Total positivity of Narayana matrices

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Abstract

We prove the total positivity of the Narayana triangles of type A and type B, and thus affirmatively confirm a conjecture of Chen, Liang and Wang and a conjecture of Pan and Zeng. We also prove the strict total positivity of the Narayana squares of type A and type B.

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1. Introduction

Let M be a (finite or infinite) matrix of real numbers. We say that M is *totally positive* (TP) if all its minors are nonnegative, and we say that it is *strictly totally positive* (STP) if all its minors are positive. Total positivity is an important and powerful concept and arises often in analysis, algebra, statistics and probability, as well as in combinatorics. See [1, 6, 7, 9, 10, 14, 15, 19] for instance.

Let $C(n,k) = \binom{n}{k}$. It is well known [15, P. 137] that the Pascal triangle

$$P = [C(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

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is totally positive. Let

$$P^{\ulcorner} = [C(n+k,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \\ 1 & 3 & 6 & 10 & \\ 1 & 4 & 10 & 20 & \\ \vdots & & & & \ddots \end{bmatrix}$$

be the Pascal square. Then $P^{\ulcorner} = PP^T$ by the Vandermonde convolution formula

$$\binom{n+k}{k} = \sum_i \binom{n}{i}\binom{k}{i}.$$

Note that the transpose and the product of matrices preserve total positivity, see [19, Propositions 1.2 and 1.4]. Hence $P^{\ulcorner}$ is also TP.

The main objective of this note is to prove the following two conjectures on the total positivity of the Narayana triangles. Let $NA(n,k) = \frac{1}{k+1}\binom{n+1}{k}\binom{n}{k}$, which are commonly known as the Narayana numbers. Let

$$N_A = [NA(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 6 & 1 & & \\ 1 & 10 & 20 & 10 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

The Narayana numbers $NA(n,k)$ have many combinatorial interpretations. An interesting one is that they appear as the rank numbers of the poset of noncrossing partitions associated to a Coxeter group of type A, see Armstrong [2, Chapter 4]. For this reason, we call N_A the Narayana triangle of type A. Chen, Liang and Wang [10] proposed the following conjecture.

Conjecture 1.1 ([10, Conjecture 3.3])**.** *The Narayana triangle N_A is TP.*

Let $NB(n,k) = \binom{n}{k}^2$, and let

$$N_B = [NB(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 9 & 9 & 1 & & \\ 1 & 16 & 36 & 16 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

We call N_B the Narayana triangle of type B since the numbers $NB(n,k)$ can be interpreted as the rank numbers of the poset of noncrossing partitions associated to a Coxeter group of type B, see also Armstrong [2, Chapter 4] and references therein. Pan and Zeng [17] proposed the following conjecture.

Conjecture 1.2 ([17, Conjecture 4.1]). *The Narayana triangle N_B is TP.*

In this note, we will prove that the Narayana triangles N_A and N_B are TP just like the Pascal triangle in a unified approach. We also prove that the corresponding Narayana squares

$$N_A^{\ulcorner} = [NA(n+k,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & & \cdots \\ 1 & 3 & 6 & 10 & & \\ 1 & 6 & 20 & 50 & & \\ 1 & 10 & 50 & 175 & & \\ \vdots & & & & \ddots & \end{bmatrix}$$

and

$$N_B^{\ulcorner} = [NB(n+k,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 4 & 9 & 16 & \\ 1 & 9 & 36 & 100 & \\ 1 & 16 & 100 & 400 & \\ \vdots & & & & \ddots \end{bmatrix}$$

are STP, as well as the Pascal square.

2. The Narayana triangles

The main aim of this section is to prove the total positivity of the Narayana triangles N_A and N_B.

Before proceeding to the proof, let us first note a simple property of totally positive matrices. Let $X = [x_{n,k}]_{n,k\geq 0}$ and $Y = [y_{n,k}]_{n,k\geq 0}$ be two matrices. If there exist positive numbers a_n and b_k such that $y_{n,k} = a_n b_k x_{n,k}$ for all n and k, then we denote $x_{n,k} \sim y_{n,k}$ and $X \sim Y$. The following result is direct by definition.

Proposition 2.1. *Suppose that $X \sim Y$. Then the matrix X is TP (resp. STP) if and only if the matrix Y is TP (resp. STP).*

Our proof of Conjectures 1.1 and 1.2 is based on the Pólya frequency property of certain sequences. Let $(a_n)_{n\geq 0}$ be an infinite sequence of real numbers, and define its Toeplitz matrix as

$$[a_{n-k}]_{n,k\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Recall that $(a_n)_{n\geq 0}$ is said to be a *Pólya frequency* (PF) sequence if its Toeplitz matrix is TP. The following is the fundamental representation theorem for PF sequences, see Karlin [15, p. 412] for instance.

Schoenberg-Edrei Theorem. *A nonnegative sequence* $(a_0 = 1, a_1, a_2, \ldots)$ *is a PF sequence if and only if its generating function has the form*

$$\sum_{n\geq 0} a_n x^n = \frac{\prod_j (1+\alpha_j x)}{\prod_j (1-\beta_j x)} e^{\gamma x},$$

where $\alpha_j, \beta_j, \gamma \geq 0$ *and* $\sum_j (\alpha_j + \beta_j) < +\infty$.

Clearly, the sequence $(1/n!)_{n\geq 0}$ is a PF sequence by Schoenberg-Edrei Theorem, which implies that the corresponding Toeplitz matrix $[a_{n-k}]_{n,k\geq 0} = [1/(n-k)!]_{n,k\geq 0}$ is TP. Also, note that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \sim \frac{1}{(n-k)!}.$$

Hence the Pascal triangle P is TP by Proposition 2.1.

Based on a classic result of Laguerre on multiplier sequences, Chen, Ren and Yang [8, Proof of Conjecture 1.1] proved that the sequence $(1/((t)_n n!))_{n\geq 0}$ is a PF sequence for any $t > 0$, where $(t)_n = t(t+1)\cdots(t+n-1)$. As an immediate corollary, we have the following result.

Lemma 2.2. *For any* $m \geq 0$*, the Toeplitz matrix* $[1/((n-k+m)!(n-k)!)]_{n,k\geq 0}$ *is TP.*

Proof. Note that

$$\frac{1}{(n-k+m)!(n-k)!} \sim \frac{1}{(m+1)_{n-k}(n-k)!}$$

and the sequence $(1/((m+1)_n n!))_{n\geq 0}$ is a PF sequence. □

We are now in a position to prove Conjectures 1.1 and 1.2. In fact, we will prove the total positivity of some general triangles composed of m-Narayana numbers, which we will recall below. Fix an integer $m \geq 0$. For any $n \geq m$ and $0 \leq k \leq n-m$, the m-Narayana number $NA_{\langle m\rangle}(n,k)$ is given by

$$NA_{\langle m\rangle}(n,k) = \frac{m+1}{n+2}\binom{n+2}{k+1}\binom{n-m}{k}. \tag{2.1}$$

When $m = 0$ we get the usual Narayana numbers $NA(n,k)$. For more information on the numbers $NA_{\langle m\rangle}(n,k)$, see [22]. It is easy to show that each row of the Narayana triangle N_A is symmetric: $NA(n,k) = NA(n,n-k)$, but

$$N_{A,\langle m\rangle} = \left[NA_{\langle m\rangle}(n,k)\right]_{n\geq m;0\leq k\leq n-m}$$

and

$$\overleftarrow{N}_{A,\langle m\rangle} = \left[NA_{\langle m\rangle}(n,n-m-k)\right]_{n\geq m;0\leq k\leq n-m}$$

are two different triangles for $m \geq 1$. We obtain the following result.

Theorem 2.3. *For any $m \geq 0$, both $N_{A,\langle m\rangle}$ and $\overleftarrow{N}_{A,\langle m\rangle}$ are TP. In particular, the Narayana triangles N_A and N_B are TP.*

Proof. Note that for $n \geq m$ and $0 \leq k \leq n-m$ we have

$$N_{A,\langle m\rangle}(n,k) = \frac{m+1}{n+2}\binom{n+2}{k+1}\binom{n-m}{k} \sim \frac{1}{(n-k-m)!(n-k+1)!}.$$

Replacing n by $n+m$, we see that the total positivity of $N_{A,\langle m\rangle}$ is equivalent to that of the Toeplitz matrix $[1/((n-k+m+1)!(n-k)!)]_{n,k\geq 0}$. By Lemma 2.2, the triangle $N_{A,\langle m\rangle}$ is TP. The total positivity of $\overleftarrow{N}_{A,\langle m\rangle}$ will follow in the same manner as $N_{A,\langle m\rangle}$. In particular, the Narayana triangle $N_A = N_{A,\langle 0\rangle}$ is TP. Since

$$N_B(n,k) = \frac{n!^2}{k!^2(n-k)!^2} \sim \frac{1}{(n-k)!^2},$$

the total positivity of N_B immediately follows from Lemma 2.2. $\square$

3. The Narayana squares

The object of this section is to prove the total positivity of the Narayana squares $N_A^{\ulcorner}$ and $N_B^{\ulcorner}$. Our proof is based on the theory of Stieltjes moment sequences.

Given an infinite sequence $(a_n)_{n\geq 0}$ of real numbers, define its Hankel matrix as

$$[a_{n+k}]_{n,k\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & a_6 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

We say that $(a_n)_{n\geq 0}$ is a *Stieltjes moment* (SM) sequence if it has the form

$$a_n = \int_0^{+\infty} x^n d\mu(x),$$

where μ is a non-negative measure on $[0, +\infty)$. The following is a classic result on the Stieltjes moment problem, see [21, Theorem 1] and [19, Theorem 4.4].

Lemma 3.1. *A sequence $(a_n)_{n\geq 0}$ is an SM sequence of some measure on $[0, +\infty)$ with infinite support if and only if one of the following conditions holds:*

(i) *for any $n \geq 0$ both $[a_{i+j}]_{0\leq i,j\leq n}$ and $[a_{i+j+1}]_{0\leq i,j\leq n}$ are strictly positive definite; or*

(ii) *the Hankel matrix $[a_{i+j}]_{i,j\geq 0}$ is STP.*

Many well-known counting coefficients are Stieltjes moment sequences, see [16]. For example, the sequence $(n!)_{n\geq 0}$ is a Stieltjes moment sequence since

$$n! = \int_0^{+\infty} x^n e^{-x} dx = \int_0^{+\infty} x^n d\left(1 - e^{-x}\right).$$

Thus the corresponding Hankel matrix $[(n + k)!]_{n,k\geq 0}$ is STP. It is known that

$$\det[(i + j)!]_{0\leq i,j\leq n} = \prod_{i=0}^{n} i!^2, \qquad \det[(i + j + 1)!]_{0\leq i,j\leq n} = (n + 1)! \prod_{i=0}^{n} i!^2,$$

see [13, Theorem 4] and the sequence A059332 in [22]. Note that

$$\binom{n+k}{k} = \frac{(n + k)!}{n!k!} \sim (n + k)!.$$

Hence the Pascal square $P^{\ulcorner}$ is also STP.

To prove that the Narayana squares are STP, we need the following result.

Lemma 3.2. *For any $s, t \geq 0$, the Hankel matrix $[(n+k+s)!(n+k+t)!]_{n,k\geq 0}$ is STP.*

Proof. Note that the submatrix of an STP matrix is still STP. Hence if the sequence $(a_n)_{n\geq 0}$ is an SM sequence of some measure with infinite support, then so is its shifted sequence $(a_{n+m})_{n\geq 0}$ for any $m \geq 0$ by (i) of Lemma 3.1. Now the sequence $(n!)_{n\geq 0}$ is an SM sequence of some measure with infinite support, so are $((n+s)!)_{n\geq 0}$ and $((n+t)!)_{n\geq 0}$ for any $s, t \geq 0$. On the other hand, the famous Schur product theorem states that if $[a_{i,j}]_{i,j=0}^n$ and $[b_{i,j}]_{i,j=0}^n$ are two strictly positive definite matrices, then so is their Hadmard product $[a_{i,j}b_{i,j}]_{i,j=0}^n$, see §4.10.4 of [19, p. 123]. As a result, if both $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are SM sequences of some measures with infinite support, then so is $(a_n b_n)_{n\geq 0}$ by (ii) of Lemma 3.1. Thus we conclude that $((n+s)!(n+t)!)_{n\geq 0}$ is an SM sequence of some measure with infinite support, and hence the Hankel matrix $[(n+k+s)!(n+k+t)!]_{n,k\geq 0}$ is STP. $\square$

Define the m-th Narayana square as

$$N^{\ulcorner}_{A,\langle m\rangle} = \left[NA_{\langle m\rangle}(n+k,k)\right]_{n\geq m; k\geq 0},$$

where $NA_{\langle m\rangle}(n,k)$ is given by (2.1). The main result of this section is as follows.

Theorem 3.3. *For any $m \geq 0$, the square $N^{\ulcorner}_{A,\langle m\rangle}$ is STP. In particular, the Narayana squares $N^{\ulcorner}_{A}$ and $N^{\ulcorner}_{B}$ are STP.*

Proof. We have

$$\begin{aligned} NA_{\langle m\rangle}(n+k,k) &= \frac{m+1}{n+k+2}\binom{n+k+2}{k+1}\binom{n+k-m}{k} \\ &\sim (n-m+k)!(n+k+1)!. \end{aligned}$$

Replacing n by $n+m$, we see that the strict total positivity of $N^{\ulcorner}_{A,\langle m\rangle}$ is equivalent to that of the Hankel matrix $[(n+k)!(n+k+m+1)!]_{n,k\geq 0}$. By Lemma 3.2, the square $N^{\ulcorner}_{A,\langle m\rangle}$ is STP. In particular, the Narayana square $N^{\ulcorner}_{A}$ is STP. The strict total positivity of $N^{\ulcorner}_{B}$ will follow in the same manner as $N^{\ulcorner}_{A,\langle m\rangle}$ in view of that

$$NB(n+k,k) = \frac{(n+k)!^2}{n!^2 k!^2} \sim (n+k)!^2,$$

so the proof is complete. $\square$

4. Remarks

There are various generalizations of classical Narayana numbers, see for instance [2, 5, 11, 12, 18]. As we mentioned before, the numbers $NA(n,k)$ (resp. $NB(n,k)$) appear as the rank numbers of the poset of generalized noncrossing partitions associated to a Coxeter group of type A (resp. B). These posets are further generalized by Armstrong [2] by introducing the notion of m-divisible noncrossing partitions for any positive integer m and any finite Coxeter group. Armstrong also showed that these generalized posets are not lattices but are still graded.

Fixing an integer $m \geq 1$, for $n \geq k \geq 0$ set

$$\begin{aligned} FNA_{\langle m \rangle}(n,k) &= \frac{1}{n+1}\binom{n+1}{k}\binom{m(n+1)}{n-k} \\ FNB_{\langle m \rangle}(n,k) &= \binom{n}{k}\binom{mn}{n-k}. \end{aligned}$$

These numbers are called the Fuss-Narayana numbers by Armstrong [2], who proved that $FNA_{\langle m \rangle}(n,k)$ (resp. $FNB_{\langle m \rangle}(n,k)$) are the rank numbers of the poset of m-divisible noncrossing partitions associated to a Coxeter group of type A (resp. B). For $m=1$, the Fuss-Narayana numbers are just the ordinary Narayana numbers.

Note that, for any $m \geq 2$, we have

$$FNA_{\langle m \rangle}(n,k) \neq FNA_{\langle m \rangle}(n,n-k), FNB_{\langle m \rangle}(n,k) \neq FNB_{\langle m \rangle}(n,n-k).$$

Now define the Fuss-Narayana triangles

$$\begin{aligned} FN_{A,\langle m \rangle} &= \left[FNA_{\langle m \rangle}(n,k)\right]_{n,k\geq 0}, \quad \overleftarrow{FN}_{A,\langle m \rangle} = \left[FNA_{\langle m \rangle}(n,n-k)\right]_{n,k\geq 0}, \\ FN_{B,\langle m \rangle} &= \left[FNB_{\langle m \rangle}(n,k)\right]_{n,k\geq 0}, \quad \overleftarrow{FN}_{B,\langle m \rangle} = \left[FNB_{\langle m \rangle}(n,n-k)\right]_{n,k\geq 0} \end{aligned}$$

and the Fuss-Narayana squares

$$\begin{aligned} FN^{\ulcorner}_{A,\langle m \rangle} &= \left[FNA_{\langle m \rangle}(n+k,k)\right]_{n,k\geq 0}, \\ FN^{\ulcorner}_{B,\langle m \rangle} &= \left[FNB_{\langle m \rangle}(n+k,k)\right]_{n,k\geq 0}. \end{aligned}$$

For $2 \leq m \leq 10$, we have verified the total positivity of the submatrices composed of the first ten rows and columns of the Fuss-Narayana triangles, as well as the strict total positivity the submatrices composed of the first ten rows and columns of the Fuss-Narayana squares. We propose the following conjecture.

Conjecture 4.1. *For any $m \geq 2$, the Fuss-Narayana triangles are TP and the Fuss-Narayana squares are STP.*

There are other symmetric combinatorial triangles, which are TP and the corresponding squares are STP. The Delannoy number $D(n,k)$ is the number of lattice paths from $(0,0)$ to (n,k) using steps $(1,0),(0,1)$ and $(1,1)$. Clearly,

$$D(n,k) = D(n-1,k) + D(n-1,k-1) + D(n,k-1),$$

with $D(0,k) = D(k,0) = 1$. It is well known that the Narayana number $NA(n,k)$ counts the number of Dyck paths (using steps $(1,1)$ and $(1,-1)$) from $(0,0)$ to $(2n,0)$ with k peaks. It is also known that $NA_{\langle m \rangle}(n,k)$ counts the number of Dyck paths of semilength n whose last m steps are $(1,-1)$ with k peaks, see Callan's note in [22]. Brenti [6, Corollar 5.15] showed that the Delannoy triangle $D = [D(n-k,k)]_{n,k\geq 0}$ and the Delannoy square $D^{\ulcorner} = [D(n,k)]_{n,k\geq 0}$ are TP by means of lattice path techniques. The following problem naturally arises.

Question 4.2. *Whether the total positivity of Narayana matrices can also be obtained by a similar combinatorial approach?*

We have seen that the Pascal square has the decomposition $P^{\ulcorner} = PP^T$. We also have $D^{\ulcorner} = P\mathrm{diag}(1,2,2^2,\ldots)P^T$ since

$$D(n,k) = \sum_j 2^j \binom{k}{j}\binom{n}{j}$$

(see [4] for instance). A natural problem is to find out the explicit (modified) Choleski decomposition of the Narayana squares $N_A^{\ulcorner}$ and $N_B^{\ulcorner}$.

Another well-known symmetric triangle is the Eulerian triangle $A = [A(n,k)]_{n,k\geq 1}$ where $A(n,k)$ is the Eulerian number, which counts the number of n-permutations with exactly $k-1$ excedances. Brenti [7, Conjecture 6.10] conjectured that the Eulerian triangle A is TP. Motivated by the strict total positivity of the Narayana squares, we pose the following conjecture. We have verified the stirct total positivity of the submatrix of $A^{\ulcorner}$ composed of its first 13 rows and columns.

Conjecture 4.3. *The Eulerian square $A^{\ulcorner} = [A(n+k,k)]_{n,k\geq 1}$ is STP.*

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