On the complexity of k-rainbow cycle colouring problems

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Abstract

An edge-coloured cycle is *rainbow* if all edges of the cycle have distinct colours. For $k \geq 1$, let $\mathcal{F}_k$ denote the family of all graphs with the property that any k vertices lie on a cycle. For $G \in \mathcal{F}_k$, a *k-rainbow cycle colouring* of G is an edge-colouring such that any k vertices of G lie on a rainbow cycle in G. The *k-rainbow cycle index* of G, denoted by $crx_k(G)$, is the minimum number of colours needed in a k-rainbow cycle colouring of G. In this paper, we restrict our attention to the computational aspects of k-rainbow cycle colouring. First, we prove that the problem of deciding whether $crx_1 = 3$ can be solved in polynomial time, but that of deciding whether $crx_1 \leq \ell$ is NP-Complete, where $\ell \geq 4$. Then we show that the problem of deciding whether $crx_2 = 3$ can be solved in polynomial time, but those of deciding whether $crx_2 \leq 4$ or 5 are NP-Complete. Furthermore, we also consider the cases of $crx_3 = 3$ and $crx_3 \leq 4$. Finally, we prove that the problem of deciding whether a given edge-colouring (with an unbounded number of colours) of a graph is a k-rainbow cycle colouring, is NP-Complete for $k = 1$, 2 and 3, respectively. Some open problems for further study are mentioned.

Keywords: rainbow cycle; k-rainbow cycle colouring; k-rainbow cycle index; polynomial time; NP-Complete

AMS Subject Classification 2010: 05C15, 05C38, 68Q25.

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1. Introduction

We follow the terminology and notations of [3] and all graphs considered here are finite and simple.

A well-known result of Dirac [9] states that for $k \geq 2$, any k specified vertices in a k-connected graph are contained in a cycle. Bondy and Lovász [2] proved that for $k \geq 2$, any $k-1$ specified vertices in a k-connected non-bipartite graph are contained in an odd cycle; and for $k \geq 3$, any k specified vertices in a k-connected graph are contained in an even cycle. Bollobás and Brightwell [1] showed the following result: if $n \geq k \geq 3$ and $d \geq 1$ are such that $s = \lceil \frac{k}{\lceil n/d \rceil - 1} \rceil \geq 3$, then for any k vertices of degree at least d in a graph of order n, there exists a cycle containing at least s of the vertices. Very recently, Liu [18] studied the following problem: For $k \geq 1$, let $\mathcal{F}_k$ denote the family of all graphs G with the property that any k vertices of G belong to a cycle. An edge-coloured cycle is *rainbow* if all edges of the cycle have distinct colours. Consider an edge-colouring of $G \in \mathcal{F}_k$ such that, any k vertices are contained in a rainbow cycle. What is the minimum number of colours in such an edge-colouring?

For $G \in \mathcal{F}_k$, a *k-rainbow cycle colouring* of G is an edge-colouring such that any k vertices of G lie on a rainbow cycle in G. A 1-rainbow cycle colouring is simply called a *rainbow cycle colouring*. An edge-coloured graph G is *k-rainbow cycle connected* if its colouring is a k-rainbow cycle colouring. The *k-rainbow cycle index* of G, denoted by $crx_k(G)$, is the minimum number of colours needed in a k-rainbow cycle colouring of G. Thus, $crx_k(G)$ is well-defined if and only if $G \in \mathcal{F}_k$. In [18], Liu studied the k-rainbow cycle index for some special classes of graphs.

This concept is related to the concept of rainbow connection number of graphs, which was introduced by Chartrand et al. [6]. An edge-coloured graph is *rainbow* if the colours of its edges are distinct. An edge-coloured graph G is *rainbow connected* if any two vertices are connected by a rainbow path. In this case, the colouring is called a *rainbow colouring* of G. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the minimum number of colours that are needed in order to make G rainbow connected. Later, Krivelevich and Yuster [11] extend the concept of rainbow connection to the vertex version. For more results on rainbow connection and rainbow vertex connection, we refer to the survey [17] and some recent papers for digraphs [14, 15].

The computational complexity of the rainbow (vertex-) connection number has been studied extensively. In [4], Caro et al. conjectured that computing the rainbow connection number is an NP-Hard problem, as well as that even deciding whether a graph has rainbow connection number 2 is NP-Complete, which was confirmed by Chakraborty et al. [5]. For the rainbow vertex-connection number, Chen et al. [8] showed that for a graph G, deciding whether the rainbow vertex connection number equals to 2 is NP-Complete. Actually, there are many other results on this topic, we refer to the recent papers [10, 16, 12] and the PhD thesis of Juho Lauri [13].

In this paper, we restrict our attention to the computational aspects of k-rainbow cycle colouring of graphs. In Section 2, we prove that the problem of deciding whether $crx_1 = 3$ can be solved in polynomial time, but that of deciding whether $crx_1 \leq \ell$ is NP-Complete, where $\ell \geq 4$. In Section 3, we show that the problem of deciding whether $crx_2 = 3$ can be solved in polynomial time, but those of deciding whether $crx_2 \leq 4$ and 5 are NP-Complete. In Section 4, we show that it is easy to check whether $crx_3 = 3$ and $crx_3 \leq 4$. In the last section, we turn to the problem of deciding whether the given edge-colouring (with an unbounded number of colours) of a graph is a k-rainbow cycle colouring and we prove that the problem is NP-Complete for $k = 1$, 2 and 3.

2. 1-Rainbow cycle index

In this section, we consider the problem of determining whether a given graph G has a 1-rainbow cycle colouring with ℓ colours, that is, determining whether $crx_1(G) \leq \ell$, where $\ell \geq 3$.

We first present a polynomial-time algorithm for the case $\ell = 3$ of the above problem.

Algorithm: Deciding Whether $crx_1(G) \leq 3$

INPUT: a graph $G = (V, E)$

OUTPUT: a 1-rainbow cycle colouring function c of G with three colours $1, 2$ and 3 or the conclusion that $crx_1(G) > 4$

1: *set* $\chi := \emptyset,\ F := \emptyset$

2: **while** $V \backslash F$ *is nonempty* **do**

3: *pick a vertex* v *from* $V \backslash F$

4: **if** *v has two adjacent neighbours u and w* **then**

5: **if** $uw \in \chi$ **then**

6: *colour vu and vw such that* $\{c(vu), c(vw)\} = \{1, 2, 3\} \backslash \{c(uw)\}$

7: *replace* χ *by* $\chi \cup \{vu, vw\}$
8: **else**
9: *set* $c(vu) := 1$, *set* $c(vw) := 2$ *and set* $c(uw) := 3$
10: *replace* χ *by* $\chi \cup \{vu, vw, uw\}$
11: **end if**
12: *replace* F *by* $F \cup \{v, u, w\}$
13: **else**
14: *return* $(crx_1(G) > 4)$
15: **end if**
16: **end while**
17: **while** $E \backslash \chi$ *is nonempty* **do**
18: *pick an edge* e *from* $E \backslash \chi$
19: *set* $c(e) := 1$
20: *replace* χ *by* $\chi \cup \{e\}$
21: **end while**
22: *return* (c)

Note that, in the above algorithm, if a vertex $v \in F$, then there is a rainbow triangle containing v. For step 5, if an edge $uw \in \chi$, then uw has been coloured and the vertices $u, w \in F$. Conversely, if $v \notin F$, then the edges adjacent to v do not belong to χ.

Moreover, the running time of the above algorithm is bounded by $\mathcal{O}(n^3 m)$. Thus, we have the following theorem:

Theorem 2.1. *Given a graph G, the problem of deciding whether $crx_1(G) \leq 3$ can be solved in polynomial time.*

For $\ell \geq 4$, the problem of deciding whether $crx_1(G) \leq \ell$ turns out to be NP-Complete. We establish its NP-completeness by reducing the following problem to it. **The ℓ-vertex-colouring problem:** given a graph G and an integer ℓ, decide whether there exists an assignment of at most ℓ colours to the vertices of G such that no pair of adjacent vertices are coloured the same, namely whether $\chi(G) \leq \ell$. It is known that this ℓ-vertex-colouring problem is NP-Complete for $\ell \geq 3$.

Theorem 2.2. *Given a graph G and an integer $\ell \geq 4$, the problem of deciding whether $crx_1(G) \leq \ell$ is NP-Complete.*

Proof. Clearly the problem belongs to NP. Now given an instance $G = (V, E)$ of the ℓ-vertex-colouring problem, we construct a graph $G' = (V', E')$ such that $\chi(G) \leq \ell$ if and only if $crx_1(G') \leq \ell$.

We start by constructing a star, with one leaf vertex corresponding to every vertex $v \in V(G)$ and an additional central vertex a. Then for every edge $v_iv_j \in E(G)$, add a v_iv_j-path of length $\ell - 2$: $v_iu^1_{i,j}u^2_{i,j} \dots u^{\ell-3}_{i,j}v_j$. For every pair v_i, v_j of vertices such that $v_iv_j \notin E(G)$, add two v_iv_j-paths of length 2: $v_iw^1_{i,j}v_j$ and $v_iw^2_{i,j}v_j$.

More formally, the vertex set V' of G' is defined as follows:

$$V' = V \cup \{a\} \cup U \cup W$$

$$U = \{u^1_{i,j}, u^2_{i,j}, \dots, u^{\ell-3}_{i,j} : \; v_iv_j \in E(G) \text{ and } i < j\}$$
$$W = \{w^1_{i,j}, w^2_{i,j} : \; v_iv_j \notin E(G) \text{ and } i < j\},$$

and the edge set E' is defined as follows:

$$E' = E_1 \cup E_2 \cup E_3$$

$$E_1 = \{av_i : \; v_i \in V\}$$
$$E_2 = \{v_iu^1_{i,j}, u^t_{i,j}u^{t+1}_{i,j}, u^{\ell-3}_{i,j}v_j : \; 1 \leq t \leq \ell - 4, v_iv_j \in E(G) \text{ and } i < j\}$$
$$E_3 = \{v_iw^1_{i,j}, w^1_{i,j}v_j, v_iw^2_{i,j}, w^2_{i,j}v_j : \; v_iv_j \notin E(G) \text{ and } i < j\}.$$

Now, if $crx_1(G') \leq \ell$, let c' be a 1-rainbow cycle colouring of G' using ℓ colours. We define the vertex-colouring c of G by $c(v_i) = c'(av_i)$, for every $v_i \in V$. Note that, for each edge $v_iv_j \in E(G)$ and the vertex $u^1_{i,j} \in U \subseteq V'$, there is only one cycle of length at most ℓ containing the vertex $u^1_{i,j}$, that is, $v_iu^1_{i,j}u^2_{i,j} \dots u^{\ell-3}_{i,j}v_jav_i$. So $c'(av_i) \neq c'(av_j)$, and thus $c(v_i) \neq c(v_j)$, that is, c is a proper ℓ-vertex-colouring.

In the other direction, assume that $\chi(G) \leq \ell$ and $c : V \to \{1, 2, \dots, \ell\}$ is a proper ℓ-vertex-colouring of G. Define the edge-colouring c' of G' as follows:

• For each edge $av_i \in E_1$, we set $c'(av_i) = c(v_i)$.

• For any $v_iv_j \in E(G)$, $c(v_i) \neq c(v_j)$, and so $c'(av_i) \neq c'(av_j)$. Except the colours $c(v_i)$ and $c(v_j)$, colour the v_iv_j-path $v_iu^1_{i,j} \dots u^{\ell-3}_{i,j}v_j$ with the remaining $(\ell - 2)$ colours such that no two edges on the path have the same colour, that is, $\{c'(av_i), c'(v_iu^1_{i,j}), \dots, c'(u^{\ell-3}_{i,j}v_j), c'(av_j)\} = \{1, 2, \dots, \ell\}$.

• Finally, for any $v_iv_j \notin E(G)$, set $c'(v_iw^1_{i,j}) = 1$, $c'(v_iw^2_{i,j}) = 2$, $c'(w^1_{i,j}v_j) = 3$ and $c'(w^2_{i,j}v_j) = 4$.

It is easy to verify that this edge-colouring c' is indeed a 1-rainbow cycle colouring of G' using ℓ colours. This completes the proof. ∎

3. 2-Rainbow cycle index

Next, we consider the problem of determining whether $crx_2(G) \leq \ell$, where $\ell \geq 3$.

In [18], Liu proved the following result for complete graphs, which will be used in the later proof. For completeness, we include the proof in [18] here.

Lemma 3.1 ([18]). *For $n \geq 3$, $crx_1(K_n) = crx_2(K_n) = 3$.*

Proof. It suffices to show that $crx_2(K_n) \leq 3$ for each $n \geq 3$. We use induction on n to show that there is a 2-rainbow cycle colouring of K_n using three colours. For $n = 3$, we simply take the rainbow-coloured K_3. Now suppose $n \geq 4$ and let u be a vertex of $G \cong K_n$. By induction, there exists a 2-rainbow cycle colouring c' for $G - u \cong K_{n-1}$, using three colours. Define a colouring c of G as follows. If $n-1$ is even, then take a perfect matching M of $G - u$, and for $vw \in M$, let $c(uv)$ and $c(uw)$ be the two colours different from $c'(vw)$. If $n-1$ is odd, then take $x_1, x_2, x_3 \in V(G - u)$ such that $c'(x_1x_2)$, $c'(x_2x_3)$ and $c'(x_3x_1)$ are distinct, and take the perfect matching M' of $G - \{u, x_, x_2, x_3\}$. For the colouring c, colour the edges from u to the vertices of M' in the same way as before, and let $c(ux_i) = c'(x_{i+1}x_{i+2})$ for $i = 1, 2, 3$ (indices are taken modulo 3). Clearly c also uses three colours. Now by induction, any two vertices of $G - u$ lie in a rainbow triangle. If v is a vertex of M ($n-1$ even) or M' ($n-1$ odd), then u and v lie in the rainbow triangle formed by u and the edge of M or M' incident with v. For $n-1$ odd, ux_ix_{i+1} is a rainbow triangle for $i = 1, 2, 3$. Hence, c is a 2-rainbow cycle colouring of K_n, and we are done by induction. ∎

For $\ell = 3$, we have the following result:

Theorem 3.1. *For a given graph G, $crx_2(G) = 3$ if and only if G is a complete graph of order at least 3.*

Proof. If G is a complete graph of order at least 3, by Lemma 3.1 we know that $crx_2(G) = 3$.

In the other direction, clearly $|V(G)| \geq 3$. Assume that G is not a complete graph, that is, there is a pair (u, v) of vertices such that $uv \notin E(G)$. Then the shortest cycle containing u and v has length at least 4. Thus, we have $crx_2(G) \geq 4$, a contradiction. ∎

Therefore, given a graph G, we can decide whether $crx_2(G) = 3$ in polynomial time. However, for $\ell = 4$, we can show that the problem is NP-Complete. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Theorem 3.2. *Given a graph G, the problem of deciding whether $crx_2(G) \leq 4$ is NP-Complete.*

Proof. Clearly the problem is in NP. We prove the NP-Completeness by reducing "the 4-vertex-colouring problem" to it. Let $G = (V, E)$ be an instance of the 4-vertex-colouring problem, where $V = \{v_1, v_2, \ldots, v_n\}$. We construct a graph $G' = (V', E')$ such that $\chi(G) \leq 4$ if and only if $crx_2(G') \leq 4$.

The vertex set V' of G' is defined as follows:

$$V' = \overline{V} \cup U \cup W^1 \cup W^2$$

$$\overline{V} = V \cup \{v_{n+1}\},\ U = \{u_{i,j} :\ v_i v_j \in E(G) \textit{ and } i < j\} \cup \{a\}$$
$$W^1 = \{w^1_{i,j}, w^2_{i,j} :\ v_i v_j \notin E(G) \textit{ and } i < j\},$$
$$W^2 = \{w^1_{i,n+1}, w^2_{i,n+1} :\ i \in [n+1]\}.$$

And the edge set E' of G' is defined as follows:

$$E' = E_1 \cup E^1_2 \cup E^2_2 \cup E^3_2 \cup E_3 \cup E^1_4 \cup E^2_4 \cup E_5$$

$$E_1 = \{xy :\ x, y \in U\}$$
$$E^1_2 = \{xy :\ x, y \in W^1\},\ E^2_2 = \{xy :\ x \in W^1, y \in W^2\}$$
$$E^3_2 = \{w^k_{i,n+1} w^l_{j,n+1} :\ 1 \leq i < j \leq n+1, 1 \leq k \leq 2, 1 \leq l \leq 2\}$$
$$E_3 = \{v_i u_{i,j}, u_{i,j} v_j :\ v_i v_j \in E(G) \textit{ and } i < j\} \cup \{a v_i :\ i \in [n]\}$$
$$E^1_4 = \{v_i w^1_{i,j}, w^1_{i,j} v_j, v_i w^2_{i,j}, w^2_{i,j} v_j :\ v_i v_j \notin E(G) \textit{ and } i < j\}$$
$$E^2_4 = \{v_i w^1_{i,n+1}, v_{n+1} w^1_{i,n+1}, v_i w^2_{i,n+1}, v_{n+1} w^2_{i,n+1} :\ i \in [n]\} \cup$$
$$\{v_n w^1_{n+1,n+1}, v_{n+1} w^1_{n+1,n+1}, v_n w^2_{n+1,n+1}, v_{n+1} w^2_{n+1,n+1}\}$$
$$E_5 = \{xy :\ x \in U, y \in W^1 \cup W^2\}.$$

Now, suppose $crx_2(G') \leq 4$ and c' is a 2-rainbow cycle colouring of G' using 4 colours. We define the vertex-colouring c of G by $c(v_i) = c'(a v_i)$, for every $v_i \in V$. Note that, $G'[\overline{V}]$ is an empty graph and moreover, for any $v_i v_j \in E(G)$, only the vertices $u_{i,j}$ and a are adjacent to both v_i and v_j. Therefore, there is only one cycle of length at most 4 containing both v_i and v_j, namely $v_iu_{i,j}v_jav_i$. So $c'(av_i) \neq c'(av_j)$, and thus $c(v_i) \neq c(v_j)$, that is, c is a proper 4-vertex-colouring.

In the other direction, assume that $\chi(G) \leq 4$ and $c : V \to \{c_1, c_2, c_3, c_4\}$ is a proper 4-vertex-colouring of G. We define the edge-colouring c' of G' as follows:

• Since $G'[U]$ and $G'[W^1]$ are complete graphs, by Lemma 3.1, we can colour E_1 and E_2^1 by c_1, c_2 and c_3 such that $c'(E_1)$ and $c'(E_2^1)$ are 2-rainbow cycle colourings of $G'[U]$ and $G'[W^1]$, respectively.

• For each edge $xy \in E_2^2$, where $x \in W^1$ and $y \in W^2$, if $y = w_{i,n+1}^1$, set $c'(xy) = c_3$; otherwise, $y = w_{i,n+1}^2$ and set $c'(xy) = c_4$, for any $i \in \{1, \ldots, n, n+1\}$.

• For the edges in E_2^3, we set $c'(w_{i,n+1}^1 w_{j,n+1}^1) = c'(w_{i,n+1}^2 w_{j,n+1}^2) = c_3$, and $c'(w_{i,n+1}^1 w_{j,n+1}^2) = c'(w_{i,n+1}^2 w_{j,n+1}^1) = c_4$, where $1 \leq i < j \leq n+1$.

• For each edge $av_i \in E_3$, set $c'(av_i) = c(v_i)$.

For any $v_iv_j \in E(G)$, since c is a proper vertex-colouring, $c(v_i) \neq c(v_j)$, and so $c'(av_i) \neq c'(av_j)$. Thus we can colour the edges $v_iu_{i,j}$ and $v_ju_{i,j}$ by $\{c_1, c_2, c_3, c_4\} \backslash \{c'(av_i), c'(av_j)\}$ such that $v_iu_{i,j}v_jav_i$ is a rainbow cycle.

• For any $v_iv_j \notin E(G) (1 \leq i < j \leq n)$, set $c'(v_iw_{i,j}^1) = c_1$, $c'(v_iw_{i,j}^2) = c_2$, $c'(w_{i,j}^1 v_j) = c_3$ and $c'(w_{i,j}^2 v_j) = c_4$.

• For the edges in E_4^2, set $c'(v_iw_{i,n+1}^1) = c_1$ and $c'(v_iw_{i,n+1}^2) = c_2$, for $1 \leq i \leq n+1$; set $c'(v_{n+1}w_{i,n+1}^1) = c_3$ and $c'(v_{n+1}w_{i,n+1}^2) = c_4$, for $1 \leq i \leq n$; set $c'(v_nw_{n+1,n+1}^1) = c_3$ and $c'(v_nw_{n+1,n+1}^2) = c_4$.

• Finally, for each edge $xy \in E_5$, where $x \in U$ and $y \in W^1 \cup W^2$, if $y = w_{i,j}^1$, set $c'(xy) = c_3$; otherwise, $y = w_{i,j}^2$ and set $c'(xy) = c_4$.

Next, we show that the 4-edge-colouring c' is a 2-rainbow cycle colouring of G'. Let x, y be any two vertices of G'.

Case 1: $x, y \in \overline{V}$.

If $\{x, y\} = \{v_i, v_j\}$, where $v_iv_j \in E(G)$, then $v_iu_{i,j}v_jav_i$ is a rainbow cycle containing x and y.

If $\{x, y\} = \{v_i, v_j\}$, where $v_iv_j \notin E(G)$, then $v_iw_{i,j}^1v_jw_{i,j}^2v_i$ is a rainbow cycle containing x and y.

If $\{x, y\} = \{v_i, v_{n+1}\}$, where $1 \leq i \leq n$, then $v_iw_{i,n+1}^1v_{n+1}w_{i,n+1}^2v_i$ is a rainbow cycle containing x and y.

Case 2: $x, y \in U$ or W^1.

$c'(E_1)$ and $c'(E_2^1)$ are indeed 2-rainbow cycle colourings of $G'[U]$ and $G'[W^1]$, respectively.

Case 3: $x, y \in W^2$ or $x \in W^1$ and $y \in W^2$.

If $\{x, y\} = \{w^1_{i,n+1}, w^2_{i,n+1}\}$ or $\{x, y\} = \{w^k_{i,n+1}, w^l_{j,n+1}\}$, where $1 \le i \ne j \le n+1, 1 \le k \le 2$ and $1 \le l \le 2$, then $w^1_{i,n+1}v_i w^2_{i,n+1} w^l_{j,n+1} w^1_{i,n+1}$ is a rainbow cycle containing x and y.

If $x \in W^1$ and $y = w^k_{i,n+1} \in W^2$, where $1 \le i \le n+1$ and $1 \le k \le 2$, then $xw^1_{i,n+1}v_i w^2_{i,n+1}x$ is a rainbow cycle containing x and y.

Case 4: $x \in U$ and $y \in \overline{V} \cup W^1 \cup W^2$.

If $y = v_i \in \overline{V}$ or $y = w^k_{i,n+1} \in W^2$, where $1 \le i \le n+1$ and $1 \le k \le 2$, then $v_i w^1_{i,n+1} x w^2_{i,n+1} v_i$ is a rainbow cycle containing x and y.

If $y = w^k_{i,j} \in W^1$, where $v_iv_j \notin E(G)$, $i < j$ and $1 \le k \le 2$, then $v_i w^1_{i,j} x w^2_{i,j} v_i$ is a rainbow cycle containing x and y.

Case 5: $x \in \overline{V}$ and $y \in W^1 \cup W^2$.

Let $x = v_i (1 \le i \le n+1)$.

If $y \in W^1$, then $v_i w^1_{i,n+1} y w^2_{i,n+1} v_i$ is a rainbow cycle containing x and y.

If $y = w^k_{j,n+1} \in W^2$, where $1 \le j \le n+1$, $i \ne j$ and $1 \le k \le 2$, then $v_i w^1_{i,n+1} w^k_{j,n+1} w^2_{i,n+1} v_i$ is a rainbow cycle containing x and y.

If $y = w^1_{i,n+1}$ or $w^2_{i,n+1} \in W^2$, choose any one vertex z from W^1 and then $v_i w^1_{i,n+1} z w^2_{i,n+1} v_i$ is a rainbow cycle containing x and y.

We have considered all the cases and so c' is indeed a 2-rainbow cycle colouring of G'. The proof is complete. ∎

By means of a similar construction as that in the proof of Theorem 3.2, we can obtain a polynomial reduction of "the 5-vertex-colouring problem" to the following problem.

Theorem 3.3. *Given a graph G, the problem of deciding whether $crx_2(G) \le 5$ is NP-Complete.*

Proof. Let $G = (V, E)$ be an instance of the 5-vertex-colouring problem, where $V = \{v_1, v_2, \ldots, v_n\}$. We can construct a graph G' similar to the graph in Theorem 3.2, except that $U = \{u^1_{i,j}, u^2_{i,j} : \ v_iv_j \in E(G) \ and \ i < j\} \cup \{a\}$ and $E_3 = \{v_i u^1_{i,j}, u^2_{i,j} v_j : \ v_iv_j \in E(G) \ and \ i < j\} \cup \{av_i : \ i \in [n]\}$. Moreover, the edges in E_1 still form a complete graph $G'[U]$ and the edges in E_5 still form a complete bipartite graph between the vertices in U and $W^1 \cup W^2$. The other vertex sets and edge sets remain unchanged.

Note that, now $G'[\overline{V}]$ is still an empty graph and for any $v_iv_j \in E(G)$, only one vertex a is adjacent to both v_i and v_j. Therefore, any cycle of length at most

5 containing both v_i and v_j must contain the edges av_i and av_j. Thus, for any 2-rainbow cycle colouring c' of G' using 5 colours, $c'(av_i) \neq c'(av_j)$. Similarly, define the vertex-colouring c of G by $c(v_i) = c'(av_i)$, for every $v_i \in V(G)$, and c is indeed a proper 5-vertex-colouring.

In the other direction, let $c : V \to \{c_1, c_2, c_3, c_4, c_5\}$ is a proper 5-vertex-colouring of G. The edge-colouring c' of G' we will define is similar to that in the proof of Theorem 3.2.

In Theorem 3.2, we colour E_1 by c_1, c_2 and c_3 such that $c'(E_1)$ is 2-rainbow cycle colouring of $G'[U]$. Now, for any $u^k_{i,j}u^l_{i',j'} \in E_1$ and $u^k_{i,j}a \in E_1$, where $\{i, j\} \neq \{i', j'\}$, $1 \leq k \leq 2$ and $1 \leq l \leq 2$, let $c'(u^k_{i,j}u^l_{i',j'})$ and $c'(u^k_{i,j}a)$ be the colours of $u_{i,j}u_{i',j'}$ and $u_{i,j}a$ in Theorem 3.2, respectively.

For each edge $av_i \in E_3$, we still set $c'(av_i) = c(v_i)$. For any $v_iv_j \in E(G)$, since $c(v_i) \neq c(v_j)$, $c'(av_i) \neq c'(av_j)$. Now colour the edges $v_iu^1_{i,j}$, $u^1_{i,j}u^2_{i,j}$ and $v_ju^2_{i,j}$ by $\{c_1, c_2, c_3, c_4, c_5\} \backslash \{c'(av_i), c'(av_j)\}$ such that $v_iu^1_{i,j}u^2_{i,j}v_jav_i$ is a rainbow cycle.

For each edge $xy \in E_5$, where $x \in U$ and $y \in W^1 \cup W^2$, if $y = w^1_{i,j}$, we still set $c'(xy) = c_3$; otherwise, $y = w^2_{i,j}$ and we still set $c'(xy) = c_4$.

The colours of the other edges remain unchanged.

Similarly, it can be verified that this edge-colouring c' is indeed a 2-rainbow cycle colouring of G' using five colours. The proof is complete. ∎

4. 3-Rainbow cycle index

For a given graph G, it is easy to check whether $crx_3(G) = 3$.

Theorem 4.1. *For a given graph G, $crx_3(G) = 3$ if and only if $G \in \{K_3, K_4\}$.*

Proof. Obviously $crx_3(K_3) = 3$. For K_4, let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Define an edge-colouring c of K_4 as follows: $c(v_1v_2) = c(v_3v_4) = 1$, $c(v_1v_3) = c(v_2v_4) = 2$ and $c(v_1v_4) = c(v_2v_3) = 3$. It is easy to check that c is a 3-rainbow cycle colouring of K_4, and so $crx_3(K_4) = 3$.

Conversely, it is clear that if $crx_3(G) = 3$, then G is a complete graph and $|V(G)| \geq 3$. Now, suppose that $G = K_n$, where $n \geq 5$. For any edge-colouring of K_n with 3 colours, since $\delta = n - 1 \geq 4$, there always exists a triangle $S \subseteq V(K_n)$, two edges of which have the same colour. Thus there is no rainbow cycle of length 3 containing S, and so $crx_3(K_n) \geq 4$, for $n \geq 5$. The proof is complete. ∎

Next, we consider the problem of deciding whether $crx_3(G) = 4$. Recall that for positive integers t_i, $1 \leq i \leq k$, the *Ramsey number* $r(t_1, t_2, \ldots, t_k)$ is the smallest integer n such that every k-edge-colouring $(E_1, E_2, \ldots, E_k)$ of K_n contains a complete subgraph on t_i vertices all of whose edges belong to E_i, for some i, $1 \leq i \leq k$. In particular, the Ramsey number $r_k = r(t_1, t_2, \ldots, t_k)$, where $t_i = 3$, $1 \leq i \leq k$, satisfies the following inequality ([3]).

Lemma 4.1. $r_k \leq \lfloor k!e \rfloor + 1$.

Now, we give the following theorem.

Theorem 4.2. *Given a graph G, if $|V(G)| \geq 66$, then $crx_3(G) > 4$.*

Proof. Let $|V(G)| = n$, and then $G \subseteq K_n$. By Lemma 4.1, $r_4 \leq \lfloor 4!e \rfloor + 1 \leq 66$. If $n \geq 66 \geq r_4$, then for any edge-colouring of K_n with 4 colours, there always exists a monochromatic copy of K_3, say with vertex set S. Thus any cycle of length at most 4 containing S must contain at least two edges with end-vertices in S, and so cannot be rainbow. Hence, $crx_3(G) \geq crx_3(K_n) > 4$. The proof is complete. ∎

Note that for $|V(G)| \leq 65$, we can use enumeration method and check all the possible colourings of $E(G)$. Though the time is bounded by a fixed integer t, it is possible that t is too big and unbearable. Thus, it is necessary to find a more effective algorithm to decide whether $crx_3(G) = 4$, for $|V(G)| \leq 65$.

For $\ell \geq 5$, the complexity of the problem of deciding whether $crx_3(G) \leq \ell$ remains unknown.

5. Rainbow cycle colouring

Suppose we are given an edge-colouring of the graph. Is it then easier to verify whether the colouring is a 1-rainbow cycle colouring? Clearly, if the number of colours is constant, then this problem becomes easy, simply by means of an exhaustive search. However, if the colouring is arbitrary, the problem becomes NP-Complete.

Theorem 5.1. *Given an edge-coloured graph G, the problem of checking whether the given colouring is a 1-rainbow cycle colouring is NP-Complete.*

We establish its NP-Completeness by reducing the following problem from [5] to it.

Lemma 5.1. *The following problem is NP-Complete: Given an edge-coloured graph G and two vertices s, t of G, decide whether there is a rainbow path connecting s and t.*

Proof of Theorem 5.1. The problem clearly belongs to NP. Now given a graph $G = (V, E)$ with two special vertices s and t and an edge-colouring c of G, we construct a graph $G' = (V', E')$ and define an edge-colouring c' of G' such that s and t are rainbow connected in G under c if and only if c' is a 1-rainbow cycle colouring of G'.

Let $V = \{v_1, v_2, \ldots, v_{n-2}, v_{n-1} = s, v_n = t\}$ be the vertex set of the original graph G. We set $V' = V \cup U \cup W$, where $U = \{u_1, u_2, \ldots, u_{n-2}\}$ and $W = \{w_1, w_2, \ldots, w_{2n-4}\}$, and $E' = E \cup \{v_i u_i, u_i v_{i+1} : i \in [n-3]\} \cup \{v_{n-2} u_{n-2}, u_{n-2} v_1\} \cup \{s w_1, w_{2n-4} t\} \cup \{w_i w_{i+1} : i \in [2n-5]\}$.

The colouring c' is defined as follows:

- All edges $e \in E$ retain the original colours, namely $c'(e) = c(e)$.
- The edges sw_1, $\{w_i w_{i+1} : i \in [2n-5]\}$ and $w_{2n-4} t$ are coloured with $(2n-3)$ new colours $c'_1, c'_2, \ldots, c'_{2n-3}$, respectively.
- Finally, set $c'(v_i u_i) = c'_{2i-1}$ and $c'(u_i v_{i+1}) = c'_{2i}$, for $i \in [n-3]$. Set $c'(v_{n-2} u_{n-2}) = c'_{2n-5}$ and $c'(u_{n-2} v_1) = c'_{2n-4}$.

Obviously, if there is a rainbow path P from s to t in G under c, then $P \cup s w_1 w_2 w_3 \ldots w_{2n-4} t$ is a rainbow cycle containing $\{s, t\} \cup W$. Moreover, $v_1 u_1 v_2 u_2 v_3 \ldots v_{n-3} u_{n-3} v_{n-2} u_{n-2} v_1$ is a rainbow cycle containing $V \backslash \{s, t\} \cup U$. Thus, c' is indeed a 1-rainbow cycle colouring of G'.

Now, in turn, assume that c' is a 1-rainbow cycle colouring of G'. Then the vertex w_1 must lie on a rainbow cycle l in G'. Obviously, any cycle containing w_1, including the cycle l, must contain the path $s w_1 w_2 w_3 \ldots w_{2n-4} t$, which uses up all the new colours: $c'_1, c'_2, \ldots, c'_{2n-3}$. Thus $l \backslash \{w_1, w_2, \ldots, w_{2n-4}\}$ is actually a rainbow path from s to t in G under c. The proof is complete. ∎

Intuitively, given an edge-coloured graph, if the colouring is arbitrary, the problem of deciding whether the colouring is a k-rainbow cycle colouring ($k \geq 2$) is not easier than that of deciding whether the colouring is a 1-rainbow cycle colouring. Actually, we can show that, for $k = 2$ and 3, the problem is indeed NP-Complete.

Firstly, we prove the following claim.

Lemma 5.2. *The first problem defined below is polynomially reducible to the second one:*

Problem 1: *Given an edge-coloured graph G, decide whether any k vertices of G are connected by a rainbow path, where $k \geq 2$.*

Problem 2: *Given an edge-coloured graph G, decide whether the given colouring is a k-rainbow cycle colouring, where $k \geq 2$.*

Proof. Given a graph $G = (V, E)$ and an edge-colouring c of G, we construct a graph $G' = (V', E')$ and define an edge-colouring c' of G' such that for the resulting edge-coloured graph the answer for Problem 2 is "yes" if and only if the answer for Problem 1 for the original edge-coloured graph is "yes".

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Now we add two adjacent vertices a and b to G, and join a and b to each vertex of $V(G)$, that is, $V'(G') = V \cup \{a, b\}$ and $E'(G') = E \cup \{av_i, bv_i : i \in [n]\} \cup \{ab\}$.

The colouring c' is defined as follows:

- All edges $e \in E$ retain the original colours, namely $c'(e) = c(e)$.
- The edges $\{av_i : i \in [n]\}$ are coloured with a new colour c'_1.
- The edges $\{bv_i : i \in [n]\}$ are coloured with a new colour c'_2.
- The edge ab is coloured with a new colour c'_3.

Firstly, assume that for any k vertices of $V(G)$, there is a rainbow path P connecting the k vertices in G. Let x and y be the ends of P. Then, obviously $xPyabx$ is a rainbow cycle in G' containing the vertices a, b and the k vertices. Thus, it is easy to see that c' is indeed a k-rainbow cycle colouring of G'.

Conversely, assume that c' is a k-rainbow cycle colouring of G'. Thus, for any k vertices of $V(G) \subseteq V'(G')$, there is a rainbow cycle C containing the k vertices. If the vertex a belongs to C, since the edges $av_1, av_2, \ldots, av_n$ have the same colour, the vertex b must belong to C. Then $C \backslash \{a, b\}$ is a rainbow path connecting the k vertices in G. If the vertices $a, b \notin V(C)$, then the deletion of any one edge from C can result in a rainbow path connecting the k vertices in G. The proof is complete. ∎

The case $k = 2$ of Problem 1 is exactly the problem in the following lemma, and its NP-Completeness has been confirmed in [5].

Lemma 5.3. *The following problem is NP-Complete: Given an edge-coloured graph G, check whether the given colouring makes G rainbow connected.*

Obviously, Problem 2 is in NP. Then Lemma 5.3, combined with Lemma 5.2, yields the following theorem.

Theorem 5.2. *Given an edge-coloured graph G, the problem of checking whether the given colouring is a 2-rainbow cycle colouring is NP-Complete.*

Next, we establish the NP-Completeness of the case $k = 3$ of Problem 1 by reducing the problem in Lemma 5.1 to it.

Lemma 5.4. *The following problem is NP-Complete: Given an edge-coloured graph G, decide whether any three vertices of G are connected by a rainbow path.*

Proof. Given a graph $G = (V, E)$ with two special vertices s and t and an edge-colouring c of G, we construct a graph $G' = (V', E')$ and define an edge-colouring c' of G' such that s and t are rainbow connected in G under c if and only if the colouring c' makes any three vertices of G' connected by a rainbow path.

Let $V = \{v_1, v_2, \ldots, v_{n-2}, v_{n-1} = s, v_n = t\}$ be the vertex set of the original graph G. We set $V' = \overline{V} \cup U \cup W \cup \{\hat{s}, \hat{t}, s, t\}$, where $\overline{V} = V \backslash \{s, t\} = \{v_1, v_2, \ldots, v_{n-2}\}$, $U = \{u_i^1, u_i^2 : i \in [n-2]\}$ and $W = \{w_i^1, w_i^2 : i \in [n-2]\}$, and $E' = E \cup \{v_i u_i^1, v_i u_i^2 : i \in [n-2]\} \cup \{v_i w_i^1, v_i w_i^2 : i \in [n-2]\} \cup \{u_i^a u_j^b, w_i^a w_j^b : i, j \in [n-2],\ a, b \in \{1, 2\}\} \cup \{u_1^1 \hat{s}, u_1^1 s, w_1^1 \hat{s}, w_1^1 s, \hat{s}s, u_{n-2}^1 \hat{t}, u_{n-2}^1 t, w_{n-2}^1 \hat{t}, w_{n-2}^1 t, \hat{t}t\}$.

The colouring c' is defined as follows:

- All edges $e \in E$ retain the original colours, namely $c'(e) = c(e)$.
- The edges $\hat{s}s$ and $\{v_i w_i^1 : i \in [n-2]\}$ are coloured with a new colour c'_1.
- The edges $\hat{t}t$ and $\{v_i u_i^2 : i \in [n-2]\}$ are coloured with a new colour c'_2.
- The edges $\{u_1^1 \hat{s}, u_1^1 s, w_1^1 \hat{s}, w_1^1 s, u_{n-2}^1 \hat{t}, u_{n-2}^1 t, w_{n-2}^1 \hat{t}, w_{n-2}^1 t\}$ and $\{v_i u_i^1 : i \in [n-2]\}$ are coloured with a new colour c'_3.
- The edges $\{u_i^1 u_i^2 : i \in \{2, 3, \ldots, n-2\}\}$ are coloured with a new colour c'_4.
- The edges $u_1^1 u_1^2$ and $\{u_i^a u_j^b : 1 \leq i < j \leq n-2,\ a, b \in \{1, 2\}\}$ are coloured with a new colour c'_5.
- The edges $\{v_i w_i^2 : i \in [n-2]\}$ are coloured with a new colour c'_6.
- The edges $\{w_i^1 w_i^2 : i \in [n-2]\}$ are coloured with a new colour c'_7.
- The edges $\{w_i^a w_j^b : 1 \leq i < j \leq n-2,\ a, b \in \{1, 2\}\}$ are coloured with a new colour c'_8.

Next, we always let $i, j, k \in [n-2]$ and $a, b \in \{1, 2\}$.

Firstly, suppose that there is a rainbow path P from s to t in G under c. Let x, y and z be any three vertices of G' and $S = \{x, y, z\}$. Now let us prove that x, y and z are connected by a rainbow path under c'.

Case 1: $S \subseteq U$ or $S \subseteq W$.

If $S = \{u_i^1, u_j^1, u_k^2\} \subseteq U$, where $i \neq j \neq k$, then $u_i^1 u_j^1 v_j w_j^1 w_k^2 v_k u_k^2$ is a rainbow path connecting S.

If $S = \{u_i^1, u_j^1, u_k^1\} \subseteq U$, where $i \neq j \neq k$ and w.l.o.g., $k \neq 1$, then $u_i^1 u_j^1 v_j w_j^1 w_k^2 v_k u_k^2 u_k^1$ is a rainbow path connecting S.

If $S = \{u_i^1, u_j^1, u_j^2\} \subseteq U$, where $i \neq j$, then $u_i^1 u_j^1 u_j^2$ is a rainbow path connecting S (if $j = 1$, let the path be $u_i^1 u_j^1 v_j u_j^2$).

The other subcases $S = \{u_i^1, u_j^2, u_k^2\}$, $S = \{u_i^2, u_j^2, u_k^2\}$ and $S \subseteq W$ are similar.

Case 2: $S \subseteq \overline{V}$.

Let $S = \{v_i, v_j, v_k\}$, where $i \neq j \neq k$, and then $v_i u_i^1 u_j^2 v_j w_j^1 w_k^2 v_k$ is a rainbow path connecting S.

Case 3: $S \subseteq \{\hat{s}, \hat{t}, s, t\}$.

Obviously, $\hat{s}sPt\hat{t}$ contains a rainbow path connecting S.

Case 4: $x, y \in U$ and $z \in \overline{V}$ or W or $\{\hat{s}, \hat{t}, s, t\}$.

Let $\{x, y\} = \{u_i^a, u_j^b\}$.

If $z = v_k \in \overline{V}$, then $u_i^a u_j^b v_j w_j^1 w_k^2 v_k$ is a rainbow path connecting S (if $k = i$ or j, let the path be $u_j^b u_i^a v_i$ or $u_i^a u_j^b v_j$).

If $z \in W$, then $u_i^a u_j^b v_j w_j^1 z$ is a rainbow path connecting S.

If $z \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$) and $a = b = 1$, then w.l.o.g., let $j \neq 1$ and $u_i^1 u_j^1 u_j^2 v_j w_j^1 w_1^1 z$ (or $u_i^1 u_j^1 u_j^2 v_j w_j^1 w_{n-2}^1 z$) is a rainbow path connecting S.

If $z \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$) and w.l.o.g., $b = 2$, then $u_i^a u_j^b v_j w_j^1 w_1^1 z$ (or $u_i^a u_j^b v_j w_j^1 w_{n-2}^1 z$) is a rainbow path connecting S.

The case $x, y \in W$ and $z \in V' \backslash W$ is similar.

Case 5: $x, y \in \overline{V}$ and $z \in U$ or W or $\{\hat{s}, \hat{t}, s, t\}$.

Let $\{x, y\} = \{v_i, v_j\}$, where $i \neq j$.

If $z \in U$ (or W), then $v_i w_i^1 w_j^2 v_j u_j^1 z$ (or $v_i u_i^1 u_j^2 v_j w_j^1 z$) is a rainbow path connecting S.

If $z \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$), then $v_i w_i^1 w_j^2 v_j u_j^2 u_1^1 z$ (or $v_i w_i^1 w_j^2 v_j u_j^2 u_{n-2}^1 z$) is a rainbow path connecting S.

Case 6: $x, y \in \{\hat{s}, \hat{t}, s, t\}$ and $z \in U$ or W or $\overline{V}$.

If $z \in U$ (or W), then $zu_1^1\hat{s}sPt\hat{t}$ (or $zw_1^1\hat{s}sPt\hat{t}$) contains a rainbow path connecting S.

If $z = v_i \in \overline{V}$, then $v_i w_i^2 w_1^1 \hat{s}sPt\hat{t}$ contains a rainbow path connecting S.

Case 7: $|U \cap S| \leq 1$, $|W \cap S| \leq 1$, $|\overline{V} \cap S| \leq 1$ and $|\{\hat{s}, \hat{t}, s, t\} \cap S| \leq 1$.

If $x = u_i^a \in U$, $y = v_j \in \overline{V}$ and $z \in W$, then $v_j u_j^b u_i^a v_i w_i^1 z$ is a rainbow path connecting S, where $a \neq b$ (if $i = j$, let the path be $u_i^a v_i w_i^1 z$).

If $x \in U$, $y \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$) and $z = w_k^a \in W$, then $yw_1^1 w_k^a v_k u_k^2 x$ (or $yw_{n-2}^1 w_k^a v_k u_k^2 x$) is a rainbow path connecting S.

If $x \in U$, $y \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$) and $z = v_j \in \overline{V}$, then $yw_1^1 w_j^2 v_j u_j^2 x$ (or $yw_{n-2}^1 w_j^2 v_j u_j^2 x$) is a rainbow path connecting S.

If $x \in \{\hat{s}, s\}$ (or $\{\hat{t}, t\}$), $y = v_j \in \overline{V}$ and $z = w_k^a \in W$, then $xu_1^1 u_j^2 v_j w_j^1 w_k^a$ (or $xu_{n-2}^1 u_j^2 v_j w_j^1 w_k^a$) is a rainbow path connecting S.

In a word, S is always connected by a rainbow path.

Conversely, assume that the colouring c' makes any three vertices of G' connected by a rainbow path. Thus, for $S = \{u_1^1, \hat{s}, \hat{t}\}$, there is a rainbow path P connecting S. Note that the edges adjacent to u_1^1 are coloured either c_3' or c_5', the edges connecting U and $V' \backslash U$ are coloured either c_2' or c_3', and the edges adjacent to $\hat{t}$ are coloured either c_2' or c_3'. It follows that u_1^1 must be one end of the path P. If $\hat{s}$ is the other end and $\hat{t}$ is an internal vertex of P, then it is easy to check that $P = \hat{s}sP't\hat{t}u_{n-2}^1 u_1^1$, where P' is exactly a rainbow path from s to t in the original graph G. Similarly, if $\hat{t}$ is the other end and $\hat{s}$ is an internal vertex of P, then $P = u_1^1 \hat{s}sP't\hat{t}$, where P' is a rainbow path from s to t in the original graph G. The proof is complete. ∎

Now from Lemma 5.4 and Lemma 5.2, we can get the following theorem. However, for $k \geq 4$, the complexity of Problem 1 and 2 remains unknown.

Theorem 5.3. *Given an edge-coloured graph G, the problem of checking whether the given colouring is a 3-rainbow cycle colouring is NP-Complete.* ∎

6. Future work

The k-rainbow cycle index, $crx_k(G)$ studied in the paper is a new topic and there are so many properties can be investigated. Furthermore, it would be interesting to study the parameter $\mathcal{F}_k$. Some properties of $\mathcal{F}_k$ are characterized by Liu in [18]. One referee pointed out to study the complexity for checking if a graph is in $\mathcal{F}_k$, for various k. Hope to come back in the future.

Acknowledgments. The authors would like to thank two anonymous referees for many helpful comments and suggestions. The authors also would like to thank Henry Liu for introducing to them the problem of the determination of the k-rainbow cycle index, and for his insightful comments and suggestions for improving the presentation of the paper.

Shasha Li was partially supported by National Natural Science Foundation of China (No. 11301480), Zhejiang Provincial Natural Science Foundation of China (No. LY18A010002), and the Natural Science Foundation of Ningbo, China (No.2017A610132). Yongtang Shi was partially supported by China–Slovenia bilateral project "Some topics in modern graph theory" (No. 12-6), the Natural Science Foundation of Tianjin (No. 17JCQNJC00300) and the National Natural Science Foundation of China. J. Tu was partially supported by the National Natural Science Foundation of China (No. 11201021), BUCT Fund for Disciplines Construction and Development (Project No. 1524). Yan Zhao was partially supported by the Natural Science Foundation of Jiangsu Province(No.BK20160573), and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China(No.16KJD110005).

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