsilon$ is not divisible by 4. By a similar argument as above, we conclude that each subgroup of T isomorphic to S_3 is contained exactly two subgroups isomorphic to A_5: one lies in Ω_1 and the other one is in Ω_1.

Define a bipartite graph Σ on $\Omega_1 \cup \Omega_2$ such that $X_1 \in \Omega_1$ and $X_2 \in \Omega_2$ are adjacent if and only if $|X_1 \cap X_2| = 6$. Then Σ is a well-defined graph of valency 10, and $\theta : [G : H] \to \Omega_i$, $Hg\iota^{i-1} \mapsto H^{g\iota^{i-1}}$, $i = 1, 2$ is an isomorphism between Σ and the graph in Example 3.5. $\square$

Example 3.7. Let $T = \mathrm{PSL}(2, 25)$. Fixes two subgroups K and H of T with $\mathbb{Z}_2 \times \mathrm{S}_3 \cong K < H \cong \mathrm{S}_5$. Then $\mathbf{N}_T(K) \cong \mathrm{D}_{24}$. Take an involution $x \in \mathbf{N}_T(K) \setminus K$. Then $\langle x, H\rangle = T$, and $\mathsf{Cos}(T, H, x)$ is a connected T-arc-transitive of order 65.

Checking the subgroups of $\mathrm{P\Sigma L}(2, 25)$, we may choose an involution $\iota \in \mathrm{P\Sigma L}(2, 25) \setminus T$ such that $\langle H, \iota\rangle = \langle\iota\rangle \times H$. Then ι normalizes $\mathbf{N}_T(K)$. Thus

$$(HxH)^\iota = (H\mathbf{N}_T(K)H)^\iota = H\mathbf{N}_T(K)H = HxH.$$

Then, by Lemma 3.1, ι induces an automorphism of $\mathsf{Cos}(T, H, x)$. It follows that $\mathrm{P\Sigma L}(2, 25) \le \mathsf{AutCos}(T, H, x)$. $\square$

Remark 3.8. Note that $\mathrm{PSL}(2, 25)$ has two conjugacy classes of subgroups isomorphic to S_5, and two classes of subgroups isomorphic to $\mathbb{Z}_2 \times \mathrm{S}_3$. It is easily shown that the graph in Example 3.7 may be reconstructed in a similar way as in part

(1) of Remark 3.6. In fact, this graph is a primitive distance transitive graph with automorphism group $\mathrm{P\Sigma L}(2, 25)$, refer to [3, pp. 381, Proposition 12.2.2]. $\square$

Example 3.9. Checking by GAP [8], we know that the first Janko group J_1 has exactly two conjugation classes of subgroups isomorphic A_5, and one of the classes consists of the subgroups having normalizer isomorphic to $\mathbb{Z}_2 \times \mathrm{A}_5$, while the other one contains only self-normalized subgroups.

Take a subgroup H of J_1 with $H \cong \mathrm{A}_5$ and $\mathbf{N}_{\mathrm{J}_1}(H) \cong \mathbb{Z}_2 \times \mathrm{A}_5$. Let $\mathrm{S}_3 \cong K < H$. Then $\mathbf{N}_{\mathrm{J}_1}(K) = K \times \langle a, b\rangle$, where a has order 5 and b is an involution with $a^b = a^{-1}$. Calculation shows that H is contained exactly in three maximal subgroups of J_1, one is $X := \mathbf{N}_{\mathrm{J}_1}(H)$, and the other two, say Y and Z, are isomorphic to $\mathrm{PSL}(2, 11)$. It is easily shown that each of these maximal subgroups contains exactly one involution centralizing K. Note that $\mathbf{N}_{\mathrm{J}_1}(K)$ contains 5 involutions centralizing K. Thus $\langle a, b\rangle$ contains at least two involutions o_1 and o_2 which lie outside of $X \cup Y \cup Z$. Then we have $\mathrm{J}_1 = \langle H, o_1\rangle = \langle H, o_2\rangle$, and so we get two connect arc-transitive graphs $\mathsf{Cos}(\mathrm{J}_1, H, o_1)$ and $\mathsf{Cos}(\mathrm{J}_1, H, o_2)$. Confirmed by GAP, o_1 and o_2 are the only involutions which are contained in $\langle a, b\rangle$ and lie outside of $X \cup Y \cup Z$, and these two involution are conjugate under $\mathbf{N}_{\mathrm{J}_1}(K)$. Thus, up to isomorphism, we get only one arc-transitive graph. $\square$

4. The structure of stabilizers

Let $\Gamma = (V, E)$ be a graph, $G \leq \mathsf{Aut}\Gamma$ and $\{u, v\} \in E$. Set $G_{uv}^{[1]} := G_u^{[1]} \cap G_v^{[1]}$. Writing G_u in group extensions, we have

$$G_u = G_{uv}^{[1]}.(G_u^{[1]})^{\Gamma(v)}.G_u^{\Gamma(u)}, \tag{4.1}$$

where $(G_u^{[1]})^{\Gamma(v)}$ is the permutation group induced by $G_u^{[1]}$ on $\Gamma(v)$. In [7, 2.3], Gardiner proved the following theorem.

Theorem 4.1. *Let $\Gamma = (V, E)$ be a connected graph, $\{u, v\} \in E$ and $G \leq \mathsf{Aut}\Gamma$. If Γ is G-locally-primitive arc-transitive, then then $G_{uv}^{[1]}$ is a p-group for some prime p.*

Note that, for $\{u, v\} \in E$, both $G_u^{[1]}$ and $G_v^{[1]}$ are normal subgroups of G_{uv}. Then

$$(G_u^{[1]})^{\Gamma(v)} \cong G_u^{[1]}/G_{uv}^{[1]} \cong (G_u^{[1]}G_v^{[1]})/G_v^{[1]} \trianglelefteq G_{uv}/G_v^{[1]} \cong G_{uv}^{\Gamma(v)} = (G_v^{\Gamma(v)})_u. \tag{4.2}$$

If $G = (V, E)$ is transitive on the arcs of Γ, then $(G_v^{\Gamma(v)})_u \cong (G_u^{\Gamma(u)})_v$. Thus, by (4.2), the following lemma holds.

Lemma 4.2. *If G is transitive on the arcs of Γ, then $(G_u^{[1]})^{\Gamma(v)}$ is isomorphic to a normal subgroup of $(G_u^{\Gamma(u)})_v$.*

Let Γ be connected and G-arc-transitive. Assume that $G_{uv}^{[1]} \neq 1$, where $\{u, v\} \in E$. Then, by the connectedness of Γ, there exists an edge $\{u', v'\} \in E$ such that $G_{uv}^{[1]} = G_{u'v'}^{[1]}$, and $G_{uv}^{[1]} \neq G_{v'w}^{[1]}$ for some $w \in \Gamma(v')$. In particular, $G_{uv}^{[1]}$ acts nontrivially on $\Gamma(w)$. Without loss of generality, we let $(u, v) = (u', v')$. Note that $G_{uv}^{[1]} \trianglelefteq G_v^{[1]} \trianglelefteq G_{vw}$. Then $G_{uv}^{[1]}$ induces a nontrivial normal subgroup of $(G_v^{[1]})^{\Gamma(w)}$. Since Γ is G-arc-transitive, $(v, w)^g = (u, v)$ for some $g \in G$. This yields $(G_v^{[1]})^{\Gamma(w)} \cong (G_u^{[1]})^{\Gamma(v)}$. Thus, if $G_{uv}^{[1]}$ is a nontrivial p-group then we may get $\mathbf{O}_p((G_u^{[1]})^{\Gamma(v)}) \neq 1$. Then the next result follows from Theorem 4.1 and Lemma 4.2. (Note that $\mathbf{O}_p(X)$ is the maximal normal p-subgroup of a group X, which is a characteristic subgroup of X.)

Theorem 4.3. *Let $\Gamma = (V, E)$ be a connected graph, $\{u, v\} \in E$ and $G \le \mathsf{Aut}\Gamma$. If Γ is G-locally-primitive arc-transitive, then $G_{uv}^{[1]}$ is a p-group for some prime p; moreover, either $G_{uv}^{[1]} = 1$, or $\mathbf{O}_p((G_u^{[1]})^{\Gamma(v)}) \ne 1$ and $\mathbf{O}_p((G_u^{\Gamma(u)})_v) \ne 1$.*

Recall that, for a positive integer s, an *s-arc* in $\Gamma = (V, E)$ is an $(s+1)$-tuple $(u_0, u_1, \cdots, u_s)$ of vertices such that $\{u_i, u_{i+1}\} \in E$ and $u_{i-1} \ne u_{i+1}$ for all possible i. Then the graph Γ is called *(G, s)-arc-transitive* if it has at least one s-arc and G acts transitively on both V and the set of s-arcs, and called *(G, s)-transitive* if it is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. For the stabilizers of s-transitive graphs, we formulate the following result from [21, 22, 23].

Theorem 4.4. *Let $\Gamma = (V, E)$ be a connected (G, s)-transitive graph with $s \ge 2$, and let $\{u, v\} \in E$. Then one of the following holds.*

(1) $G_{uv}^{[1]} = 1$ *and* $s \le 3$;
(2) $G_{uv}^{[1]}$ *is a non-trivial p-group,* $\mathrm{PSL}(n, p^e) \trianglelefteq G_u^{\Gamma(u)}$, $|\Gamma(u)| = \frac{p^{en}-1}{p^e-1}$, *and either* $n \ge 3$ *and* $s \in \{2, 3\}$, *or* $n = 2$, $s \ge 4$ *and one of the following holds:*
   (i) $s = 4$ *and* $G_u = [p^{2e}]{:}(\mathbb{Z}_a.\mathrm{PGL}(2, p^e)).O$, *where* $a = \frac{p^e-1}{(3,p^e-1)}$ *and O is of order a divisor of* $(3, p^e - 1)e$;
   (ii) $s = 5$, $p = 2$ *and* $G_u = [2^{3e}]{:}\mathrm{GL}(2, 2^e).\mathbb{Z}_b$, *where b is a divisor of e;*
   (iii) $s = 7$, $p = 3$ *and* $G_u = [3^{5e}]{:}\mathrm{GL}(2, 3^e).\mathbb{Z}_b$, *where b is a divisor of e.*

Based on Theorems 4.3 and 4.4, we produce here a description for the stabilizers of locally-primitive arc-transitive graphs of valency $2k$, where k is a prime no less than 5.

Theorem 4.5. *Let $\Gamma = (V, E)$ be a connected G-locally-primitive arc-transitive graph. Assume that Γ is (G, s)-transitive and of valency $2k$, where k is a prime no less than 5. Then one of the following holds.*

(1) $s = 1$, $k = 5$ *and one of the following holds:*
   (i) $G_u \cong \mathrm{A}_5$ *or* S_5;
   (ii) $\mathbf{O}_3(G_u) \ne 1$ *and* $G_u = \mathbf{O}_3(G_u).O.\mathrm{A}_5$, *where* $O \le \mathbb{Z}_2$;
   (iii) $\mathbf{O}_3(G_u) \ne 1$ *and* $G_u = \mathbf{O}_3(G_u).O.\mathrm{S}_5$, *where* $O \le \mathbb{Z}_2^2$;
   (iv) $\mathbf{O}_2(G_u) \ne 1$ *and* $G_u = \mathbf{O}_2(G_u).O.\mathrm{S}_5$, *where* $O \trianglelefteq \mathrm{S}_3$.
(2) $s \in \{2, 3\}$, $k = 11$, $G_u \cong \mathrm{M}_{22}$, $\mathrm{M}_{22}.2$ *or* $(\mathrm{PSL}(3, 4) \times \mathrm{M}_{22}).O$, *where $|O|$ is a divisor of 4.*
(3) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{PSL}(2, p^e)$, $k = \frac{p^e+1}{2}$ *for an odd prime p and a power e of 2, and one of the following holds:*
   (v) $G_{uv}^{[1]} = 1$ *and* $s \in \{2, 3\}$, *where* $v \in \Gamma(u)$;
   (vi) $s = 4$ *and* $G_u = [p^{2e}]{:}(\mathbb{Z}_a.\mathrm{PGL}(2, p^e)).R$, *where* $a = \frac{p^e-1}{(3,p^e-1)}$ *and $|R|$ is a divisor of* $(3, p^e - 1)e$;
   (vii) $s = 7$, $p = 3$ *and* $G_u = [3^{5e}]{:}\mathrm{GL}(2, 3^e).\mathbb{Z}_b$, *where b is a divisor of e.*
(4) $s \in \{2, 3\}$, $G_u \cong \mathrm{A}_{2k}$, S_{2k} *or* $(\mathrm{A}_{2k-1} \times \mathrm{A}_{2k}).O$, *where $|O|$ is a divisor of 4.*

Proof. By the assumption, $G_u^{\Gamma(u)}$ is a primitive permutation group of degree $2k$. Since a soluble primitive permutation group has degree a power of some prime, noting that

$k \neq 2$, we know that $G_u^{\Gamma(u)}$ is insoluble. Then $G_u^{\Gamma(u)}$ is known by [19, Corollary 1.2]. Let $v \in \Gamma(u)$. Up to permutation isomorphism, one of the following holds:

(a) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_5$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{S}_3$ and $k = 5$;
(b) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{M}_{22}$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{PSL}(3,4)$ and $k = 11$;
(c) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{PSL}(2,p^e)$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathbb{Z}_p^e{:}\mathbb{Z}_{\frac{p^e-1}{2}}$ and $k = \frac{p^e+1}{2}$, where p is an odd prime, and e is a power of 2;
(d) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_{2k}$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{A}_{2k-1}$.

For (b), (c) and (d), since $G_u^{\Gamma(u)}$ is 2-transitive, $s \geq 2$. For (c), by Theorem 4.4, part (3) of this theorem follows. We next lay out the remainder argument in three cases.

Case 1. Let $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_5$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{S}_3$ and $k = 5$. In this case, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$, and so Γ is not $(G,2)$-arc-transitive. Thus $s = 1$. We next shows that part (1) of this theorem occurs.

Suppose that $G_u^{\Gamma(u)} = \mathrm{soc}(G_u^{\Gamma(u)})$. Then, by Theorem 4.3, $G_{uv}^{[1]}$ is a 3-group. By Lemma 4.2, $(G_u^{[1]})^{\Gamma(v)}$ is isomorphic to a normal subgroup of S_3. If $(G_u^{[1]})^{\Gamma(v)} = 1$ then $G_u^{[1]} = G_v^{[1]}$, and so $G_u^{[1]} = 1$ by the connectedness of Γ, yielding $G_u \cong \mathrm{A}_5$. Now let $(G_u^{[1]})^{\Gamma(v)} \cong \mathbb{Z}_3$ or S_3. Noting that $G_u/G_u^{[1]} \cong \mathrm{A}_5$, it follows that $\mathbf{O}_3(G_u) = \mathbf{O}_3(G_u^{[1]}) = G_{uv}^{[1]}.\mathbb{Z}_3$. Then (4.1) implies that $G_u = \mathbf{O}_3(G_u).\mathbb{Z}_l.\mathrm{A}_5$, where $l \leq 2$.

Suppose that $G_u^{\Gamma(u)} \cong \mathrm{S}_5$. Then $(G_u^{\Gamma(u)})_v \cong \mathbb{Z}_2 \times \mathrm{S}_3$, and by Theorem 4.3, $G_{uv}^{[1]}$ is a 2-group or a 3-group. If $(G_u^{[1]})^{\Gamma(v)} = 1$ then $G_u \cong \mathrm{S}_5$. Thus suppose further that $(G_u^{[1]})^{\Gamma(v)} \neq 1$. Then $(G_u^{[1]})^{\Gamma(v)} \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, S_3, $\mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathrm{S}_3$. If $G_{uv}^{[1]}$ is a nontrivial 2-group then $(G_u^{[1]})^{\Gamma(v)} \cong \mathbb{Z}_2$, $\mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathrm{S}_3$, and so $\mathbf{O}_2(G_u) \neq 1$ and $G_u = \mathbf{O}_2(G_u).O.\mathrm{S}_5$, where $O \trianglelefteq \mathrm{S}_3$. If $G_{uv}^{[1]}$ is a nontrivial 3-group then $(G_u^{[1]})^{\Gamma(v)} \cong \mathbb{Z}_3$, $\mathbb{Z}_6$, S_3 or $\mathbb{Z}_2 \times \mathrm{S}_3$, yielding $\mathbf{O}_3(G_u) \neq 1$ and $G_u = \mathbf{O}_3(G_u).O.\mathrm{S}_5$, where $O \leq \mathbb{Z}_2^2$. Assume that $G_{uv}^{[1]} = 1$ then $G_u = N.\mathrm{S}_5$, where $1 \neq N \trianglelefteq \mathbb{Z}_2 \times \mathrm{S}_3$. Then either $\mathbf{O}_2(G_u) = \mathbf{O}_2(N) \cong \mathbb{Z}_2$ and $N/\mathbf{O}_2(G_u)$ is isomorphic to a normal subgroup of S_3, or $\mathbf{O}_3(G_u) = \mathbf{O}_3(N) \cong \mathbb{Z}_3$ and $N/\mathbf{O}_2(G_u) \lesssim \mathbb{Z}_2^2$. Then one of (iii) and (iv) follows.

Case 2. Let $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{M}_{22}$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{PSL}(3,4)$ and $k = 11$. In this case, $s \geq 2$. By Theorem 4.3 or 4.4, $G_{uv}^{[1]} = 1$, and so $(G_u^{[1]})^{\Gamma(v)} \cong G_u^{[1]}$, and $s \leq 3$. Then $G_u^{[1]}$ is isomorphic to a normal subgroup of $(G_u^{\Gamma(u)})_v$. If $G_u^{\Gamma(u)} \cong \mathrm{M}_{22}$ then $G_u^{[1]} = 1$ or $G_u^{[1]} \cong \mathrm{PSL}(3,4)$, and so $G_u \cong \mathrm{M}_{22}$ or $\mathrm{PSL}(3,4) \times \mathrm{M}_{22}$. If $G_u^{\Gamma(u)} \cong \mathrm{M}_{22}.2$ then $G_u^{[1]} = 1$ or $\mathrm{PSL}(3,4) \lesssim G_u^{[1]} \lesssim \mathrm{PSL}(3,4).2$, and so $G_u \cong \mathrm{M}_{22}$ or $(\mathrm{PSL}(3,4) \times \mathrm{M}_{22}).O$, where $|O| = 2$ or 4. Thus part (2) of this theorem follows.

Case 3. Let $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_{2k}$, $\mathrm{soc}(G_u^{\Gamma(u)})_v \cong \mathrm{A}_{2k-1}$. Then a similar argument as in Case 2 yields part (4) of this theorem. □

Note that $|V| = |G : G_u| = |G|/|G_u|$. Considering the orders of the stabilizers G_u listed in Theorem 4.5, we have the following simple facts.

Corollary 4.6. *Let $\Gamma = (V,E)$ be a connected graph of valency* 10 *and square-free order, $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is G-locally-primitive arc-transitive but not $(G,2)$-arc-transitive. Then neither 5^3 nor r^2 is a divisor of $|G|$, where r is a prime no less than 7; moreover, for $u \in V$, one of the following holds:*

(i) *$G_u \cong \mathrm{A}_5$ or S_5, and neither 2^5 nor 3^3 is a divisor of $|G|$;*

(ii) $\mathbf{O}_3(G_u) \neq 1$ *and* $G_u = \mathbf{O}_3(G_u).O.\mathrm{A}_5$ *for some* $O \leq \mathbb{Z}_2$, *and* 2^5 *is not a divisor of* $|G|$;

(iii) $\mathbf{O}_3(G_u) \neq 1$ *and* $G_u = \mathbf{O}_3(G_u).O.\mathrm{S}_5$ *for some* $O \leq \mathbb{Z}_2^2$, *and* 2^7 *is not a divisor of* $|G|$;

(iv) $\mathbf{O}_2(G_u) \neq 1$ *and* $G_u = \mathbf{O}_2(G_u).O.\mathrm{S}_5$ *for some* $O \trianglelefteq \mathrm{S}_3$, *and* 3^4 *is not a divisor of* $|G|$.

Corollary 4.7. *Let* $\Gamma = (V, E)$ *be a connected graph of valency* 10 *and square-free order,* $\{u, v\} \in E$ *and* $G \leq \mathsf{Aut}\Gamma$. *Assume that* Γ *is* (G, s)*-transitive, where* $s \geq 2$. *Then one of the following holds.*

(1) $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{PSL}(2, 9)$, *and one of the following holds:*
  - (i) $G_{uv}^{[1]} = 1$ *and* $s \in \{2, 3\}$, *where* $v \in \Gamma(u)$;
  - (ii) $s = 4$ *and* $G_u = [3^4]{:}(\mathbb{Z}_8.\mathrm{PGL}(2, 3^2)).\mathbb{Z}_b$, *where* $b \leq 2$;
  - (iii) $s = 7$ *and* $G_u = [3^{10}]{:}\mathrm{GL}(2, 3^2).\mathbb{Z}_b$, *where* $b \leq 2$.

(2) $s \in \{2, 3\}$ *and* $G_u \cong \mathrm{A}_{10}$, S_{10} *or* $(\mathrm{A}_9 \times \mathrm{A}_{10}).O$, *where* $O \leq \mathbb{Z}_2^2$.

5. Graphs arising from almost simple groups

Let $\Gamma = (V, E)$ be a connected graph of valency 10 and square-free order, and let $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is G-locally-primitive arc-transitive, and that G is an almost simple group with socle $\mathrm{soc}(G) = T$. Let $u \in V$. By Corollaries 4.6 and 4.7, $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_5$, $\mathrm{PSL}(2, 9)$ or A_{10}.

Since $|V|$ is square-free, T is not semiregular on V. Then, by Lemma 2.4, $T_u^{\Gamma(u)}$ is transitive on $\Gamma(u)$. In particular, T has at most two orbits on V. Since T is normal in G, all orbits of T have the same length which is a divisor of $|V|$. Thus $|T : T_u| = |V|$ or $\frac{|V|}{2}$, and so $|T : T_u|$ is square-free. Moreover, $|T : T_u| \geq 11$ as Γ has valency 10. Since $T_u \trianglelefteq G_u$, we have $T_u^{\Gamma(u)} \trianglelefteq G_u^{\Gamma(u)}$. It follows that $\mathrm{soc}(T_u^{\Gamma(u)}) = \mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_5$, $\mathrm{PSL}(2, 9)$ or A_{10}. In particular, T_u is primitive on $\Gamma(u)$, and if T is transitive on V then Γ is T-locally-primitive arc-transitive.

5.1. Assume that $\mathrm{soc}(G_u^{\Gamma(u)}) = \mathrm{soc}(T_u^{\Gamma(u)}) \cong \mathrm{A}_5$. Then, by Corollary 4.6, the order of T satisfies the following conditions: neither 5^3 nor p^2 is a divisor of $|T|$, where p is a prime no less that 7; and one of 2^7 and 3^4 is not a divisor of $|T|$.

Lemma 5.1. *Up to isomorphism,* T *is one of the following simple groups:*

$\mathrm{PSL}(2, r)$, *where* r *is a prime with* $r \equiv \pm 1 \pmod 5$;
$\mathrm{PSL}(2, 25)$, $\mathrm{PSL}(3, 4)$, $\mathrm{PSp}(4, 4)$, $\mathrm{G}_2(4)$, A_7, M_{22}, M_{23}, M_{24} *and* J_1.

Proof. Noting that A_{10} has order divisible by both 2^7 and 3^4, if T is an alternating simple group, then T is one of A_7, A_8 and A_9. For the 26 sporadic simple groups, checking their orders, we have that T is one of M_{11}, M_{12}, M_{22}, M_{23}, M_{24} and J_1. Recall that T has a subgroup T_u, which has square-free index in T and possesses a composition factor A_5. Then, employing the Atlas [5], T is one of A_7, M_{22}, M_{23}, M_{24} and J_1.

Now let T be a simple group of Lie type over the finite field $\mathbb{F}_q$ of order q, where $q = r^f$ for some prime r. Consider the orders of simples group of Lie type, refer to [12, pp. 170, Tables 5.1 A and B]. We conclude that either $r \leq 5$, or $T = \mathrm{PSL}(2, r)$

with $r \equiv \pm 1 \pmod 5$. If $r = 5$ then the only possibility is that $T = \mathrm{PSL}(2, 25)$. Thus we next assume that $r \in \{2, 3\}$.

Case 1. Assume first that T has Lie rank 1. Note that the group ${}^2\mathrm{B}_2(q)$ is excluded as its order is not divisible by 3, and ${}^2\mathrm{G}_2(q)$ is excluded as it has order not divisible by 5. Thus, up to isomorphism, $T = \mathrm{PSL}(2, q)$ or $\mathrm{PSU}(3, q)$.

Let $T = \mathrm{PSL}(2, q)$. Since T is simple, $f \geq 2$. By [10, II. 8.27], we conclude that $T_u \cong \mathrm{A}_5$ or S_5. Recalling that $|T : T_u| \geq 11$, we have $f \geq 4$. This yields that $|T : T_u|$ is not square-free, a contradiction.

Let $T = \mathrm{PSU}(3, q)$. Then $q > 2$, and $|T| = \frac{1}{(3,q+1)} q^3(q^2 - 1)(q^3 + 1)$. If $r^{2f} - 1$ has no primitive prime divisor then $(r, f) = (3, 1)$ or $(2, 3)$. The group $\mathrm{PSU}(3, 3)$ is excluded as its order is not divisible by 5, and the group $\mathrm{PSU}(3, 8)$ is excluded as its order is divisible by $2^7 \cdot 3^4$. Let t be a primitive prime divisor of $r^{2f} - 1$. Then t is also a divisor of $r^f + 1$. Since $|T|$ is divisible by $\frac{(r^f+1)^2}{(3,r^f+1)}$ but not by p^2 for some prime $p \geq 7$, we know that $t \in \{3, 5\}$. This yields that $(r, f) = (2, 2)$ or $(3, 2)$. By the Atlas [5], both $\mathrm{PSU}(3, 4)$ and $\mathrm{PSU}(3, 9)$ have no insoluble subgroup of square-free index, a contradiction.

Case 2. Assume that T has Lie rank at least 2. Then either $T = \mathrm{PSL}(3, q)$, or $|T|$ is divisible by $\frac{(q^2-1)^2}{2}$.

Suppose that $|T|$ is divisible by $\frac{(q^2-1)^2}{2} = \frac{(r^{2f}-1)^2}{2}$. Recalling that $|T|$ has no divisor a square of some prime no less that 7, by Corollary 2.9, (r, f) is one of $(2, 1)$, $(2, 2)$, $(3, 1)$ and $(3, 2)$. Since $|T|$ is not divisible by 5^3 or $2^7 \cdot 3^4$, we conclude that T is one of $\mathrm{PSL}(4, 2)$, $\mathrm{PSL}(5, 2)$, $\mathrm{PSU}(4, 2)$, $\mathrm{PSp}(4, 3)$, $\mathrm{PSp}(4, 4)$, $\mathrm{G}_2(3)$ and $\mathrm{G}_2(4)$. Employing the Atlas [5], among these groups, only $\mathrm{G}_2(4)$ has a subgroup group which has square-free index and a composition factor A_5.

Let $T = \mathrm{PSL}(3, q)$. Then $|T|$ is divisible by $\frac{(r^f-1)^2}{(3,r^f-1)}$. Since $|T|$ is not divisible by p^2 for prime $p \geq 7$, by Corollary 2.9, $q = 2$, 4, 16, 3, 9 or 3^4. The groups $\mathrm{PSL}(3, 2)$ and $\mathrm{PSL}(3, 3)$ are excluded as their orders are not divisible by 5, and the groups $\mathrm{PSL}(3, 9)$ and $\mathrm{PSL}(3, 3^4)$ are excluded as their orders are divisible by $2^7 \cdot 3^4$. Suppose that $T \cong \mathrm{PSL}(3, 16)$. Then, by [2, pp. 378, Table 8.3], T_u is contained in the stabilizer of some projective point (or line) in $\mathrm{PSL}(3, 16)$. It follows that $\mathrm{PSL}(2, 16)$ has an insoluble subgroup of square-free index, which is impossible. Thus we have $T \cong \mathrm{PSL}(3, 4)$. Then the lemma follows. $\square$

Lemma 5.2. $T \neq \mathrm{PSL}(3, 4)$.

Proof. Suppose that $T = \mathrm{PSL}(3, 4)$. Recall that $T_u^{\Gamma(u)} \cong \mathrm{A}_5$ or S_5. Since $|T : T_u|$ is square-free, checking the subgroups of T, we know that $T_u = \mathbb{Z}_2^4{:}\mathrm{A}_5$; in particular, $\mathbf{O}_2(T_u)$ has order 2^4. Noting that $\mathbf{O}_2(T_u) \leq \mathbf{O}_2(G_u)$, by Corollary 4.6, $G_u^{\Gamma(u)} \cong \mathrm{S}_5$. Checking the subgroup of G, the only possibility is that $|G : G_u| = 21$. This yields that G is 2-transitive on V, and hence $\Gamma \cong \mathsf{K}_{21}$, a contradiction. $\square$

Lemma 5.3. $T \neq \mathrm{PSp}(4, 4)$.

Proof. Suppose that $T = \mathrm{PSp}(4, 4)$. By a similar argument as in the proof of Lemma 5.2, we have $|\mathbf{O}_2(T_u)| \geq 2^5$, $T_u^{\Gamma(u)} \cong \mathrm{A}_5$ and $G_u^{\Gamma(u)} \cong \mathrm{S}_5$; in particular, $G \neq T$. Then $G = T.\mathbb{Z}_2$ or $T.\mathbb{Z}_4$. Recalling T_u is primitive on $\Gamma(u)$, by Corollary 4.6, T is intransitive on V, and thus T has two orbits of equal length on V. In particular, $|T : T_u|$ is odd. Then $T_u \cong \mathbb{Z}_2^6{:}\mathrm{A}_5$ or $\mathbb{Z}_2^6{:}(\mathbb{Z}_3 \times \mathrm{A}_5)$. Thus $\mathbb{Z}_2^6 \cong \mathbf{O}_2(T_u) \le \mathbf{O}_2(G_u)$, this implies that $G_u \cong \mathbb{Z}_2^6{:}\mathrm{A}_5{:}\mathbb{Z}_2$ or $\mathbb{Z}_2^6{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$; in particular, $G_u \le T.\mathbb{Z}_2 := X$. If $G = X$, then $|G : G_u| = |T : T_u|$, a contradiction. Then we have $G = T.\mathbb{Z}_4$, both T and X has two orbits on V, say U_1 and U_2, which have length 85 or 255.

Let H_1 and H_2 be maximal subgroups of X with $X_u \le H_1 \cong \mathbb{Z}_2^6{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$ and $X_v \le H_2 \cong \mathbb{Z}_2^6{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$, where $v \in \mathit{\Gamma}(u)$. Then $B_1 := u^{H_1}$ and $B_2 := v^{H_2}$ are blocks of X on U_1 and U_2, respectively. Set $\mathcal{B}_i = \{B_i^x \mid x \in X\}$, $i = 1, 2$. Then $|\mathcal{B}_1| = |\mathcal{B}_2| = 85$, and X acts primitively (and faithfully) on both $\mathcal{B}_1$ and $\mathcal{B}_2$. Moreover, each of these two actions is equivalent to the action of X on the point set $\mathcal{P}$ or the line set $\mathcal{L}$ of the symplectic generalized quadrangle of order $(4, 4)$.

By [10, II.9.15], for $\mathbf{p} \in \mathcal{P}$, the stabilizer $X_{\mathbf{p}}$ has three orbits on $\mathcal{P}$, which have sizes 1, 20 and 64 respectively. Note that G has a element interchanging $\mathcal{P}$ and $\mathcal{L}$. It follows that, for $L \in \mathcal{L}$, the stabilizer X_L also has three orbits on $\mathcal{L}$ with sizes 1, 20 and 64 respectively. Let $\mathbf{p} \in \mathcal{P}$ and $\mathcal{L}_1$ be the set of lines containing $\mathbf{p}$. Then $\mathcal{L}_1$ is a $X_{\mathbf{p}}$-orbit of size 5, and the points on these 5 lines form two $X_{\mathbf{p}}$-orbits (on $\mathcal{P}$) of lengths 1 and 20. Thus the third orbits of $X_{\mathbf{p}}$ on $\mathcal{P}$ consists of the points which are not contained in $\mathbf{p}^{\perp}$ and lie on the remain 80 lines. By [10, II.9.11], we conclude that every line $L \in \mathcal{L} \setminus \mathcal{L}_1$ is not contained in $\mathbf{p}^{\perp}$. It follows that $X_{\mathbf{p}}$ is transitive on $\mathcal{L} \setminus \mathcal{L}_1$, and thus $X_{\mathbf{p}}$ has two orbits on $\mathcal{L}$.

Assume that actions of X on $\mathcal{B}_1$ and on $\mathcal{B}_2$ are equivalent. Then $X_{B_1} = H_1$ has three orbits on $\mathcal{B}_2$, say Δ_1, Δ_2 and Δ_3, which have sizes 1, 20 and 64 respectively. Since $|H_1 : X_u| = 1$ or 3, by Lemma 2.1, X_u is transitive on every Δ_i. Note that $G_u = X_u$, and $\mathit{\Gamma}(u)$ is a G_u-orbits. It follows that there is some i such that $\mathit{\Gamma}(u) \subseteq \cup_{B \in \Delta_i} B$ and $B \cap \mathit{\Gamma}(u) \ne \emptyset$ for each $B \in \Delta_i$. Since $|B| = 1$ or 3 for $B \in \mathcal{B}$, we have $|\mathit{\Gamma}(u)| \le 3$ or $|\mathit{\Gamma}(u)| \ge 20$, a contradiction.

Now, with out loss of generality, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{P}$ and $\mathcal{L}$, respectively. Then X_{B_1} has two orbits on $\mathcal{B}_2$, say Λ_1 and Λ_2, which have sizes 5 and 80, respectively. Again by Lemma 2.1, Λ_1 and Λ_2 are X_u-orbits. Since $\mathit{\Gamma}(u)$ is an X_u-orbit of length 10, we have $\mathit{\Gamma}(u) \subseteq \cup_{B \in \Lambda_1} B$. It follows that $|\mathit{\Gamma}(u) \cap B| = 2$ for each $B \in \Lambda_1$. Thus a $G_u = X_u$ has a nontrivial block $\mathit{\Gamma}(u) \cap B$, which contradicts that $G_u^{\mathit{\Gamma}(u)}$ is primitive. $\square$

We next exclude the groups $\mathrm{G}_2(4)$, M_{22}, M_{23} and M_{24}. Note that each of these groups has order divisible by 2^7. Thus, if T is one of these groups then (iv) of Corollary 4.6 holds; in particular, $G_u^{\mathit{\Gamma}(u)} \cong \mathrm{S}_5$. Recall that $\mathit{\Gamma}(u)$ is a T_u-orbit, where $u \in V$.

Lemma 5.4. *$T \ne \mathrm{G}_2(4)$.*

Proof. Let $T = \mathrm{G}_2(4)$. Then, checking the maximal subgroups of T and G, we know that $G = T.\mathbb{Z}_2$ and, letting M be a maximal subgroup of T with $T_u \le M$, we have $|T : M| = 1365$. In particular, T has two orbits on V, and each T-orbit has length $1365|M : T_u|$. Noting that the prime divisors of $|M|$ is 2, 3 and 5, it follows that $T_u = M$. Let U and W be the T-orbits on V. Then T acts primitively and equivalently on U and W. Then T_u has four orbits on W with sizes 1, 20, 320 and 1024, refer to [25]. This contradicts that $|\mathit{\Gamma}(u)| = 10$. $\square$

Lemma 5.5. *$T \ne \mathrm{M}_{22}$, M_{23}, M_{24}.*

Proof. Let $T = \mathrm{M}_{22}$. If $T_u^{\Gamma(u)} \cong \mathrm{A}_5$ then T is transitive on V, and so Γ is T-arc-transitive, which contradicts (iv) of Corollary 4.6. Thus $T_u^{\Gamma(u)} \cong \mathrm{S}_5$. It follows that $T_u \cong \mathbb{Z}_2^4{:}\mathrm{S}_5$. In particular, T is primitive on each of its orbits on V. Suppose that that T is transitive on V. Then T_u has exactly 4 orbits on V, which have sizes 1, 30, 60 and 140, respectively, refer to [25]. This contradicts that $|\Gamma(u)| = 10$. Suppose that T has two orbits, say U and W, on V. Then the actions of T on both U and W are primitive and equivalent. This implies that $|\Gamma(u)| > 10$, again a contradiction.

Let $T = \mathrm{M}_{23}$. Then $G = T$. Take a maximal subgroup M of G with $G_u \leq M$. Then $M \cong \mathrm{M}_{22}$, $\mathbb{Z}_2^4{:}\mathrm{A}_7$ or $\mathbb{Z}_2^4{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$. If $G_u = M$ then $G_u \cong \mathbb{Z}_2^4{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$ and $|G : G_u| = 1771$; however, in this case, G_u has no orbit of size 10 (refer to [25]), a contradiction. Checking the subgroups of M, we have $G_u \cong \mathbb{Z}_2^4{:}\mathrm{S}_5$, and $G_{uv} \cong \mathbb{Z}_2^4{:}\mathrm{D}_{12}$ for $v \in \Gamma(u)$. By the information given for M_{23} in the Atlas [5], we may let $M \cong \mathbb{Z}_2^4{:}\mathrm{A}_7$ or $\mathbb{Z}_2^4{:}(\mathbb{Z}_3 \times \mathrm{A}_5){:}\mathbb{Z}_2$. Confirmed by GAP, we have $\mathbf{N}_G(G_{uv}) = \mathbf{N}_M(G_{uv})$. Since Γ is connected, $G = \langle G_u, \mathbf{N}_G(G_{uv})\rangle \leq M$, a contradiction.

Finally, let $G = T = \mathrm{M}_{24}$. Take a maximal subgroup M of G with $G_u \leq M$. Then $|G : M|$ is square-free, and so $M \cong \mathbb{Z}_2^4{:}\mathrm{A}_8$ or $\mathbb{Z}_2^6{:}\mathbb{Z}_3\mathrm{S}_6$. Checking the subgroups of A_8, it has no subgroup with a composition factor A_5 and square-free index. Thus $M \cong \mathbb{Z}_2^6{:}\mathbb{Z}_3\mathrm{S}_6$, and hence $G_u = \mathbb{Z}_2^6{:}(\mathbb{Z}_3\mathrm{S}_5)$, and $G_{uv} \cong \mathbb{Z}_2^6{:}(\mathbb{Z}_3\mathrm{D}_{12})$. Confirmed by GAP, we get a similar argument as above. □

Theorem 5.6. *Let $\Gamma = (V, E)$ be a connected graph of valency* 10 *and square-free order, and let $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is G-locally-primitive arc-transitive, and G is almost simple. If Γ is not $(G, 2)$-arc-transitive then one of the following holds.*

(1) $G = \mathrm{PSL}(2, r)$ *for a prime r with $r \equiv \pm 1 \pmod 5$ and $r \equiv \pm 1 \pmod{12}$, and Γ is isomorphic to the graph constructed in* Example 3.4*; or*
$G = \mathrm{PGL}(2, r)$ *for a prime r with $r \equiv \pm 1 \pmod 5$ and $r \not\equiv \pm 1 \pmod{12}$, and Γ is bipartite and isomorphic to the graph constructed in* Example 3.5.

(2) $G = \mathrm{PSL}(2, 25)$ *or* $\mathrm{P\Sigma L}(2, 25)$*, and Γ is isomorphic to the graph given in* Example 3.7.

(3) $G = \mathrm{A}_7$ *or* S_7*, and Γ is the complement graph of $L(\mathsf{K}_7)$, where $L(\mathsf{K}_7)$ is the line graph of the complete graph K_7 of order* 7.

(4) $G = \mathrm{J}_1$*, and Γ is isomorphic to the graph in* Example 3.9.

Proof. Assume that Γ is not $(G, 2)$-arc-transitive. Then $\mathrm{soc}(G_u^{\Gamma(u)}) = \mathrm{soc}(T_u^{\Gamma(u)}) \cong \mathrm{A}_5$ and, by Lemmas 5.1-5.5, we may let T be one of $\mathrm{PSL}(2, r)$, $\mathrm{PSL}(2, 25)$, A_7 and J_1.

(1) Let $T = \mathrm{PSL}(2, r)$. Checking the subgroups of $\mathrm{PSL}(2, r)$ and $\mathrm{PGL}(2, r)$ (see [10, II. 8.27] and [4, Theorem 2]), we have $G_u = T_u \cong \mathrm{A}_5$, and so $G_{uv} = T_{uv} \cong \mathrm{S}_3$. Let $\epsilon = \pm 1$ be such that 3 is a divisor of $r + \epsilon$. Let Z be the subgroup of G_{uv} or order 3. Then $G_{uv} \leq \mathbf{N}_G(G_{uv}) \leq \mathbf{N}_G(Z) \leq M := \mathbf{N}_{\mathrm{PGL}(2,r)}(Z) \cong \mathrm{D}_{2(r+\epsilon)}$, and so $\mathbf{N}_G(G_{uv}) \leq \mathbf{N}_M(G_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. On the other hand, $G_{uv} \leq \mathbf{N}_T(G_{uv}) \leq N := \mathbf{N}_T(Z) \cong \mathrm{D}_{r+\epsilon}$, and so $\mathbf{N}_T(G_{uv}) = \mathbf{N}_N(G_{uv})$.

Assume that $r + \epsilon$ is divisible by 4. Then $\mathbf{N}_T(G_{uv}) = \mathbf{N}_N(G_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Thus we have $\mathbf{N}_G(G_{uv}) = \langle o\rangle \times T_{uv}$, where o is the involution lying the center of $\mathbf{N}_T(G_{uv})$. In particular, since Γ is connected, $G = \langle G_u, \mathbf{N}_G(G_{uv})\rangle \leq T$, and so $G = T$. Noting that $T_u x T_u = T_u o T_u$ for each $x \in \mathbf{N}_G(G_{uv}) \setminus G_{uv}$, by Lemma 3.2, Γ is isomorphic to the graph constructed in Example 3.4.

Assume that $r+\epsilon$ is not divisible by 4. Then $\mathbf{N}_T(G_{uv}) = \mathbf{N}_N(G_{uv}) = G_{uv}$. By Lemma 3.3, $G \neq T$, and so $G = \mathrm{PGL}(2,r)$ in this case. Then $\mathbf{N}_G(G_{uv}) = \mathbf{N}_M(G_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Write $\mathbf{N}_G(G_{uv}) = \langle \iota \rangle \times G_{uv}$. Then $\iota \in G \setminus T$, and Γ is isomorphic to the graph constructed in Example 3.5.

(2) Let $T = \mathrm{PSL}(2,25)$. Then $\mathrm{PSL}(2,25) \leq G \leq \mathrm{P\Gamma L}(2,25)$. Inspect the subgroups of G, refer to [5]. We know that $T_u \cong \mathrm{A}_5$ or S_5, and either $G_u = T_u$, or $G = \mathrm{P\Sigma L}(2,25)$ and G_u is isomorphic to one of $\mathbb{Z}_2 \times \mathrm{A}_5$, S_5 and $\mathbb{Z}_2 \times \mathrm{S}_5$.

Suppose that $T_u \cong \mathrm{A}_5$. Then $T_{uv} \cong \mathrm{S}_3$, and $|T : T_u| = 130$ is even. Recall that $|T : T_u| = |V|$ or $\frac{|V|}{2}$. Since $|V|$ is square-free, we have $|V| = |T : T_u|$, that is, T is transitive on V, and so Γ is T-arc-transitive. Since Γ is connected, by Lemma 3.2, there is $x \in \mathbf{N}_T(T_{uv})$ with $\langle x, T_u \rangle = T$. Let Z be the subgroup of T_{uv} or order 3. Then $T_{uv} \leq \mathbf{N}_T(T_{uv}) \leq M := \mathbf{N}_T(Z) \cong \mathrm{D}_{24}$, and so $\mathbf{N}_T(T_{uv}) = \mathbf{N}_M(T_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Take a maximal subgroup H of T with $T_u < H \cong \mathrm{S}_5$. Then $\mathbf{N}_H(T_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$, yielding $\mathbf{N}_T(T_{uv}) = \mathbf{N}_H(T_{uv}) \leq H$. Thus $T = \langle x, T_u \rangle \leq H$, which is impossible.

Let $G_u = T_u \cong \mathrm{S}_5$. Then $G_{uv} \cong \mathbb{Z}_2 \times \mathrm{S}_3$. Checking the subgroups of G, we have $\mathbf{N}_G(G_{uv}) = \mathbf{N}_T(G_{uv}) \cong \mathrm{D}_{24}$. Since Γ is connected, $G = \langle G_u, \mathbf{N}_G(G_{uv}) \rangle \leq T$, and so $G = T$. It follows that Γ is isomorphic to the graph given in Example 3.7.

Now let $G_u \neq T_u$. Then $G = \mathrm{P\Sigma L}(2,25)$ and $G_u \cong \mathbb{Z}_2 \times \mathrm{S}_5$, and so $|G : G_u| = |T : T_u| = 65$. Thus Γ is T-arc-transitive, and again Γ is isomorphic to the graph in Example 3.7.

(3) Let $T = \mathrm{A}_7$. By Corollary 4.6 and checking the subgroup of G, we know that one of the following cases occurs: $G_u = T_u \cong \mathrm{A}_5$ or S_5; $G_u \cong \mathrm{S}_5$, $T_u \cong \mathrm{A}_5$ and $G = \mathrm{S}_7$; $G_u \cong \mathbb{Z}_2 \times \mathrm{S}_5$, $T_u \cong \mathrm{S}_5$ and $G = \mathrm{S}_7$. These cases yield that the action of G acting on V is equivalent to that on the 2-subsets or ordered pairs of the set $\{1,2,3,4,5,6,7\}$.

If V is the set of ordered pairs of the set $\{1,2,3,4,5,6,7\}$ then it easily to show that $G_u^{\Gamma(u)}$ is not primitive, a contradiction. Thus we may let V be the set of 2-subsets of $\{1,2,3,4,5,6,7\}$. Then Γ is either the line graph $L(\mathsf{K}_7)$ of the complete graph K_7 or the complement graph of $L(\mathsf{K}_7)$. Recalling that $G_u^{\Gamma(u)}$ is primitive, we conclude that Γ is the complement graph $L(\mathsf{K}_7)$.

(4) Let $T = \mathrm{J}_1$. Then $G = T$ and $G_u \cong \mathrm{A}_5$ or $\mathbb{Z}_2 \times \mathrm{A}_5$. By Corollary 4.6, we have $G_u \cong \mathrm{A}_5$, and so $|V| = |G : G_u| = 2926 = 2 \cdot 7 \cdot 11 \cdot 19$. Let $v \in \Gamma(u)$. Then $G_{uv} \cong \mathrm{S}_3$.

Checking by GAP [8], we know that J_1 has exactly two conjugacy classes of subgroups isomorphic A_5, and one of them consists of the subgroups having normalizer isomorphic to $\mathbb{Z}_2 \times \mathrm{A}_5$, while the other one contains only self-normalized subgroups. Moreover, if $\mathbf{N}_G(G_u) \cong \mathbb{Z}_2 \times \mathrm{A}_5$ then $\mathbf{N}_G(G_{uv}) \cong \mathrm{D}_6 \times \mathrm{D}_{10}$, and if $\mathbf{N}_G(G_u) = G_u$ then $\mathbf{N}_G(G_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$.

Let $\mathbf{N}_G(G_u) = G_u$. Take a maximal subgroup M of G with $G_u < M \cong \mathrm{PSL}(2,11)$. Then we have $\mathbf{N}_M(G_{uv}) \cong \mathbb{Z}_2 \times \mathrm{S}_3$. This implies that $\mathbf{N}_G(G_{uv}) = \mathbf{N}_M(G_{uv})$, and so $\langle G_u, \mathbf{N}_G(G_{uv}) \rangle \leq M \neq T$, which contradicts the connectedness of Γ. Therefore, $\mathbf{N}_G(G_u) \cong \mathbb{Z}_2 \times \mathrm{A}_5$ and $\mathbf{N}_G(G_{uv}) \cong \mathrm{D}_6 \times \mathrm{D}_{10}$, and then Γ is isomorphic the graph given in Example 3.9. $\square$

5.2. Assume that $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{PSL}(2,9)$. Then, by Corollary 4.7, the order of T satisfies the following conditions: neither 5^3 nor p^2 is a divisor of $|T|$, where p is a prime no less that 7. Moreover, letting Γ be (G,s)-transitive, then either $G_{uv}^{[1]}=1$ for $v\in\Gamma(u)$, or one of the following occurs:

(a) $s=4$, $|T|$ is not divisible by 2^{10} or 3^8;
(b) $s=7$, $|T|$ is not divisible by 2^{10} or 3^{14}.

For the case where $G_{uv}^{[1]}=1$, by (4.1), Lemma 4.2 and Corollary 4.7, we have

(c) $s\in\{2,3\}$, $|T|$ is not divisible by 2^{11} or 3^6.

Lemma 5.7. *T is isomorphic to one of the following simple groups:*

A_{10}*,* A_{11} *and* M_n*, where* $n\in\{11,12,22,23,24\}$*;*
$\mathrm{PSL}(3,9)$*,* $\mathrm{PSL}(3,3^4)$*,* $\mathrm{PSU}(4,9)$*,* $\mathrm{PSp}(4,9)$ *and* $\mathrm{G}_2(9)$*.*

Proof. For the alternating simple groups and sporadic simple groups, checking their orders, we know that T is one of A_n (with $n\le 13$), M_n (with $n\in\{11,12,22,23,24\}$), J_2 and J_3. Note that $|T:T_u|$ is square-free and no less than 11, and that T_u has a composition factor $\mathrm{PSL}(2,9)$. Checking the insoluble subgroups of those groups, we conclude that T is one of A_{10}, A_{11} and M_n, where $n\in\{11,12,22,23,24\}$.

Now let T be a simple group of Lie type over $\mathbb{F}_q$, where $q=r^f$ for some prime r. Suppose that $T\cong\mathrm{PSL}(2,q)$. Checking the subgroups of $\mathrm{PSL}(2,q)$, we know that $r=3$, $T_u\cong\mathrm{PSL}(2,9)$ or $\mathrm{PGL}(2,9)$. In this case, $|T:T_u|$ is divisible by 9, a contradiction. Thus we assume further that $T\not\cong\mathrm{PSL}(2,q)$. Consider the orders of simple groups of Lie type. We conclude that $r\in\{2,3,5\}$. If $r=5$ then, since $|T|$ is not divisible by 5^3, we have $T\cong\mathrm{PSL}(2,25)$, a contradiction. Thus $r\in\{2,3\}$. By a similar argument as in the proof of Lemma 5.1, noting the limits on $|T|$, we conclude that T is isomorphic to one of the following groups: $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(3,3^4)$, $\mathrm{PSL}(4,2)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSL}(5,2)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(3,4)$, $\mathrm{PSU}(3,8)$, $\mathrm{PSU}(3,9)$, $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(4,3)$, $\mathrm{PSU}(5,2)$, $\mathrm{PSU}(5,3)$, $\mathrm{PSp}(4,3)$, $\mathrm{PSp}(4,4)$, $\mathrm{PSp}(4,9)$, $\mathrm{PSp}(6,2)$, $\mathrm{PSp}(6,3)$, $\Omega(7,3)$, $\mathrm{G}_2(3)$ and $\mathrm{G}_2(9)$. Recall that $|T:T_u|$ is square-free and T_u has a composition factor $\mathrm{PSL}(2,9)$. Employing the Atlas [5], we know that T is isomorphic to one of $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(3,3^4)$, $\mathrm{PSU}(4,9)$, $\mathrm{PSp}(4,9)$ and $\mathrm{G}_2(9)$. $\square$

Theorem 5.8. *Let $\Gamma=(V,E)$ be a connected graph of valency 10 and square-free order, and let $G\le\mathsf{Aut}\Gamma$. Assume that G is almost simple and $\mathrm{soc}(G_u^{\Gamma(u)})\cong\mathrm{PSL}(2,9)$ for $u\in V$. Then either*

(1) *$G=\mathrm{M}_{11}$ and $\Gamma\cong\mathsf{K}_{11}$, the complete graph of order 11; or*
(2) *$G=\mathrm{PSL}(3,9).\mathbb{Z}_2$ or $\mathrm{P\Gamma L}(3,9).\mathbb{Z}_2$, and Γ is the point-line incidence graph of the projective plane $\mathrm{PG}(2,9)$.*

Proof. Let $\mathrm{soc}(G)=T$. Then T is know by Lemma 5.7.

Case 1. Assume that $T=\mathrm{A}_{10}$, A_{11} or M_n for $n\in\{11,12,22,23,24\}$. Then $|G|$ is not divisible by 3^6. By Corollary 4.7, $G_{uv}^{[1]}=1$, and so $(G_u^{[1]})^{\Gamma(v)}\cong G_u^{[1]}$. By Lemma 4.2 and checking the subgroups of $G_u^{\Gamma(u)}$ in the Atlas [5], either $G_u^{[1]}=1$ or $\mathbf{O}_3(G_u^{[1]})\cong\mathbb{Z}_3^2$. The latter case yields that $|G|$ is divisible by 3^4.

Suppose that $T=\mathrm{A}_{10}$. If G_u is faithful on $\Gamma(u)$ then $\mathrm{PSL}(2,9)\lesssim G_u\lesssim\mathrm{P\Gamma L}(2,9)$; in this case, $|G:G_u|$ has a divisor 4, a contradiction. This implies that $\mathbf{O}_3(G_u^{[1]})\cong\mathbb{Z}_3^2$, and so $|G:G_u|$ is coprime to 3. Let M be a maximal subgroup of G with $G_u\le M$. Then both $|G:M|$ and $|M:G_u|$ are square-free and coprime to 3. Checking the maximal subgroups of G, we have $M \cong \mathrm{S}_9$ or A_9. Thus A_9 or S_9 contains a subgroup having square-free index and a composition factor $\mathrm{PSL}(2,9)$, which is impossible by checking the subgroups of A_9 and S_9. Similarly, the group A_{11} is excluded.

Suppose that $T = \mathrm{M}_{11}$, M_{12}, M_{22}, M_{23} or M_{24}. Then $|G|$ is not divisible by 3^4. It follows that $G_u^{[1]} = 1$, and so $\mathrm{PSL}(2,9) \lesssim G_u \lesssim \mathrm{P\Gamma L}(2,9)$. Since $|G : G_u|$ is square-free, we know that $|G|$ is not divisible by 2^7. This yields that $T = \mathrm{M}_{11}$ or M_{12}. Let $T = \mathrm{M}_{12}$. Checking the subgroups of M_{12} and $\mathrm{M}_{12}.\mathbb{Z}_2$, we know that $G = \mathrm{M}_{12}$ and G_u is maximal in G with index 66. Thus G is a primitive permutation group of rank 3 with subdegrees 1, 20 and 45, refer to [25]. This contradicts the fact that $|\Gamma(u)| = 10$. Therefore, $G = T = \mathrm{M}_{11}$, and so $G_u \cong \mathrm{M}_{10}$ or A_6. If $G_u \cong \mathrm{A}_6$, then both G_u and $\mathbf{N}_G(G_{uv})$ are contained in a same maximal subgroup isomorphic to M_{10}, which is contradicts Lemma 3.2. (Confirmed by GAP!) Thus $G_u \cong \mathrm{M}_{10}$ and $\Gamma \cong \mathsf{K}_{11}$.

Case 2. Let $T = \mathrm{PSL}(3,9)$, $\mathrm{PSL}(3,3^4)$, $\mathrm{PSU}(4,9)$, $\mathrm{PSp}(4,9)$ or $\mathrm{G}_2(9)$, and let Γ be (G,s)-transitive. Then $s \geq 2$.

(1) Assume that $T = \mathrm{PSL}(3,9)$. Then $|G|$ is divisible by 3^6 but not by 3^7, by Corollary 4.7, we have $s = 4$ and $|G_u| = 2^7 \cdot 3^6 \cdot 5$ or $2^8 \cdot 3^6 \cdot 5$. Recall that T has at most two orbits on V.

Suppose that T is transitive on V. Recalling that $\mathrm{soc}(T_u^{\Gamma(u)}) \cong \mathrm{PSL}(2,9)$, we know that Γ is $(T,2)$-arc-transitive. Since $|T|$ is divisible by 3^6, by Corollary 4.7, we have $|T_u| = 2^7 \cdot 3^6 \cdot 5$. Checking the subgroups of $\mathrm{PSL}(3,9)$, we get $T_u \cong \mathbb{Z}_3^4{:}\mathrm{GL}(2,9)$. Thus the action of T on V is equivalent to that on the points or the lines of the projective plane $\mathrm{PG}(2,9)$. Then T is 2-transitive on V, and so $\Gamma \cong \mathsf{K}_{91}$, a contradiction.

Now let T have two orbits on V, say U and W. Then $|G : G_U| = 2$, $|U| = |W| = |T : T_u|$ is odd, and $G_U = TG_u$ for $u \in U$. Note that $|G : T|$ is a divisor of 4. Since $|G : T| = |G : TG_u||TG_u : T| = |G : TG_u||G_u : T_u|$, we have $|G_u : T_u| \leq 2$, and so $|T_u| = 2^6 \cdot 3^6 \cdot 5$ or $2^7 \cdot 3^6 \cdot 5$. It implies that $|T_u| = 2^7 \cdot 3^6 \cdot 5$ as $|T : T_u|$ is odd. Checking the subgroups of T and G, we conclude that $T_u \cong \mathbb{Z}_3^4{:}\mathrm{GL}(2,9)$, $|U| = |W| = 91$, and $G = \mathrm{PSL}(3,9).\mathbb{Z}_2$ or $\mathrm{P\Gamma L}(3,9).\mathbb{Z}_2$. This implies that the actions of T on U and W are equivalent to the actions on the point set and the line set of $\mathrm{PG}(2,9)$, respectively. It is easily shown that the stabilizer of a line of $\mathrm{PG}(2,9)$ has two orbits on the point set, which have length 10 and 81, respectively. It follows that Γ is the point-line incidence graph of $\mathrm{PG}(2,9)$.

(2) Assume that T is one $\mathrm{PSL}(3,3^4)$, $\mathrm{PSU}(4,9)$, $\mathrm{PSp}(4,9)$ and $\mathrm{G}_2(9)$. Noting that $|G|$ is divisible by 3^8, by Corollary 4.7, we have $s = 7$ and $G_u = [3^{10}]{:}\mathrm{GL}(2,9).\mathbb{Z}_b$, where $b \leq 2$. In particular, $|G|$ is divisible by 3^{12}, and so the group $\mathrm{PSp}(4,9)$ is excluded. Let M be a maximal subgroup of T with $T_u \leq M$. Since $|T : T_u|$ is square-free, both $|M : T_u|$ and $|T : M|$ are square-free.

Suppose that $T = \mathrm{PSL}(3,3^4)$. Check the maximal subgroups of T, refer to [2, Tables 8.3 and 8.4], we conclude that $M \cong \mathbb{Z}_3^8{:}\mathrm{GL}(2,3^4)$. Let N be the maximal soluble normal subgroup of M. Then $M/N \cong \mathrm{PGL}(2,3^4)$. Since $|M : T_u|$ is square-free, $|M/N : (T_uN/N)|$ is square-free. Moreover, T_uN/N has a composition factor $\mathrm{PSL}(2,9)$ as $T_uN/N \cong T_u/(T_u \cap N)$. Thus $\mathrm{PGL}(2,3^4)$ has a subgroup of square-free index, which has a composition factor $\mathrm{PSL}(2,9)$. This is impossible by checking the subgroups of $\mathrm{PGL}(2,3^4)$, refer to [4].

For $T=\mathrm{PSU}(4,9)$, checking the maximal subgroups of T (refer to [2, Tables 8.10 and 8.11]), M is the stabilizer of some totally singular subspace of dimension 1. In particular, $|T:M|=(9^3+1)(9^2+1)$, which is not square-free, a contradiction.

Now let $T=\mathrm{G}_2(9)$. Check the maximal subgroups of $\mathrm{G}_2(9)$, refer to [11]. We have $M\cong[3^{10}]{:}\mathrm{GL}(2,9)$, and so $|T:M|=\frac{9^6-1}{9-1}=2\cdot5\cdot7\cdot13\cdot73$. Then $|T:T_u|$ is even, and hence T is transitive on V. It implies that Γ is not a bipartite group. Noting that $|G:T|$ is a divisor of 4, every Sylow 3-subgroup of G has order 3^{12}. This implies that $|V|=|G:G_u|$ is coprime to 3. Note that $|V|=|T:T_u|=|T:M||M:T_u|$. Since $|V|$ is square-free and each prime divisor of $|M|$ is less than 7, we conclude that $|M:T_u|=1$, that is, $M=T_u$. Thus T and hence G is primitive on V; however, by [14], there is no vertex-primitive 7-transitive graph, a contradiction. $\square$

5.3. Assume that $\mathrm{soc}(G_u^{\Gamma(u)})\cong\mathrm{A}_{10}$. Then $G_u\cong\mathrm{A}_{10}$, S_{10} or $(\mathrm{A}_9\times\mathrm{A}_{10}).O$, where $O\leq\mathbb{Z}_2^2$. Since G/T is soluble, $\mathrm{soc}(G_u)$ is contained in T. Since $|T:T_u|$ is square-free, $|T|$ is not divisible by one of 2^{17}, 3^{10}, 5^5, 7^4 and p^2, where p is a prime no less than 11. In particular, $|T|$ is not divisible by r^{17} for an arbitrary prime r.

Note that either $|T_u|$ is divisible by $2^7\cdot3^4\cdot5^2$ but not by 2^9, or $|T_u|$ is divisible by $2^{13}\cdot3^8\cdot5^3\cdot7^2$ but not by 2^{16}. Checking the orders of the sporadic simple groups, we know that T is not a sporadic simple group. For the alternating simple groups, we have $T=\mathrm{A}_n$ with $11\leq n\leq19$.

Now let T be a simple group of Lie type over $\mathbb{F}_q$, where $q=r^f$ for some prime r. Recall that $|T|$ is not divisible by r^{17}. This excludes most of the exceptional simple groups of Lie type except for $\mathrm{G}_2(q)$, ${}^2\mathrm{B}_2(q)$, ${}^2\mathrm{G}_2(q)$ and ${}^3\mathrm{D}_4(q)$. Checking the subgroups of these exceptions (refer to [24, Table 4.1, Theorems 4.1-4.3]), none of them has a subgroup isomorphic A_{10}. Thus T is one of the classical simple groups of Lie type. By [12, Proposition 5.3.7], T has dimension no less than 8. It follows that T is one of $\mathrm{PSp}(8,r)$, $\Omega(9,r)$ and $\mathrm{P}\Omega^{\pm}(8,r)$. Noting that $|T|$ is not divisible by p^{10} for odd prime p, we have $r=2$, and so $T=\mathrm{PSp}(8,2)$, $\mathrm{P}\Omega^-(8,2)$ or $\mathrm{P}\Omega^+(8,2)$. These three groups have orders divisible by 2^{12} but not by 3^8. Thus $|T_u|$ is divisible by 2^{11} but not by 3^8, which is impossible. Then we have the following result.

Theorem 5.9. *Let $\Gamma=(V,E)$ be a connected graph of valency 10 and square-free order, and let $G\leq\mathsf{Aut}\Gamma$. Assume that G is almost simple and $\mathrm{soc}(G_u^{\Gamma(u)})\cong\mathrm{A}_{10}$ for $u\in V$. Then one of the following holds:*

(1) *$G=\mathrm{A}_{11}$ or S_{11}, and $\Gamma\cong\mathsf{K}_{11}$;*
(2) *$G=\mathrm{S}_{11}$, and Γ is isomorphic to the standard double cover of K_{11};*
(3) *$G=\mathrm{A}_{19}$ or S_{19}, and $\Gamma\cong\mathrm{O}_{10}$, the Odd graph of valency 10.*

Proof. By the foregoing argument, we have $T=\mathrm{A}_n$ with $11\leq n\leq19$.

Assume first that $G_u\cong\mathrm{A}_{10}$ or S_{10}. Then $\mathrm{A}_{10}\lesssim T_u\lesssim\mathrm{S}_{10}$. Since $|T:T_u|$ is square-free, either $T=\mathrm{A}_{11}$ and $T_u\cong\mathrm{A}_{10}$, or $T=\mathrm{A}_{12}$ and $T_u\cong\mathrm{S}_{10}$. Suppose that $T_u\cong\mathrm{S}_{10}$. Then $|T:T_u|$ is even, and so T is transitive on V, which implies that Γ is T-arc-transitive. Moreover, T_u is in fact the stabilizer of some 2-subset or 10-subset in the natural action of T acting on a 12-set, refer to [20]. It follows that T_u has exactly three orbits on V, which have sizes 1, 20 and 45 respectively. This implies that Γ is not of valency 10, a contradiction. Thus we have $T=\mathrm{A}_{11}$ and $T_u\cong\mathrm{A}_{10}$.

If T is transitive on V, then $\Gamma \cong \mathsf{K}_{10}$. Assume that T is intransitive on V. Then $G = \mathrm{S}_{11}$, and T is 2-transitive on each of it orbits. This implies that Γ is isomorphic to the standard double cover of K_{11}.

Suppose that $\mathrm{soc}(G_u) \cong \mathrm{A}_9 \times \mathrm{A}_{10}$. Recalling that $\mathrm{soc}(G_u) \leq T$, we have $T = \mathrm{A}_{19}$, and $T_u \cong \mathrm{A}_9 \times \mathrm{A}_{10}$ or $(\mathrm{A}_9 \times \mathrm{A}_{10}){:}\mathbb{Z}_2$. Since $|T : T_u|$ is square-free, we have $T_u \cong (\mathrm{A}_9 \times \mathrm{A}_{10}){:}\mathbb{Z}_2$. In particular, $|T : T_u|$ is even, and so Γ is T-arc-transitive. By [20], T_u is in fact the stabilizer of some 9-subset or 10-subset in the natural action of T acting on a 19-set. Up to the equivalence of group actions, we may identify V as the set of 9-subsets of a 19-set. Then T_u has ten orbits on V, says, $U_i = \{u_i \mid |u \cap u_i| = i\}$, $0 \leq i \leq 9$. Since Γ has valency 10, we have $\Gamma(u) = U_0$, and so $\Gamma \cong \mathrm{O}_{10}$. $\square$

6. The classification

Let $\Gamma = (V, E)$ be a connected graph of valency 10 and square-free order, and $G \leq \mathsf{Aut}\Gamma$. Assume that Γ is G-locally-primitive arc-transitive. Let $u \in V$. By Corollaries 4.6 and 4.7, $\mathrm{soc}(G_u^{\Gamma(u)}) \cong \mathrm{A}_5$, $\mathrm{PSL}(2,9)$ or A_{10}. In particular, G is insoluble.

Lemma 6.1. *Let $N \trianglelefteq G$. Then either N is semiregular on V and N has square-free order, or N is transitive on E and* $\mathrm{soc}(N_u^{\Gamma(u)}) = \mathrm{soc}(G_u^{\Gamma(u)})$. *In particular, if N is soluble then N is semiregular and has at least three orbits on V.*

Proof. Since N is normal in G and G is transitive on V, all N-orbits on V have the same length, which is a divisor of $|V|$. Noting that $|V| = |G : G_u|$ is square-free, if N is semiregular on V then $|N|$ is square-free. If $N_u \neq 1$ then, by Lemma 2.2, $N_u^{\Gamma(u)} \neq 1$, and then the first part of this lemma follows by noting that $N_u^{\Gamma(u)} \trianglelefteq G_u^{\Gamma(u)}$.

Now let N be soluble. Then N is semiregular on V. Suppose that N has at most two orbits on V. By Lemma 2.3, $G_u \cong G_u^{\Gamma(u)}$. Note that $X := NG_u$ is the set-stabilizer in G of an N-orbit containing u. Then $|G : X| \leq 2$; in particular, $X \trianglelefteq G$. Note that $X/\mathbf{C}_X(N) = \mathbf{N}_X(N)/\mathbf{C}_X(N) \lesssim \mathsf{Aut}(N)$. Since N has square-free order, $\mathsf{Aut}(N)$ is soluble, see [13, Lemma 2.2] for example. It follows that $\mathbf{C}_X(N)$ is insoluble as X is insoluble, this yields that $\mathrm{soc}(G_u) \leq \mathbf{C}_X(N)$. Noting that $\mathrm{soc}(G_u)$ is a nonabelian simple group, it is easily shown that $\mathrm{soc}(G_u)$ is a characteristic subgroup of X, and so $\mathrm{soc}(G_u)$ is normal in G. This implies that $\mathrm{soc}(G_u)$ acts trivially on V, a contradiction. This completes the proof. $\square$

Since Γ has square-free order, Γ is not the complete bipartite graph of order 20. Recall that the soluble radical of G is the maximal soluble normal subgroup. Then the next lemma follows from Lemmas 2.5, 6.1 and [16, Theorem 4].

Lemma 6.2. *Let M be the soluble radical of G. Then M has square-free order and at least three orbits on V, Γ is a cover of Γ_M and $G = M{:}X$ for some almost simple subgroup of G.*

By Lemma 6.2 and the argument given in Section 5, we have the following lemma.

Lemma 6.3. *Assume that G has no soluble normal subgroups. Then G is almost simple; in particular, the pair (G, Γ) is known as in Theorems* 5.6, 5.8 *and* 5.9.

Theorem 6.4. *Assume that G has nontrivial soluble radical M. Then one of the following statements holds.*

(1) $G = \mathbb{Z}_2 \times \mathrm{PSL}(2,r)$ *for a prime r with* $r \equiv \pm 1 \,(\mathsf{mod}\ 5)$, $r \equiv \pm 1 \,(\mathsf{mod}\ 12)$ *and* $r^2 \not\equiv 1 \,(\mathsf{mod}\ 16)$, *and Γ is isomorphic to the standard double cover of the graph in* Example 3.4.

(2) $G = \mathbb{Z}_2 \times \mathrm{PSL}(2,25)$ *or* $\mathbb{Z}_2 \times \mathrm{P\Sigma L}(2,25)$, *and Γ is isomorphic to the standard double cover of the graph given in Example* 3.7.

(3) $G = \mathbb{Z}_2 \times \mathrm{A}_7$ *or* $\mathbb{Z}_2 \times \mathrm{S}_7$, *$\Gamma$ is the standard double cover of the complement graph of* $L(\mathsf{K}_7)$.

(4) $G = \mathbb{Z}_2 \times \mathrm{M}_{11}$, $\mathbb{Z}_2 \times \mathrm{A}_{11}$ *or* $\mathbb{Z}_2 \times \mathrm{S}_{11}$, *and Γ is the standard double cover of* K_{11}.

Proof. By Lemma 6.2, $G = M{:}X$ for $X < G$, M is semiregular on V and Γ is a normal cover of $\Sigma := \Gamma_M$. Denote by $\mathcal{B}$ the vertex set of Σ, that is, the set of M-orbits on V. Then $|V| = |M||\mathcal{B}|$. Since $|V|$ is square-free, if $|\mathcal{B}|$ is even then $|M|$ is odd.

We identify X with a subgroup of $\mathsf{Aut}\Sigma$. Noting that X is almost simple, by Lemma 2.5 and 6.3, the pair (X, Σ) is known. In particular, X, X_B, T_B and $|\mathcal{B}|$ are listed in Table 1, where $B \in \mathcal{B}$ and $T = \mathrm{soc}(X)$.

X	X_B	T_B	$\lvert\mathcal{B}\rvert$	$\lvert M\rvert$
$\mathrm{PSL}(2,r)$	A_5	A_5	$\frac{r(r^2-1)}{120}$	
$\mathrm{PGL}(2,r)$	A_5	A_5	$\frac{r(r^2-1)}{60}$	Odd
$\mathrm{PSL}(2,25)$	S_5	S_5	65	
$\mathrm{P\Sigma L}(2,25)$	$\mathbb{Z}_2 \times \mathrm{S}_5$	S_5	65	
A_7, S_7	S_5, $\mathbb{Z}_2 \times \mathrm{S}_5$	S_5	21	
J_1	A_5	A_5	2926	Odd
M_{11}	M_{10}	M_{10}	11	
A_{11}, S_{11}	A_{10}, S_{10}	A_{10}	11	
S_{11}	A_{10}	A_{10}	22	Odd
A_{19}, S_{19}	$(\mathrm{A}_9 \times \mathrm{A}_{10}){:}\mathbb{Z}_2$, $\mathrm{S}_9 \times \mathrm{S}_{10}$	$(\mathrm{A}_9 \times \mathrm{A}_{10}){:}\mathbb{Z}_2$	92378	Odd
$\mathrm{PSL}(2,9).\mathbb{Z}_2$	$\mathbb{Z}_3^4{:}\mathrm{GL}(2,9)$	$\mathbb{Z}_3^4{:}\mathrm{GL}(2,9)$	182	Odd
$\mathrm{P\Gamma L}(3,9).\mathbb{Z}_2$	$\mathbb{Z}_3^4{:}\Gamma\mathrm{L}(2,9)$	$\mathbb{Z}_3^4{:}\mathrm{GL}(2,9)$	182	Odd

TABLE 1. Candidates for (X, X_B)

Set $N = MT$. Then $N \trianglelefteq G$, and so $\mathbf{C}_N(M) \trianglelefteq G$ and $M\mathbf{C}_N(M) \trianglelefteq G$. Since $|M|$ is square-free, $\mathsf{Aut}(M)$ is soluble. Note that $N/\mathbf{C}_N(M) = \mathbf{N}_N(M)/\mathbf{C}_N(M) \lesssim \mathsf{Aut}(M)$. It follows that $T \leq \mathbf{C}_N(M)$, and so $M\mathbf{C}_N(M) = M{\times}T$. This implies that T is a characteristic subgroup of $M\mathbf{C}_N(M)$, yielding $T \trianglelefteq G$. Noting that $|T|$ has order divisible by 4, T is not semiregular on V, and so T has at most two orbits on V, see Lemma 2.4,.

Suppose that $|M|$ is odd. Recalling that T has at most two orbits on V, we conclude that M fixes each T-orbit on V. Let U be a T-orbit on V, and choose $u \in U$ and $u \in B \in \mathcal{B}$. Then $B \subseteq U$, MT_B fixes B setwise, and both M and T_B are transitive on B. Since $MT_B = M{\times}T_B$, by [6, Theorem 4.2A], both M and T_B induce two regular permutation groups on B. In particular, T_B has a normal subgroup of odd index $|B| = |M| \neq 1$, which is impossible.

Now let $|M|$ be even. Then $|\mathcal{B}|$ is odd as $|V| = |M||\mathcal{B}|$. In particular, T is transitive on $\mathcal{B}$, Σ is T-arc-transitive, and Γ is MT-arc-transitive. Take $B \in \mathcal{B}$ and $u \in B$. Then $|\mathcal{B}| = |T : T_B|$. Since B is a block of G on V, we have $T_u \leq T_B$. Consider the transitive action of $M \times T_B$ on B. Since T_B is normal in $M \times T_B$, all T_B-orbits on B have the same length $|T_B : T_u|$. On the other hand, by [6, Theorem 4.2A], T_B induces a semiregular permutation group on B. It follows that T_u is the kernel of T_B acting on B; in particular, T_u is normal in T_B and of square-free index. Since $|\mathcal{B}|$ is odd, checking Table 1, we have $T_B \cong \mathrm{A}_5$, S_5, M_{10} or A_{10}. Then T_B and $\mathrm{soc}(T_B)$ are the only normal subgroups of T_B with square-free index. It follows that $|T_B : T_u| = 1$ or 2. Let t be the number of T on V. Then $t \leq 2$, and $|V| = t|T : T_u| = t|T : T_B||T_B : T_u|$. Since $|V| = |M||\mathcal{B}|$ and $|\mathcal{B}| = |T : T_B|$, we get $|M| = t|T_B : T_u|$, and so $|T_B : T_u| = |M|$ or $\frac{|M|}{2}$. It follows that $|M| = 2$. Since $MT = M \times T$, by Lemma 2.7, Γ is the standard double cover of Σ. Then our theorem follows. $\square$

Note that, for each graph Γ involved in Theorems 5.6, 5.9, 5.8 and 6.4, all possible candidates for G, which make Γ a G-locally primitive arc-transitive graph, have been determined. Then $\mathsf{Aut}\Gamma$ is just the maximal candidate for G. Thus Theorem 1.1 follows from Lemma 6.3 and Theorem 6.4.

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