Critical $(P_6, banner)$-Free Graphs

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November 3, 2018

Abstract

Given two graphs H_1 and H_2, a graph is (H_1, H_2)-free if it contains no induced subgraph isomorphic to H_1 or H_2. Let P_t and C_t be the path and the cycle on t vertices, respectively. A banner is the graph obtained from a C_4 by adding a new vertex and making it adjacent to exactly one vertex of the C_4. In this paper, we show that there are finitely many k-critical $(P_6, banner)$-free graphs for $k = 4$ and $k = 5$. For $k = 4$, we characterize all such graphs. Our results generalize previous results on k-critical (P_6, C_4)-free graphs for $k = 4$ and $k = 5$.

1 Introduction

All graphs in this paper are finite and simple. A *k-coloring* of a graph G is a function $\phi : V(G) \longrightarrow \{1, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever u and v are adjacent in G. Equivalently, a k-coloring of G can be viewed as a partition of $V(G)$ into k stable sets. We say that G is *k-colorable* if it admits a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable.

A graph G is *k-chromatic* if $\chi(G) = k$. We say that G is *k-critical* if it is k-chromatic and $\chi(G - e) < \chi(G)$ for any edge $e \in E(G)$. For instance, K_2 is the only 2-critical graph and odd cycles are the only 3-critical graphs. A graph is *critical* if it is k-critical for some integer $k \geq 1$. Critical graphs were first defined and studied by Dirac [8, 9, 10] in the early 1950s, and then by Gallai and Ore [11, 12, 20] among many others, and more recently by Kostochka and Yancey [18].

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A natural question about critical graphs is that can we *characterize* all k-critical graphs? We refer to this question as the *characterization problem*. The answer is yes for $k = 1, 2, 3$. For $k \geq 4$ the answer is unknown and the problem seems intractable even for $k = 4$ since there are many different ways one can construct 4-critical graphs. Therefore, researchers have investigated the characterization problem when graphs belong to some special graph class. To properly present known results in this respect, we first state the definition of critical graphs with respect to a graph class given in [3]. Let $\mathcal{H}$ be a set of graphs. A graph G is *$\mathcal{H}$-free* if it does not contain any member in $\mathcal{H}$ as an induced subgraph. In case that $\mathcal{H}$ consists of a single graph H or two graphs H_1 and H_2, we simply write H-free and (H_1, H_2)-free instead of $\{H\}$-free and $\{H_1, H_2\}$-free, respectively. A graph G is called *k-critical $\mathcal{H}$-free* if G is $\mathcal{H}$-free and k-chromatic, and any proper $\mathcal{H}$-free subgraph G' of G has $\chi(G') < k$. Note that when $\mathcal{H} = \emptyset$ this definition coincides with that of the usual k-critical graphs. A graph G is *k-vertex-critical* if $\chi(G) = k$ and $\chi(G-v) < k$ for any $v \in V(G)$. For a set $\mathcal{H}$ of graphs and a graph G, we say that G is *k-vertex-critical $\mathcal{H}$-free* if it is k-vertex-critical and $\mathcal{H}$-free. Observe that a k-critical $\mathcal{H}$-free graph is k-vertex-critical $\mathcal{H}$-free. The importance of k-critical $\mathcal{H}$-free graphs is manifested in the following theorem.

Theorem 1 (Folklore). *Let G be $\mathcal{H}$-free and $k \geq 3$ be an integer. Then G is $(k-1)$-colorable if and only if it contains no k-critical $\mathcal{H}$-free graphs as a subgraph.* □

Let K_n be the complete graph on n vertices. Let P_t and C_t denote the path and the cycle on t vertices, respectively. The answer to the characterization problem with respect to graph classes is known for many graph classes. For example, the only k-critical perfect graph (see the definition in section 2) is the complete graph K_k, and there are no 5-critical planar graphs by the Four Color Theorem. Another class of graph that has been extensively studied recently is the class of P_t-free graphs. In [3], it was shown that there are exactly six 4-critical P_5-free graphs. This result was later generalized to P_6-free graphs [5]: there are 24 4-critical P_6-free graphs. In the same paper, an infinite family of 4-critical P_7-free graphs was constructed. Randerath and Schiermeyer [21] have shown that the only 4-critical (P_6, C_3)-free graph is the well-known Grötzsch graph. Hell and Huang [14] proved that there are four 4-critical (P_6, C_4)-free graphs (and they are (a)-(d) in Figure 1). The result in [14] was generalized by Goedgebeur and Schaudt [13] to (P_t, C_4)-free graphs for $t = 7$ and $t = 8$ in which they proved that there are 17 4-critical (P_7, C_4)-free graphs and there are 94 4-critical (P_8, C_4)-free graphs. In the same paper, Goedgebeur and Schaudt [13] also determined all 6 4-critical (P_7, C_5)-free graphs. It was also known [15] that there are eight 5-critical (P_5, C_5)-free graphs.

Another line of research is to determine whether the set of all k-critical $\mathcal{H}$-free graphs is finite. We refer to this question as the *finiteness problem.* It is not hard to see that the set of k-critical $\mathcal{H}$-free graphs is finite if and only if the set of k-vertex-critical $\mathcal{H}$-free graphs is finite (see [15] for a proof). The finiteness problem is meaningful because the finiteness of the set has a fundamental algorithmic implication.

Theorem 2 (Folklore)**.** *If the set of all k-critical $\mathcal{H}$-free graphs is finite, then there is a polynomial-time algorithm to determine whether an $\mathcal{H}$-free graph is $(k-1)$-colorable.* □

For two graphs G and H, we use $G+H$ to denote the *disjoint union* of G and H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. For $s, r \geq 1$, let $K_{r,s}$ be the complete bipartite graph with one part of size r and the other part of size s. For the class of H-free graphs, it was shown in [4, 5] that the set of 4-critical H-free graphs is finite if and only if H is a subgraph of P_6, $2P_3$ or $P_4 + rP_1$ for some $r \geq 1$. The finiteness was also shown for many other classes, e.g., for $(P_5, \overline{P_5})$-free graphs [7], (P_6, C_4)-free graphs [14], $(P_t, K_{s,r})$-free graphs [17] for any $t, s, r \geq 1$.

Our Contributions. A *banner* is the graph obtained from a C_4 by adding a new vertex and making it adjacent to exactly one vertex of the C_4. In this paper, we show that there are finitely many k-critical $(P_6, banner)$-free graphs for $k = 4$ and $k = 5$. For $k = 4$, we characterize all such graphs. The results generalize the previous results on k-critical (P_6, C_4)-free graphs [14, 16] for $k = 4, 5$. We remark that the result for $k = 4$ (Theorem 4) is in fact implied by the main result of [5]. In [5], a complete list of 80 4-vertex-critical P_6-free graphs is given with the aid of a computer program. One can filter all graphs in the list that are not banner-free and thus obtain a complete list of all 4-vertex-critical $(P_6, banner)$-free graphs. By testing whether each of these graphs is 4-critical $(P_6, banner)$-free, one can determine all 4-critical $(P_6, banner)$-free graphs. Here we give an independent proof that is computer-free.

The remainder of the paper is organized as follows. We present some preliminaries in section 2 and give structural properties around an induced C_5 in a $(P_6, banner)$-free graph in section 3. We then show that there are finitely many 4-critical and 5-critical $(P_6, banner)$-free graphs in section 4 and section 5, respectively. We conclude our paper in section 6.

2 Preliminaries

For general graph theory notation we follow [1]. The *complement* of a graph G is denoted by $\overline{G}$. For $k \geq 4$, an induced cycle of length k is also called a *k-hole*. A k-hole is an *odd hole* (respectively *even hole*) if k is odd (respectively even). A *k-antihole* is the complement of a k-hole. Odd and even antiholes are defined analogously.

Let $G = (V, E)$ be a graph. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of neighbors of v. For a set $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$ and $N_G[X] = N(X) \cup X$. The *degree* of v, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We shall omit the subscript G when the context is clear. The *minimum degree* of G over all vertices in G is denoted by $\delta(G)$. For $x \in V$ and $S \subseteq V$, we denote by $N_S(x)$ the set of neighbors of x that are in S, i.e., $N_S(x) = N_G(x) \cap S$. For $X, Y \subseteq V$, we say that X is *complete* (resp. *anticomplete*) to Y if every vertex in X is adjacent (resp. nonadjacent) to every vertex in Y. A vertex subset $S \subseteq V$ is *stable* if no two vertices in S are adjacent. A *clique* is the complement of a stable set. A vertex subset $K \subseteq V$ is a *clique cutset* if $G - K$ has more connected components than G and K is a clique. A vertex is *universal* in G if it is adjacent to all other vertices in G. For $S \subseteq V$, the subgraph *induced* by S is denoted by $G[S]$. We say that a vertex w *distinguishes* two vertices u and v if w is adjacent to exactly one of u and v. Two nonadjacent vertices u and v are said to be *comparable* if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.

The following lemma is well-known in the study of k-critical graphs.

Lemma 1 (Folklore)**.** *Let G be a k-critical graph. Then the following holds: (i) $\delta(G) \geq k - 1$; (ii) G contains no clique cutsets; (iii) G contains no pair of comparable vertices.*

The *clique number* of G, denoted by $\omega(G)$, is the size of a largest clique in G. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Another result we use is the well-known Strong Perfect Graph Theorem.

Theorem 3 (The Strong Perfect Graph Theorem [6])**.** *A graph is perfect if and only if it does not contain any odd hole or odd antihole as an induced subgraph.*

3 Structure Around a 5-Hole

Let $G = (V, E)$ be a graph and H be an induced subgraph of G. We partition $V \setminus V(H)$ into subsets with respect to H as follows: for any $X \subseteq V(H)$, we denote by $S(X)$ the set of vertices in $V \setminus V(H)$ that have X as their neighborhood among $V(H)$, i.e.,

$$S(X) = \{v \in V \setminus V(H) : N_{V(H)}(v) = X\}.$$

For $0 \leq j \leq |V(H)|$, we denote by S_j the set of vertices in $V \setminus V(H)$ that have exactly j neighbors in $V(H)$. Note that $S_j = \bigcup_{X \subseteq V(H):|X|=j} S(X)$. We say that a vertex in S_j is a *j-vertex*.

Let G be a (P_6, banner)-free graph and $C = 1, 2, 3, 4, 5$ be an induced C_5 in G. We partition $V \setminus C$ with respect to C as above. All indices below are modulo five. Clearly, $S(\{i, i+2\}) = S(\{i-2, i, i+2\}) = \emptyset$ since G is banner-free. In the following, we shall write $S(i, i+2)$ for $S(\{i, i+2\})$, $S(i-2, i, i+2)$ for $S(\{i-2, i, i+2\})$, etc. We now prove a number of useful properties of $S(X)$ using the fact that G is (P_6, banner)-free. All properties are proved for $i = 1$ due to symmetry.

(1) $S(i-1, i, i+1)$ and $S(i-1, i-2, i+2, i+1)$ are cliques.

If $S(5, 1, 2)$ contains two nonadjacent vertices x and y, then $\{x, y, 5, 2, 3\}$ induces a banner. Similarly, if $S(2, 3, 4, 5)$ contains two nonadjacent vertices x and y, then $\{x, y, 5, 1, 3\}$ induces a banner. □

(2) $S(i)$ is complete to $S(i+2)$ and anticomplete to $S(i+1)$. Moreover, if neither $S(i)$ nor $S(i+2)$ is empty then both sets are cliques.

Let $x \in S(1)$, $y \in S(2)$ and $z \in S(3)$. If $xz \notin E$, then $x, 1, 5, 4, 3, z$ induces a P_6. If $xy \in E$, then $\{x, y, 1, 2, 3\}$ induces a banner. This proves the first part of the claim. If $z, z' \in S(3)$ are not adjacent, then $\{x, z, z', 1, 3\}$ induces a banner. □

(3) $S(i, i+1)$ is complete to $S(i+1, i+2)$. Moreover, if neither $S(i, i+1)$ nor $S(i+1, i+2)$ is empty then both sets are cliques.

Let $x \in S(1, 2)$ and $y \in S(2, 3)$. If $xy \notin E$, then $x, 1, 5, 4, 3, y$ induces a P_6. If $y, y' \in S(2, 3)$ are not adjacent, then $\{y, y', x, 1, 3\}$ induces a banner. □

(4) If $x \in S(i-2, i-1)$ and $y \in S(i+1, i+2)$ are adjacent, then x and y are universal vertices in $S(i-2, i-1)$ and $S(i+1, i+2)$, respectively.

If $z \in S(4, 5)$ is not adjacent to x, then either $\{z, x, y, 5, 3\}$ induces a banner or $z, 4, x, y, 2, 1$ induces a P_6 depending on whether $zy \in E$. This shows that x is universal in $S(4, 5)$. By symmetry, y is universal in $S(2, 3)$. □

(5) $S(i)$ is anticomplete to $S_2 \setminus S(i-2, i+2)$. Moreover, if a vertex in $S(i-2, i+2)$ has a neighbor in $S(i)$ then it is universal in $S(i-2, i+2)$.

Let $x \in S(1)$. If x is adjacent to $y \in S(1,2)$, then $x, y, 2, 3, 4, 5$ induces a P_6. If x is adjacent to $y \in S(2,3)$, then $\{x, y, 5, 1, 2\}$ induces a banner. This proves the first part of the claim. The proof of the second part is similar to **(4)**. □

(6) $S(i)$ is anticomplete to $S(i+1, i+2, i+3)$.

If $x \in S(1)$ and $y \in S(2,3,4)$ are adjacent, then $\{x, y, 5, 1, 2\}$ induces a banner. □

(7) $S(i-2, i+2)$ is anticomplete to $S(i-1, i, i+1)$.

If $x \in S(5,1,2)$ and $y \in S(3,4)$ are adjacent, then $\{x, y, 5, 2, 3\}$ induces a banner. □

(8) Either $S(i)$ or $S(i+1, i+2)$ is empty. By symmetry, either $S(i)$ or $S(i-1, i-2)$ is empty.

If $x \in S(1)$ and $y \in S(2,3)$, then $xy \notin E$ by **(5)**. So, $x, 1, 5, 4, 3, y$ induces a P_6. □

(9) If none of $S(i-1, i)$, $S(i, i+1)$ or $S(i+2, i-2)$ is empty, then $S(i-2, i+2)$ is complete to $S(i-1, i) \cup S(i, i+1)$.

Let $x \in S(5,1)$, $y \in S(1,2)$ and $z \in S(3,4)$. Note that $xy \in E$ by **(3)**. If $xz \notin E$, then either $z, 4, 5, x, y, 2$ induces a P_6 or $\{x, y, z, 2, 3\}$ induces a banner depending on whether $zy \in E$. By symmetry, $zy \in E$. Since the argument works for every edge xy between $S(5,1)$ and $S(1,2)$, the claim follows. □

(10) If both $S(i-1)$ and $S(i+1)$ are nonempty, then $S_2 = \emptyset$, and if both $S(i)$ and $S(i+1)$ are nonempty, then $S_2 = S(i, i+1)$.

Let $x \in S(5)$ and $y \in S(2)$. By **(9)**, $S(5,1) = S(1,2) = S(3,4) = \emptyset$. If $S(2,3)$ contains a vertex z, then z is not adjacent to y but adjacent to x by **(5)** and **(2)**. This implies that $z, 2, y, x, 5, 4$ induces a P_6. So, $S(2,3) = \emptyset$ and by symmetry $S(4,5) = \emptyset$. This proves the first part of the claim. The second part follows directly from **(9)**. □

(11) Let $x \in S(i-1, i, i+1)$. If both $S(i+1, i+2)$ and $S(i-2, i-1)$ are nonempty, then x is either complete or anticomplete to $S(i-2, i-1) \cup S(i+1, i+2)$. If x is complete to $S(i-2, i-1) \cup S(i+1, i+2)$, both $S(i+1, i+2)$ and $S(i-2, i-1)$ are cliques.

Suppose that $y \in S(2,3)$ and $z \in S(4,5)$. Assume that $xy \in E$. If $xz \notin E$, then either $1, x, y, 3, 4, z$ induces a P_6 or $\{x, y, z, 3, 4\}$ induces a banner depending on whether $yz \in E$. This shows that x is either complete or anticomplete to each pair $\{y, z\}$ of vertices where $y \in S(2,3)$ and $z \in S(4,5)$. This proves the first part of the claim. The second part of the claim follows from the fact that G is banner-free. $\square$

(12) If $S(i)$ is not anticomplete to $S(i-2, i+2)$, then $S_1 = S(i)$.

Let $x \in S(1)$ have a neighbor in $y \in S(3,4)$. By **(9)**, $S(2) = S(5) = \emptyset$. If $S(3)$ contains a vertex z, then z is adjacent to x but not to y by **(2)** and **(5)**. Then $\{x, y, z, 1, 3\}$ induces a banner. This shows that $S(3) = \emptyset$. By symmetry, $S(4) = \emptyset$. $\square$

(13) $S(C \setminus \{i\})$ is complete to $S(i-1) \cup S(i) \cup S(i+1)$ and anticomplete to $S(i-2) \cup S(i+2)$.

Let $x \in S(C \setminus \{1\})$. If x is not adjacent to a vertex $y \in S(5) \cup S(1) \cup S(2)$, then $\{x, y, 5, 1, 2\}$ induces a banner. Similarly, if x is adjacent to a vertex $y \in S(3) \cup S(4)$, then $\{x, y, 5, 1, 2\}$ induces a banner. $\square$

(14) $S(C \setminus \{i\})$ is complete to $S(i-2, i-1) \cup S(i+1, i+2)$ and anticomplete to $S_2 \setminus (S(i-2, i-1) \cup S(i+1, i+2))$.

Let $x \in S(C \setminus \{1\})$. If x is not adjacent to a vertex $y \in S(2,3) \cup S(4,5)$, then $\{x, y, 5, 1, 2\}$ induces a banner. This proves that $S(C \setminus \{1\})$ is complete to $S(2,3) \cup S(4,5)$. Suppose that x is adjacent to $y \in (3,4) \cup S(1,2) \cup S(5,1)$. By symmetry, we may assume that $y \in (3,4) \cup S(1,2)$. If $y \in S(3,4)$, then $\{x, y, 5, 1, 2\}$ induces a banner, and if $y \in S(1,2)$, then $\{x, y, 5, 1, 3\}$ induces a banner. This proves that $S(C \setminus \{1\})$ is anticomplete to $(3,4) \cup S(1,2) \cup S(5,1)$. $\square$

(15) $S(C \setminus \{i\})$ is complete to $S(i+1, i+2, i+3)$.

If $x \in S(C \setminus \{1\})$ and $y \in S(2,3,4)$ are not adjacent, then $\{x, y, 5, 1, 2\}$ induces a banner. $\square$

(16) Let $x, y \in S_5$ with $xy \notin E$. Then any vertex in S_0 is either complete or anticomplete to $\{x, y\}$, any vertex in $S_1 \cup S_2$ is complete to $\{x, y\}$, and any vertex in $S_3 \cup S_4$ is adjacent to at least one of x and y.

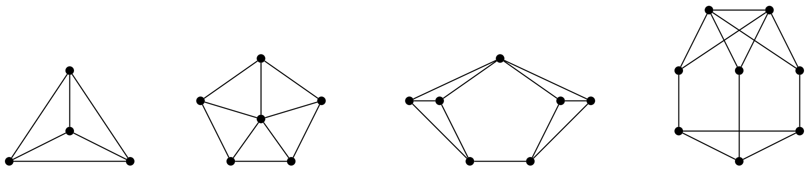
Let $t \in S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$. If $t \in S_0$ and t is adjacent to exactly one of x and y, then $\{t, x, y, 1, 3\}$ induces a banner. This proves the first claim.

Assume that $t \in S_1 \cup S_2 \cup S_3 \cup S_4$. Since t is adjacent to at least one but not all vertices on C, there exists a vertex $i \in C$ such that t is adjacent to i but not to $i+2$. If t is adjacent to neither x nor y, then $\{t, i, i+2, x, y\}$ induces a banner. This proves the third claim.

If $t \in S_1 \cup S_2$, there exists a vertex $j \in C$ such that t is adjacent to neither j nor $j+2$. If t is adjacent to exactly one of x and y, then $\{t, j, j+2, x, y\}$ induces a banner. This proves the second claim. $\square$

4 4-Critical Graphs

In this section we determine all 4-critical $(P_6, banner)$-free graphs. Let $\mathcal{F}_4$ be the set of graphs shown in Figure 1.



(a) K_4. (b) W_5. (c) The Hajós graph. (d) F.

Figure 1: All 4-critical $(P_6, banner)$-free graphs.

Theorem 4. *A graph is 4-critical $(P_6, banner)$-free if and only if it belongs to $\mathcal{F}_4$.*

Proof. It is straightforward to verify that all graphs in $\mathcal{F}_4$ are 4-critical $(P_6, banner)$-free. Let G be a 4-critical $(P_6, banner)$-free graph. We show that $G \in \mathcal{F}_4$. If G contains any member in $\mathcal{F}_4$ as a subgraph, then G is isomorphic to this subgraph by the definition of k-critical graphs and we are done. Hence, we assume in the following that G contains no subgraph isomorphic to a member in $\mathcal{F}_4$. We show that this is impossible.

Note first that $\overline{C_7}$ contains the Hajós graph as a subgraph, and so G is $\overline{C_7}$-free. Since G is (C_7, K_4)-free, it contains an induced $C = C_5 = 1, 2, 3, 4, 5$. For otherwise G is perfect by Theorem 3 and hence 3-colorable. This contradicts that G is 4-chromatic. We partition $V(G)$ with respect to C. Since G contains no W_5, it follows that $S_5 = \emptyset$. Since G contains no subgraph isomorphic to the Hajós graph, there are at most two nonempty $S(i-1, i, i+1)$ and $S(j-1, j, j+1)$ with $|i-j| = 1$. Moreover, $|S(i-1, i, i+1)| \leq 1$ since G is K_4-free, and so $|S_3| \leq 2$.

Claim 1. $S_0 = \emptyset$.

Proof of Claim 1. Suppose that $S_0 \neq \emptyset$. Let A be an arbitrary connected component of S_0. Since G is $(P_6, banner)$-free, A is anticomplete to $S_1 \cup S_2 \cup S_4$. Moreover, since G is P_6-free, any vertex in S_3 is either complete or anticomplete to A. Since G has no clique cutset by Lemma 1, it follows that S_3 contains two vertices and are complete to A. Without loss of generality, assume that $S(5,1,2) = \{x\}$ and $S(1,2,3) = \{y\}$. Note that $xy \notin E$ since G is K_4-free. If A contains two nonadjacent vertices a and b, then $\{a,b,x,y,3\}$ induces a banner. So, A is a clique.

If S_0 has a second connected component B, then the above argument shows that $\{x,y\}$ is complete to B. Then $\{a,b,x,y,3\}$ induces a banner where $a \in A$ and $b \in B$. This shows that $S_0 = A$. Since G is K_4-free, $|S_0| \leq 2$. Let $a \in A$ and b be a possible neighbor of a in A. Note that $N(a) \subseteq \{b,x,y\}$. Since G is 4-critical $(P_6, banner)$-free, $G-a$ admits a 3-coloring ϕ. Observe that x and y are colored alike under ϕ because $\{x,y,1,2\}$ induces a diamond. Hence, ϕ can be extended to a 3-coloring of G by assigning a color from $\{1,2,3\}$ that does not appear on $N(a)$. This contradicts that G is 4-chromatic. □

Claim 2. *$S_4 = \emptyset$.*

Proof of Claim 2. Suppose that $S_4 \neq \emptyset$. Let $x \in S(C \setminus \{1\})$. Since G contains no subgraph isomorphic to the Hajós graph, $S(C \setminus \{i\}) = \emptyset$ for $i \neq 1$. By the fact that G is K_4-free and **(1)**, $S_4 = \{x\}$. By **(15)** and K_4-freeness of G, $S(2,3,4) \cup S(3,4,5) = \emptyset$. If S_3 contains a vertex y, then $C \cup \{x,y\}$ contains the Hajós graph as a subgraph, a contradiction. So, $S_3 = \emptyset$.

By **(14)** and K_4-freeness of G, $S(2,3) \cup S(4,5) = \emptyset$. If $S(1)$ contains a vertex y, then $xy \in E$ by **(13)**. But then $\{x,y,1,3,5\}$ induces a banner. So, $S(1) = \emptyset$. Since G is 4-critical, $d(1) \geq \delta(G) \geq 3$ by Lemma 1. By symmetry, we may assume that $S(5,1) \neq \emptyset$. By **(8)**, $S(2) = S(4) = \emptyset$.

Case 1. $S(1,2) \neq \emptyset$. By **(8)**, $S(3) = S(5) = \emptyset$ and so $S_1 = \emptyset$. By **(3)** and K_4-freeness of G, $S(5,1) = \{y\}$ and $S(1,2) = \{z\}$ and $yz \in E$. If $S(3,4) \neq \emptyset$, then $S(3,4)$ is complete to $S(5,1) \cup S(1,2)$ by **(9)**. If $S(3,4)$ contains two nonadjacent vertices u and v, then $\{y,u,v,1,3\}$ induces a banner, a contradiction. So, $S(3,4)$ is a clique and hence contains at most one vertex since G is K_4-free. It follows that $8 \leq |G| \leq 9$ and G is 3-colorable. This contradicts that G is 4-chromatic.

Case 2. $S(1,2) = \emptyset$. Note that $S_1 = S(3) \cup S(5)$ and $S_2 = S(5,1) \cup S(3,4)$. Moreover, $S(5,1)$ and $S(3,4)$ are stable sets since G is K_4-free. Since $S_2 \neq \emptyset$, either $S(3) = \emptyset$ or $S(5) = \emptyset$ by **(10)**. Suppose first that $S(3) = \emptyset$. By **(13)**, x is complete to $S(5)$ and so $S(5)$ is a stable set since G is K_4-free. Furthermore, $S(5)$ is anticomplete to S_2 by **(5)** and x ia anticomplete to

$S(3,4)$ by **(14)**. Then G can be partitioned into three stable sets $\{1,x\} \cup S(3,4)$, $\{2,4\} \cup S(5,1) \cup S(5)$ and $\{3,5\}$. This contradicts that G is 4-chromatic. Assume now that $S(5) = \emptyset$. Let $X \subseteq S(3)$ be the set of vertices that have a neighbor in $S(5,1)$ and $Y = S(3) \setminus X$. We show that X and Y are stable sets. Suppose that X contains two adjacent vertices x_1 and x_2. By definition, x_1 has a neighbor $y \in S(5,1)$. By **(5)** and K_4-freeness of G, $S(5,1) = \{y\}$ and so y is also adjacent to x_2. But now $\{x, y, x_1, x_2, 3, 4, 5\}$ induces the Hajós graph, a contradiction. This proves that X is stable. Let A be an arbitrary connected component of Y. Note that since G is P_6-free, any vertex in X is either complete or anticomplete to A. If A has no neighbor in X, $\{3\}$ is a clique cutset separating A from the rest of the graph. This contradicts Lemma 1. So, A has a neighbor $x' \in X$. Since G is K_4-free, A is a singleton. This proves that Y is stable. Now G can be partitioned into three stable sets $\{1,x\} \cup S(3,4) \cup X$, $\{2,4\} \cup S(5,1) \cup Y$ and $\{3,5\}$. This contradicts that G is 4-chromatic.

In either case we get a contradiction. This completes the proof of the claim. □

Claim 3. $S_3 = \emptyset$.

Proof of Claim 3. Recall that $|S_3| \leq 2$. Suppose first that $|S_3| = 2$. Without loss of generality, assume that $S(5,1,2) = \{x\}$ and $S(1,2,3) = \{y\}$. Note that $xy \notin E$ since G is K_4-free. Furthermore, $S(4) = \emptyset$ for otherwise any vertex $z \in S(4)$ starts an induced $P_6 = z, 4, 3, y, 1, x$ since $zx, zy \notin E$ by **(6)**. If x is not adjacent to a vertex $z \in S(4,5)$, then $z, 5, x, 1, y, 3$ induces a P_6. So, x is complete to $S(4,5)$. If $S(4,5)$ contains two nonadjacent vertices z, z', then $\{x, z, z', 4, 2\}$ induces a banner, a contradiction. So, $S(4,5)$ is a clique and so contains at most one vertex since G is K_4-free. By symmetry, y is complete to $S(3,4)$ and $S(3,4)$ contains at most one vertex.

By Lemma 1, $d(4) \geq \delta(G) \geq 3$. So, either $S(3,4) \neq \emptyset$ or $S(4,5) \neq \emptyset$. By symmetry, we assume that $S(3,4) \neq \emptyset$ and let $S(3,4) = \{z\}$. Note that $yz \in E$. Suppose first that $S(4,5) = \emptyset$. Then $N(4) = \{3, 5, z\}$. Since G is 4-critical, $G - 4$ admits a 3-coloring ϕ. Observe that vertices 2, 5 and z are colored alike under ϕ. So, we can extend ϕ to G by assigning to the vertex 4 a color from $\{1,2,3\}$ that does not appear on $N(4)$. This contradicts that G is 4-chromatic. Therefore, $S(4,5) \neq \emptyset$ and let $S(4,5) = \{w\}$. Recall that $xw \in E$. Moreover, $yw \notin E$ by **(7)**. By the fact that $S(4) = \emptyset$ and **(8)**, $S_1 = \emptyset$. We claim that $S_2 = \{w, z\}$. Suppose not. Let $u \in S_2 \setminus \{w, z\}$. By symmetry, we may assume that $u \in S(1,2) \cup S(2,3)$. If $u \in S(1,2)$, then u is adjacent to z but not to y by **(9)** and K_4-freeness of G. But now

$\{u, y, z, 2, 4\}$ induces a banner. If $u \in S(2,3)$, then u is adjacent to z and x but not to y by **(3)**, **(11)** and K_4-freeness of G. Then either $\{x, y, w, z, u\}$ or $\{y, w, u, 3, 4\}$ induces a banner depending on whether $wu \in E$. We have shown that $S_2 = \{w, z\}$. Now G can be partitioned into three stable sets $\{4, x, y\}$, $\{1, 3, w\}$ and $\{2, 5, z\}$. This contradicts that G is 4-chromatic.

Therefore, $|S_3| = 1$ and we assume that $S(5, 1, 2) = \{x\}$. If $y \in S(3, 4)$ is adjacent to $z \in S(5, 1)$, then either $\{x, y, z, 4, 5\}$ induces a banner or $\{x, z, 5, 1\}$ induces a K_4 depending on whether $xz \in E$. This shows that $S(3, 4)$ is anticomplete to $S(5, 1)$ and to $S(1, 2)$ by symmetry.

Case 1. $S(1) \neq \emptyset$. Applying the argument in Claim 1 to $C' = C \setminus \{1\} \cup \{x\}$ shows that x is complete to $S(1)$. By **(8)**, $S(2,3) = S(4,5) = \emptyset$. Since G is F-free, $S(1)$ is anticomplete to $S(3,4)$. By **(5)** and **(7)**, $\{3, 4\}$ would be a clique cutset if $S(3, 4) \neq \emptyset$. So, $S(3, 4) = \emptyset$. Since $d(3), d(4) \geq 3$, $S(3)$ and $S(4)$ are nonempty and so $S_2 = \emptyset$ by **(8)**. Since G is K_4-free, $S(1)$ is a singleton and let $S(1) = \{y\}$. If $S(3)$ contains two vertices z, z', then $\{x, y, z, z', 1, 2, 3\}$ induces the Hajós graph, a contradiction. So, $S(3)$ is a singleton and let $S(3) = \{z\}$. Similarly, $S(4)$ is a singleton and let $S(4) = \{w\}$. By **(2)** and K_4-freeness of G, $|S(i)| \leq 2$ for $i = 2, 5$. If $S(5)$ contains two vertices u, u', then $\{x, y, u, u', z, 1, 5\}$ induces the Hajós graph. So, $S(5)$ contains at most one vertex. Similarly, $S(2)$ contains at most one vertex. If x is adjacent to a vertex $t \in S(2)$, then $\{x, y, w, t, 4\}$ induces a banner. So, x is anticomplete to $S(2)$ and to $S(5)$ by symmetry. It can be easily seen that G is now 3-colorable. This contradicts that G is 4-chromatic.

Case 2. $S(1) = \emptyset$. We first show that $S(3, 4) = \emptyset$. Suppose not. Let $y \in S(3, 4)$. By **(7)** and K_4-freeness of G, $S(3, 4)$ is stable and anticomplete to x. By **(5)** and Lemma 1, either $S(2, 3) \neq \emptyset$ or $S(4, 5) \neq \emptyset$. By symmetry, we may assume that $S(4, 5)$ contains a vertex z. By **(8)**, $S_1 = S(4)$. If x is adjacent to a vertex in $S(2, 3) \cup S(4, 5)$, then this vertex and 1 are 3-vertices with respect to $C' = C \setminus \{1\} \cup \{x\}$. This reduces to the previous case that $|S_3| = 2$. So, x is anticomplete to $S(2, 3) \cup S(4, 5)$. If $S(5, 1)$ contains a vertex t, then either $\{t, x, 1, 5\}$ induces a K_4 or $x, 1, t, z, y, 3$ induces a P_6 depending on whether $tx \in E$. So, $S(5, 1) = \emptyset$. Since G is K_4-free, x is anticomplete to $S(1, 2)$. We have shown that $N(x) = \{1, 2, 5\}$. If $S(2, 3)$ contains a vertex t, then $G - x$ admits a 3-coloring ϕ with $\phi(2) = \phi(5)$, and thus we can extend ϕ to G by assigning to x a color from $\{1, 2, 3\}$ that does not appear on $N(x)$. This contradicts that G is 4-chromatic. Hence, $S(2, 3) = \emptyset$ and so $N(3) = \{2, 4, y\}$. Since G is 4-critical, $G - 3$ admits a 3-coloring ϕ. Observe that the vertex 2 and y are colored alike under ϕ. So, we can extend ϕ to G by assigning to 3 a color from $\{1, 2, 3\}$ that does not appear on $N(3)$. This contradicts that G is 4-chromatic. This proves that $S(3, 4) = \emptyset$.

By (8) and $\delta(G) \geq 3$, it follows that either $S(2,3)$ and $S(4,5)$ are nonempty or $S(3)$ and $S(4)$ are nonempty. Suppose first that $S(3)$ and $S(4)$ are nonempty. Then $S_2 = \emptyset$ by (8). Since $\{3\}$ does not separate $S(3)$, $S(5) \neq \emptyset$. Similarly, $S(2) \neq \emptyset$. Let $u_i \in S(i)$ for $i \neq 1$. By (2) and K_4-freeness of G, each $S(i)$ is a clique of size at most two for $i \neq 1$. Moreover, $|S(2)| + |S(4)| = 3$ and $|S(3)| + |S(5)| = 3$ since $\delta(G) \geq 3$. Suppose that $|S(2)| = 2$. Then $|S(4)| = 1$. If $|S(3)| = 2$, then $\{u_5, 2, 3\} \cup S(3) \cup S(2)$ induces the Hajós graph. So, $|S(3)| = 1$ and thus $|S(5)| = 2$. But now $S(2) \cup S(5)$ induces a K_4. This shows that $|S(2)| = 1$. By symmetry, $|S(5)| = 1$ and hence $|S(3)| = |S(4)| = 2$. Note that x is anticomplete to $S(2) \cup S(5)$ since G is $(W_5, banner)$-free. Now G can be partitioned into three stable sets $\{2, 5, u_3, u_4\}$, $\{1, 4, u_2, u_3'\}$, $\{3, x, u_4', u_5\}$ where $u_j' \in S(j)$ for $j = 3, 4$.

Now assume that $S(2,3)$ and $S(4,5)$ are nonempty and $S(3)$ and $S(4)$ are empty. Let $y \in S(2,3)$ and $z \in S(4,5)$. Recall that x is anticomplete to $S(2,3) \cup S(4,5)$. We show that $S(1,2) = \emptyset$. If $S(1,2)$ contains a vertex w, then w, y, z induces a triangle by (3) and (9). Moreover, $xw \notin E$ since G is K_4-free. Now $\{w, x, y, z, 5, 1, 2\}$ contains the Hajós graph as a subgraph, a contradiction. So, $S(1,2) = \emptyset$ and by symmetry $S(5,1) = \emptyset$. Since $\{4,5\}$ is not a clique cutset, $S(4,5)$ has a neighbor in $S(2)$. Similarly, $S(2,3)$ has a neighbor in $S(5)$. But this contradicts (10).

In either case we get a contradiction. This completes the proof of the claim. $\square$

By Claim 1-Claim 3, $V(G) = C \cup S_1 \cup S_2$. Note that $S(i, i+1)$ is stable since G is K_4-free. Suppose first that $S_1 = \emptyset$. If there are at most two nonempty $S(i, i+1)$, one can easily see that G is 3-colorable, a contradiction. On the other hand, it follows from (3) and (9) that if there are four nonempty $S(i, i+1)$,then G contains a K_4, a contradiction. So, there are exactly three nonempty $S(i, i+1)$. By (3) and (9), each nonempty $S(i, i+1)$ is a singleton. It can be easily seen that G is 3-colorable, a contradiction.

Therefore, $S_1 \neq \emptyset$ and assume that $x \in S(1)$. By (8), $S(2,3) = S(4,5) = \emptyset$. Moreover, it follows from (8) and Lemma 1 that either $S(4)$ and $S(5)$ are nonempty or $S(5,1)$ and $S(3,4)$ are nonempty. Suppose that $S(5,1)$ and $S(3,4)$ are nonempty. By (8), $S(2) = \emptyset$. So, $S(1,2) \neq \emptyset$ since $d(2) \geq 3$. This implies that $S_1 = S(1)$ and $S_2 = S(5,1) \cup S(1,2) \cup S(3,4)$. By (3) and (9), each nonempty $S(i, i+1)$ is a singleton and S_2 induces a triangle uvw where $u \in S(5,1)$, $v \in S(1,2)$ and $w \in S(3,4)$. By (5), $xu, xv \notin E$. If $xw \in E$, then $\{x, u, w, 4, 5\}$ induces a banner. So, $xw \notin E$. We have shown that $S(1)$ is anticomplete to S_2. But then $\{1\}$ is a clique cutset, contradicting Lemma 1. So, $S(4)$ and $S(5)$ are nonempty. By symmetry, $S(2)$ and $S(3)$

are nonempty. So, $S(i)$ is nonempty for each i and $S_2 = \emptyset$. By K_4-freeness of G and **(2)**, $|S(i)| \leq 2$ for each i and $5 \leq |S_1| \leq 7$. It can be seen that G is 3-colorable if $|S_1| \leq 6$. If $|S_1| = 7$, we may assume that $u_i \in S(i)$ for each i and $u_1' \in S(1)$ and $u_2' \in S(2)$. Then $\{1, 2, u_1, u_1', u_2, u_2', u_4\}$ induces the Hajós graph, a contradiction. □

5 5-Critical Graphs

In this section, we show that there are finitely many 5-critical $(P_6, banner)$-free graphs.

Theorem 5. *There are finitely many 5-critical $(P_6, banner)$-free graphs.*

We denote by $G+u$ the graph obtained from a 4-critical $(P_6, banner)$-free graph G by adding a new vertex u and making u adjacent to each vertex in G. Observe that each $G \in \mathcal{F}_4$ is 4-critical. Thus, $G + u$ is 5-critical $(P_6, banner)$-free graphs for any $G \in \mathcal{F}_4$. Therefore, it suffices to prove Theorem 5 for 5-critical $(P_6, banner)$-free graphs that are not isomorphic to $G + u$ for any $G \in \mathcal{F}_4$. We divide the proof of the theorem into Lemma 2 and Lemma 3 below. It follows from Lemma 2 and Lemma 3 that if a 5-critical $(P_6, banner)$-free graph contains an induced $\overline{C_7}$ or C_5, then G has finite order. The remaining case is that G is perfect and so G is isomorphic to K_5. Therefore, the theorem follows.

5.1 7-Antihole

Lemma 2. *Let G be a 5-critical $(P_6, banner)$-free graph. If G contains an induced $\overline{C_7}$, then G has finite order.*

Proof. Let $C = 1, 2, 3, 4, 5, 6, 7$ be an induced $\overline{C_7}$ such that $ij \in E$ if and only if $|i - j| > 1$. All indices are modulo seven. Since G is $\overline{C_7} + u$-free, $S_7 = \emptyset$. Let $x \in S(X)$ for some $X \subseteq C$ with $|X| = 3$. By symmetry, $X = \{i, i+1, i+2\}$, or $X = \{i, i+1, i+3\}$, or $X = \{i-2, i, i+2\}$ or $X = \{i-3, i, i+3\}$ for some i. We show that there is an induced $C_4 \subseteq C$ containing exactly one neighbor of x and this C_4 together with x gives an induced banner in G. Specifically, if $X = \{i, i+1, i+2\}$ or $X = \{i-2, i, i+2\}$, then $\{i-1, i, i+3, i-3, x\}$ induces a banner. If $X = \{i, i+1, i+3\}$ or $X = \{i-3, i, i+3\}$, then $\{i+2, i+3, i-2, i-1, x\}$ induces a banner. This shows that $S_3 = \emptyset$. Similarly, $S_1 = S_2 = \emptyset$.

We now show that $S_0 = \emptyset$. Suppose not. By the connectivity of G, some vertex $a \in A$ has a neighbor $n \in S_i$ for some $4 \le i \le 6$. It can be readily seen that there exists an index i such that n is adjacent to $i-3$ and $i+3$ but not to some j where $j \in \{i-1, i, i+1\}$. Now $\{i-3, i+3, j, n, a\}$ induces a banner, a contradiction. This proves that $S_0 = \emptyset$.

Let $X \subseteq C$ be an arbitrary set with $4 \le |X| \le 6$. Since $4 \le |X| \le 6$, there exists an index $i \in C$ such that $i, i+1 \in X$ but $i+2 \notin X$. If X contains two nonadjacent vertices x and y, then $\{i, i+1, i+2, x, y\}$ induces a banner. So, X is a clique. Since G is K_5-free, $|S(X)| \le 2$. Since $G = C \cup S_4 \cup S_5 \cup S_6$, $|G| \le 7 + 2(\binom{7}{4} + \binom{7}{5} + \binom{7}{6}) = 133$. $\square$

5.2 5-Hole

Lemma 3. *Let G be a 5-critical $(P_6, banner)$-free graph. If G contains an induced C_5, then G has finite order.*

Proof. Let $C = 1, 2, 3, 4, 5$ is an induced C_5. We partition $V \setminus C$ with respect to C. By **(1)**, $S(i-1, i, i+1)$ and $S(C \setminus \{i\})$ are cliques. Since G is K_5-free, each such set has size at most two. So, $|S_i| \le 10$ for $i = 3, 4$. since G contains no $W_5 + u$, it follows that S_5 is stable.

Claim 4. $|S_5| \le 20$.

Proof of Claim 4. If $|S_5| \le 1$, then the claim holds. So, assume that $|S_5| \ge 2$. By **(16)**, each vertex in S_0 is either complete or anticomplete to S_5, $S_1 \cup S_2$ is complete to S_5, and any vertex in $S_3 \cup S_4$ is adjacent to all but at most one vertex in S_5. If $|S_5| > |S_3 \cup S_4|$, then there exists a vertex $w \in S_5$ that is complete to $S_3 \cup S_4$. Let $u \in S_5$ with $u \neq w$. Then $N(u) \subseteq N(w)$, contradicting Lemma 1. So, $|S_5| \le |S_3 \cup S_4| \le 20$. $\square$

Claim 5. $|S_0| \le 120$.

Proof of Claim 5. Let A be a connected component of S_0. Since G is $(P_6, banner)$-free, A is anticomplete to $S_1 \cup S_2 \cup S_4$. Moreover, any vertex in S_3 is either complete or anticomplete to A. Let $X = N(A) \cap S_3$ and $Y = N(A) \cap S_5$. Suppose that $x \in X$ and $y \in Y$ are not adjacent. By definition, y has a neighbor $a \in A$. Note that $ax \in E$. Since $x \in S_3$, there exists $i \in C$ such that $ix \in E$ but $(i+2)x \notin E$. But then $\{a, i, i+2, x, y\}$ induces a banner. This shows that X and Y are complete.

Since G has no clique cutsets, $N(A) = X \cup Y$ contains two nonadjacent vertices s and t. Then either s and t are in S_3 or s and t are in S_5. We say that A is of *first type* if one can choose s and t such that both of them are in S_3 and of *second type* if the choice is not possible. Let n_1 and n_2 be the number of connected components of first and second type, respectively.

Suppose first that A is of first type. By **(1)**, no $S(i-1, i, i+1)$ contains both s and t. So, there exists an index $i \in C$ such that i is adjacent to exactly one of s and t. If A contains two nonadjacent vertices a and b, then $\{a, b, s, t, i\}$ induces a banner. So, A is a clique and so has size at most three. Moreover, the pigeonhole principle shows that if $n_1 > 40 \geq \binom{5}{2}|S(i-1, i, i+1)||S(j-1, j, j+1)|$, there are two such connected components A_1 and A_2 that correspond to the same pair of vertices s and t. Then $\{a_1, a_2, s, t, i\}$ induces a banner where $a_i \in A_i$ for $i = 1, 2$. So, $n_1 \leq 40$.

Now assume that A is of second type. Then X is a clique. Since $s, t \in S_5$, $|S_5| \geq 2$. We first show that A is complete to S_5. Let $S \subseteq A$ be the set of vertices that are complete to A and $T = A \setminus S$. By **(16)**, T is anticomplete to S_5. There is nothing to prove if T is empty. So, assume that T is nonempty. By the connectivity of A, there is an edge $xy \in E$ such that $x \in S$ and $y \in T$. But now $\{x, y, s, t, 1\}$ induces a banner. This proves that A is complete to S_5. Since G is 5-critical, $G - A$ and $N[A]$ admit a 4-coloring. Clearly, S_5 is monochromatic in any 4-coloring of $G - A$. Moreover, since any two vertices in S_5 have the same neighbors in $N[A]$ there exists a 4-coloring of $N[A]$ such that S_5 is monochromatic. Therefore, we can combine the two 4-colorings of $G - A$ and $G - N[A]$ by permuting colors. This contradicts that G is 5-chromatic. This shows that $n_2 = 0$, i.e., there is no connected component of second type.

Therefore, $|S_0| \leq 3 \cdot n_1 \leq 120$. □

Claim 6. *For each i, $|S(i)| \leq 771$.*

Proof of Claim 6. We prove the claim for $i = 1$. If $S(3) \cup S(4) \neq \emptyset$, then the claim follows from **(2)** and K_5-freeness of G. So, $S(3) \cup S(4) = \emptyset$. Suppose that $S(C \setminus \{1\})$ contains a vertex q. If $S(1)$ contains two nonadjacent vertices x and y, then $\{1, x, y, q, 3\}$ induces a banner. So, $S(1)$ is a clique and thus has size at most three.

In the following, we assume that $S(C \setminus \{1\}) = \emptyset$. Let $X \subseteq S(1)$ be the set of vertices that have a neighbor in $S(3,4)$ and $Y = S(3,4) \setminus X$. By **(5)**, each vertex $x \in N(X) \cap S(3,4)$ is universal in $S(3,4)$. Since G is K_5-free, $|N(X) \cap S(3,4)| \leq 2$. Let x and y be the possible vertices in $N(X) \cap S(3,4)$. Then $X \subseteq N(\{x,y\}) \cap S(1)$. If $N(x) \cap S(1)$ contains two nonadjacent vertices z and w, then $\{x,z,w,1,4\}$ induces a banner. So, $N(x) \cap S(1)$ is a clique. Similarly, $N(y) \cap S(1)$ is a clique. Therefore, $|X| \leq |N(x) \cap S(1)| + |N(y) \cap S(1)| \leq 6$.

We next bound Y by showing that the number of connected components of Y is bounded and the size of each connected component is also bounded. Let A be an arbitrary connected component of Y. Note first that since G is P_6-free, any vertex in $X \cup S_3$ is either complete or anticomplete to A. By **(13)**, each vertex in S_4 is either complete or anticomplete to A. Moreover, $N(A) \subseteq \{1\} \cup X \cup S_3 \cup S_4 \cup S_5$. Let $N_1 = N(A) \cap (\{1\} \cup X \cup S_3 \cup S_4)$ and $N_2 = N(A) \cap S_5$. Since G has no clique cutset, $N(A)$ contains two nonadjacent vertices s and t. We say that A is of *first type* if it is possible to choose s and t such that $s, t \in \{1\} \cup X \cup S_3 \cup S_4$, of *second type* if it is possible to choose s and t such that one of s and t belongs to S_5 and the other does not, and of *third type* if the previous two choices are not possible. Let n_1, n_2 and n_3 be the number of connected components of first, second or third type, respectively.

Suppose first A is of first type. By **(1)**, s and t do not belong to the same $S(i-1,i,i+1)$ or $S(C \setminus \{i\})$. It can be readily verified that there exists an index $i \in C \setminus \{1\}$ such that i is adjacent to exactly one of s and t. If A contains two nonadjacent vertices a and b, then $\{a,b,s,t,i\}$ induces a banner. So, any connected component of first type is a clique and so has size at most two. If $n_1 > 27^2 \geq |\{1\} \cup X \cup S_3 \cup S_4|^2$, then the pigeonhole principle implies that there are two such connected components A_1 and A_2 that correspond to the same pair of vertices s and t. Then $\{a_1, a_2, s, t, i\}$ induces a banner where $a_i \in A_i$ for $i = 1, 2$. So, $n_1 \leq 27^2$.

Suppose now that A is of second type. We may assume that $s \in S_5$ and $t \notin S_5$. Let $a \in A$ be a neighbor of s. Then $ta \in E$. Clearly, $t \in X \cup S_3 \cup S_4$. We first show that t must be in X. Suppose not. Then $t \in S_3 \cup S_4$. Since $S(C \setminus \{1\}) = \emptyset$, there exists $i \in C \setminus \{1\}$ such that $i+2 \neq 1$ and $it \in E$, $(i+2)t \notin E$. Then $\{a, i, s, t, i+2\}$ induces a banner. This proves that $t \in X$. By **(16)** and $st \notin E$, $S_5 = \{s\}$. If $n_2 > |X|$, then the pigeonhole principle implies that there are two such connected components A_1 and A_2 that correspond to the same pair of vertices s and t. Then $\{a_1, a_2, s, t, 2\}$ induces a banner where $a_i \in A_i$ is a neighbor of s for $i = 1, 2$. So, $n_2 \leq |X| \leq 6$. It remains to bound the size of A. Let $A_1 = N(s) \cap A$ and $A_2 = A \setminus A_1$. If A_1 contains two nonadjacent vertices a and b, then $\{s, t, a, b, 2\}$ induces a banner. So, A_1 is a clique and thus has at most two vertices. Recall that

t has a neighbor in $S(3,4)$, say x. Then $sx \notin E$ for otherwise $\{t, a, s, x, 2\}$ induces a banner. Let K be an arbitrary connected component of A_2. By connectivity, K has a neighbor $k \in A_1$. If k distinguishes an edge bc in K, then $b, c, k, s, 3, x$ induces a P_6. So, k is complete to K. Since $\{k, t, 1\}$ is complete to K, it follows that K is a singleton. We have proved that A_2 is a stable set. Note that only vertices in A_1 can distinguish two vertices in A_2. So, $|A_2| \leq 2^{|A_1|} = 4$ for otherwise G would contains a pair of comparable vertices. So, $|A| = |A_1| + |A_2| \leq 6$.

Finally, assume that A is of third type. Then s and t are in S_5, and so $|S_5| \geq 2$. Moreover, N_1 is a clique and is complete to N_2. By **(16)**, S_5 is complete to A and so $N_2 = S_5$. A similar argument in Claim 5 shows that $n_3 = 0$, i.e., there is no connected component of third type.

Therefore, $|S(1)| = |X| + |Y| \leq 6 + 2 \cdot 27^2 + 6 \cdot 6 = 771$. □

Claim 7. *For each i, $|S(i, i+1)| \leq 2^{13} + 8$.*

Proof of Claim 7. We prove the claim for $i = 3$. By **(3)** the K_5-freeness of G, the claim follows if $S(2,3) \cup S(4,5) \neq \emptyset$. Let $X \subseteq S(3,4)$ be the set of vertices that have a neighbor in $S_1 \cup (S_2 \setminus S(3,4))$ and $Y = S(3,4) \setminus X$. By **(4)** and **(5)**, each vertex in X is a universal vertex in $S(3,4)$. So, $|X| \leq 2$. If $X \neq \emptyset$, Y is stable since G is K_5-free. By **(14)**, no vertex in S_4 can distinguish two vertices in Y. By **(16)**, if a vertex in S_5 distinguishes two vertices in Y, S_5 contains at most one vertex. The pigeonhole principle implies that if $|S(3,4)| > 2^{11}$ then G contains a pair of comparable vertices. This contradicts Lemma 1. So, $|Y| \leq 2^{11}$ and thus $|S(3,4)| \leq 2 + 2^{11}$. In the following, we assume that $X = \emptyset$. In other words, $S(3,4)$ is anticomplete to $S_1 \cup (S_2 \setminus S(3,4))$. Similarly, if $S(C \setminus \{i\})$ for $i = 2, 5$ is nonempty, then $S(3,4)$ is a stable set by **(14)**. Then the pigeonhole principle shows that $|S(3,4)| \leq 2^{|S_3|+1} \leq 2^{11}$. So, $S(C \setminus \{2\}) = S(C \setminus \{5\}) = \emptyset$ and thus $S(3,4)$ is anticomplete to S_4.

Suppose first that $|S_5| \geq 2$. By **(16)**, S_5 is complete to $S(3,4)$ and thus $S(3,4)$ is stable. Then the pigeonhole principle shows that $|S(3,4)| \leq 2^{|S_3|} \leq 2^{10}$. Now assume that $|S_5| \leq 1$. Let $X \subseteq S(3,4)$ be the set of vertices that have a neighbor in $S(1,2,3) \cup S(4,5,1)$ and $Y = S(3,4) \setminus X$. Since G is banner-free, the set of vertices in $S(3,4)$ that are neighbors of a vertex in $S(1,2,3) \cup S(4,5,1)$ is a clique. So, $|X| \leq |S(1,2,3) \cup S(4,5,1)| \cdot 2 = 8$. We next bound Y. Let A be an arbitrary connected component of Y. Note that $N(A) \subseteq \{3,4\} \cup X \cup S(2,3,4) \cup S(3,4,5) \cup S_5$. Since G has no clique cutset, $N(A)$ contains two nonadjacent vertices s and t. Since $|S_5| \leq 1$, one of s and t is in $X \cup S(2,3,4) \cup S(3,4,5)$. It can be seen that s is complete to A since G is P_6-free. Since G is K_5-free, A is a singleton. This shows that Y is stable. Note that only vertices in $X \cup S(2,3,4) \cup S(3,4,5) \cup S_5$ can distinguish two vertices in Y. So, $|Y| \leq 2^{|X \cup S(2,3,4) \cup S(3,4,5) \cup S_5|} \leq 2^{13}$. Therefore, $|S(3,4)| = |X| + |Y| \leq 8 + 2^{13}$. □

The lemma follows from Claim 4-Claim 7. □

6 Conclusion

We have determined all 4-critical $(P_6, banner)$-free graphs and proved that there are finitely many 5-critical $(P_6, banner)$-free graphs. Our results generalize previous results on 4-critical (P_6, C_4)-free graphs and on the polynomial-time algorithm for 4-coloring $(P_6, banner)$-free graphs [14, 16]. It is still open whether there are finitely many k-critical $(P_6, banner)$-free graphs for fixed $k \geq 6$. In our proof for the case $k = 5$ we heavily rely on the fact that S_5 is stable which allows us to find comparable vertices in many cases of the proof. This fact simply does not hold anymore when $k \geq 6$. Therefore, to prove finiteness for $k \geq 6$ requires new techniques to handle S_5. In fact, it is not even known whether k-coloring $(P_6, banner)$-free graphs can be solved in polynomial time for any $k \geq 5$. Determining the complexity of k-coloring $(P_6, banner)$-free graphs could be a good first step to attack the problem. On the other hand, polynomial time algorithms for 4-coloring are also known for other subclasses of P_6-free graphs, for instance (P_6, chair)-free graphs [2] and (P_6, bull)-free graphs [19]. It would be interesting to see if there are finitely many 5-critical graphs with respect to the class of (P_6, chair)-free graphs or of (P_6, bull)-free graphs.

Acknowledgments. Shenwei Huang is partially supported by the National Natural Science Foundation of China (11801284). Tao Li is partially supported by the National Natural Science Foundation (61872200) and the National Key Research and Development Program of China (2018YFB1003405, 2016YFC0400709). Yongtang Shi was partially supported by National Natural Science Foundation of China (Nos. 11771221, 11811540390), Natural Science Foundation of Tianjin (No. 17JCQNJC00300), and the China-Slovenia bilateral project "Some topics in modern graph theory" (No. 12-6).

We thank one of the reviewers for pointing out a mistake in the proof of Theorem 4 and for his/her many constructive suggestions that improve the presentation of the paper greatly.

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