Proof of a Conjecture of Reiner-Tenner-Yong on Barely Set-valued Tableaux

Neil J.Y. Fan[1], Peter L. Guo[2] and Sophie C.C. Sun[3]

[1]Department of Mathematics
Sichuan University, Chengdu, Sichuan 610064, P.R. China

[2,3]Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China

[1]fan@scu.edu.cn, [2]lguo@nankai.edu.cn, [3]suncongcong@mail.nankai.edu.cn

Abstract

Barely set-valued tableaux were introduced by Reiner, Tenner and Yong in their study of the probability distribution of edges in the Young lattice of partitions. We prove a generalization of a conjecture of Reiner, Tenner and Yong on the number of barely set-valued tableaux. To do this we apply results of Chan, Haddadan, Hopkins and Moci on jaggedness of shapes.

1 Introduction

Set-valued semistandard Young tableaux were introduced by Buch [1] in his study of the Littlewood-Richardson rule for stable Grothendieck polynomials. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an integer partition, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. The Young diagram of λ is a left-justified array of squares with λ_i squares in row i. If no confusion arises, we do not distinguish a partition and its Young diagram. A Young diagram is also called a shape. A set-valued semistandard Young tableau of shape λ is an assignment of finite sets of positive integers into the squares of λ such that the sets in each row (respectively, column) are weakly (respectively, strictly) increasing. Here, for two sets A and B of positive integers, write $A \leq B$ if $\max A \leq \min B$ and $A < B$ if $\max A < \min B$. When the set in each square contains a single integer, a set-valued semistandard Young tableau becomes an ordinary semistandard Young tableau.

A barely set-valued semistandard Young tableau is a set-valued semistandard Young tableau such that exactly one square is assigned two integers and each of the remaining squares is occupied by a single integer, see Figure 1.1 for an example. Barely set-valued tableaux arose in the work of Reiner, Tenner and Yong [5] on the probability distribution of the edges in the Young lattice of partitions.

A flagged set-valued semistandard Young tableau, defined by Knutson, Miller and Yong [4], is a set-valued semistandard Young tableau such that each value in row i

1	2	2	3
24	4	4	
5	6		
7	7		

Figure 1.1: A barely set-valued semistandard Young tableau.

does not exceed a positive integer ϕ_i. We use $\mathrm{BSSYT}(\lambda, k)$ (respectively, $\mathrm{SYT}(\lambda, k)$) to represent the set of barely set-valued semistandard Young tableaux (respectively, ordinary semistandard Young tableaux) such that each value in row i does not exceed $\phi_i = k + i$.

When λ is a rectangular staircase shape $\delta_d(b^a)$, namely, the Young diagram obtained from the staircase shape $\delta_d = (d-1, d-2, \ldots, 1)$ by replacing each square by an $a \times b$ rectangle, Reiner, Tenner and Yong [5] posed the following conjecture.

Conjecture 1.1 (Reiner, Tenner and Yong [5, Conjecture 6.4′])**.** *For any positive integers a, b, d and k,*

$$|\mathrm{BSSYT}(\delta_d(b^a), k)| = \frac{kab(d-1)}{(a+b)} |\mathrm{SYT}(\delta_d(b^a), k)|. \tag{1.1}$$

For $d = 2$, Reiner, Tenner and Yong showed that the above conjecture is true by employing the RSK algorithm as well as Stanley's hook content formula for semistandard Young tableaux.

The objective of this paper is to prove Conjecture 1.1. In fact, we prove a generalization of this conjecture in Theorem 3.1 by applying results of Chan, Haddadan, Hopkins and Moci [2] on jaggedness of shapes.

Reiner, Tenner and Yong [5] showed that Conjecture 1.1 can be reformulated in terms of polynomials defined on 0-Hecke words of permutations. As usual, we use s_i $(1 \le i \le n-1)$ to denote the simple transposition that swaps i and $i+1$. An expression of w as a product of simple transpositions is called reduced if it consists of a minimum number of simple transpositions. The length of w, denoted $\ell(w)$, is the number of simple transpositions in a reduced expression of w. If we write $w = w_1 w_2 \cdots w_n$ in one-line notation, then the length $\ell(w)$ equals the number of inversions of w, that is,

$$\ell(w) = |\{(i, j) \mid 1 \le i < j \le n, w_i > w_j\}|.$$

For a permutation w and a simple transposition s_i, define a product $w * s_i$ to be w if $\ell(ws_i) < \ell(w)$ and ws_i otherwise. Given a sequence $S = (s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$ of simple transpositions, let $H(S) = s_{i_1} * s_{i_2} * \cdots * s_{i_\ell}$. If $H(S) = w$, then S is called a 0-Hecke word of w of length ℓ. It is easily seen that a 0-Hecke word of w of length $\ell(w)$ is a reduced expression of w. Reiner, Tenner and Yong [5] defined the polynomial $FK(w,\ell)$ by

$$FK(w,\ell) = \sum_{(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})} (x+i_1)(x+i_2)\cdots(x+i_\ell),$$

where the sum ranges over 0-Hecke words of w of length ℓ. When w is the longest permutation $w_0 = n(n-1)\cdots 1$, the polynomial $FK(w_0, \ell(w_0))$ was considered by Fomin and Kirilov [3].

A permutation w is called a dominant permutation if it is 132-avoiding, that is, if there are no indices $i_1 < i_2 < i_3$ such that $w_{i_1} < w_{i_3} < w_{i_2}$. Equivalently, w is dominant if and only if the Lehmer code $(c_1(w), c_2(w), \ldots, c_n(w))$ of w is a weakly decreasing sequence, where, for $1 \le i \le n$, $c_i(w)$ is the number of inversions of w at position i, namely,

$$c_i(w) = |\{j \mid i < j, w_i > w_j\}|.$$

Reiner, Tenner and Yong [5] found that Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.2 (Reiner, Tenner and Yong [5, Conjecture 6.4])**.** *Let w be a dominant permutation whose Lehmer code is a rectangular staircase shape $\lambda = \delta_d(b^a)$. Then*

$$\frac{FK(w, \ell(w)+1)}{FK(w, \ell(w))} = \binom{\ell(w)+1}{2}\left(\frac{4x}{d(a+b)}+1\right). \tag{1.2}$$

2 The formula of Chan-Haddadan-Hopkins-Moci

In this section, we shall give an overview of a formula of Chan, Haddadan, Hopkins and Moci [2] for the expected jaggedness of a subshape in a Young diagram under a toggle-symmetric distribution. For the purpose of this paper, we only need the case when the Young diagram is a balanced shape and the toggle-symmetric distribution is the weak distribution on the subshapes of a balanced shape.

Let us begin with the necessary terminology. Given a finite poset $(P, \le)$, an order ideal I of P is a subset of P such that if $p \in I$ and $q \in P$ with $q \le p$, then $q \in I$. Let $J(P)$ denote the set of order ideals of P. We say that an element $p \in P$ can be toggled into I if p is a minimal element not in I, and that p can be toggled out of I if p is a maximal element in I. For each $p \in P$, the indicator random variables $\mathcal{T}_p^+$ and $\mathcal{T}_p^-$ on $J(P)$ are defined as follows. For an order ideal I of P, set $\mathcal{T}_p^+(I) = 1$ if p can be toggled into I, and $\mathcal{T}_p^+(I) = 0$ otherwise. Similarly, set $\mathcal{T}_p^-(I) = 1$ if p can be toggled out of I, and $\mathcal{T}_p^-(I) = 0$ otherwise. The jaggedness of I, denoted $\mathrm{jag}(I)$, is defined to be the total number of elements in P which can be toggled into I or toggled out of I.

We say that a distribution on $J(P)$ is *toggle-symmetric* if for every $p \in P$, the expected value $\mathbb{E}(\mathcal{T}_p^+)$ of the random variable $\mathcal{T}_p^+$ equals the expected value $\mathbb{E}(\mathcal{T}_p^-)$ of the random variable $\mathcal{T}_p^-$, that is, for each $p \in P$,

$$\sum_{I \in J(P)} \mathbf{Prob}(I) \cdot \mathcal{T}_p^+(I) = \sum_{I \in J(P)} \mathbf{Prob}(I) \cdot \mathcal{T}_p^-(I).$$

To a Young diagram λ, one can associate a poset structure on the squares of λ. For two squares B and B' of λ, we say that $B \leq B'$ if B occurs weakly northwest of B'. It is readily seen that a subset of squares of λ forms an order ideal with respect to the above poset structure if and only if it is a subshape of λ.

A corner of a shape μ is a square in μ such that the squares immediately below and to the right are *not* in μ. While an outside corner of μ is a square out of μ such that the squares immediately above and to the left are in μ. We assume that the square just to the right of the first row and the square just below the first column are also outside corners. Suppose that μ is a subshape of λ. By a proper outside corner of μ we mean an outside corner of μ contained in λ. Clearly, a square of λ can be toggled out of μ if and only if it is a corner of μ, while a square of λ can be toggled into μ if and only if it is a proper outside corner of μ. Thus the jaggedness $\mathrm{jag}(\mu)$ of μ equals the total number of corners and proper outside corners of μ.

For example, the jaggedness of the subshape $(3,3,2,1)$ of the diagram $(4,4,3,1)$ in Figure 2.2 equals 5, since it has three corners and two proper outside corners, which are depicted by solid squares and open squares respectively.

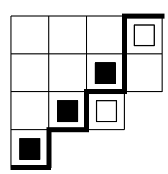

Figure 2.2: Corners and proper outside corners.

Chan, Haddadan, Hopkins and Moci [2] found a formula for the expected jaggedness of a subshape for a general skew Young diagram, which turns out to have a closed form when it is a balanced Young diagram. A Young diagram λ is called a *balanced shape* if the northwest turning point of each outside corner of λ lies on the main anti-diagonal of λ. For example, Figure 2.3 illustrates two balanced shapes, where the dashed lines represent the main anti-diagonals.

Theorem 2.1 (Chan, Haddadan, Hopkins and Moci [2, Corollary 3.8])**.** *For a balanced Young diagram λ with r rows and c columns and for any toggle-symmetric distribution, the expected jaggedness of a subshape of λ equals*

$$\frac{2rc}{r+c}.$$

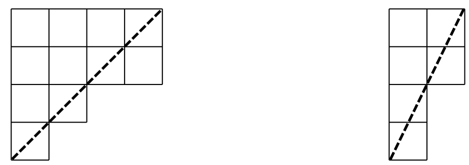

Figure 2.3: Balanced Young diagrams.

We conclude this section with a description of a specific toggle-symmetric distribution, called the *weak distribution*, see [2, Definition 2.2]. A reverse plane partition of shape λ is an assignment of nonnegative integers into the squares of λ such that the integers in each row and each column are weakly increasing, see Stanley [6, Chapter 7]. Given a positive integer k, let $\mathrm{RPP}(\lambda, k)$ denote the set of reverse plane partitions of shape λ with every entry not exceeding k.

To define the weak distribution, consider the pairs (P, i) with $P \in \mathrm{RPP}(\lambda, k)$ and $i \in \{1, 2, \ldots, k\}$. A pair (P, i) determines a subshape of λ, denoted $\alpha(P, i)$, which consists of squares of P occupied by the entries strictly less than i. The subshape $\alpha(P, i)$ is also called an induced subshape. Let

$$Q(\lambda, k) = \{(P, i) \,|\, P \in \mathrm{RPP}(\lambda, k), 1 \leq i \leq k\}. \tag{2.1}$$

Assume that the pairs (P, i) in $Q(\lambda, k)$ are generated uniformly. Then we are led to the weak distribution of subshapes of λ, that is, a subshape μ occurs with probability

$$\frac{|\{(P, i) \in Q(\lambda, k) \,|\, \alpha(P, i) = \mu\}|}{|Q(\lambda, k)|}. \tag{2.2}$$

Chan, Haddadan, Hopkins and Moci [2, Lemma 2.8] showed that the weak distribution is indeed a toggle-symmetric distribution. Hence, in the case when λ is a balanced shape, the expected jaggedness under the weak distribution can be computed by the formula in Theorem 2.1, and so the following relation holds.

Theorem 2.2. *For a balanced shape λ with r rows and c columns, we have*

$$\frac{\sum_{\mu} |\{(P, i) \in Q(\lambda, k) \,|\, \alpha(P, i) = \mu\}| \cdot \mathrm{jag}(\mu)}{|Q(\lambda, k)|} = \frac{2rc}{r + c},$$

where μ ranges over the subshapes of λ.

3 Proof of the conjecture

In this section, we present a proof of Conjecture 1.1. In fact, we establish the following relation on $|\mathrm{BSSYT}(\lambda, k)|$ and $|\mathrm{SYT}(\lambda, k)|$ for a balanced shape λ.

Theorem 3.1. *For any positive integer k and a balanced shape λ with r rows and c columns, we have*

$$|\mathrm{BSSYT}(\lambda, k)| = \frac{krc}{(r+c)}|\mathrm{SYT}(\lambda, k)|. \tag{3.1}$$

As observed by Chan, Haddadan, Hopkins and Moci [2], a rectangular staircase shape is a balanced shape. Moreover, a rectangular staircase shape $\delta_d(b^a)$ has $a(d-1)$ rows and $b(d-1)$ columns. Thus Theorem 3.1 specializes to Conjecture 1.1.

To prove Theorem 3.1, we find the following representation of a barely set-valued tableau.

Theorem 3.2. *There is a bijection between the set* $\mathrm{BSSYT}(\lambda, k)$ *and the set of triples* (P, i, C)*, where* $1 \leq i \leq k$*,* P *is a reverse plane partition in* $\mathrm{RPP}(\lambda, k)$*, and* C *is a corner of the induced subshape* $\alpha(P, i)$*.*

Proof. Let T be a barely set-valued tableau in $\mathrm{BSSYT}(\lambda, k)$. We construct a reverse plane partition $P \in \mathrm{RPP}(\lambda, k)$, an integer i $(1 \leq i \leq k)$ and a corner C in the induced subshape $\alpha(P, i)$. For each entry in T, if it is in the t-th row, then subtract it by t. This results in a tableau T' with every entry not exceeding k in which each row and each column are weakly increasing. Assume that B is the square of T containing two entries, say, a and b with $a < b$, and assume that B is in the r-th row of T. Then the entries of T' in the square B are $a - r$ and $b - r$. Define P to be the reverse plane partition in $\mathrm{RPP}(\lambda, k)$ obtained from T' by deleting the entry $b - r$ in B.

We next determine the integer i and the corner C in the induced subshape $\alpha(P, i)$. Notice that $r \leq a < b$. So we have $1 \leq b - r \leq k$. Set $i = b - r$. We choose the corner C of $\alpha(P, b-r)$ to be the square B. This is feasible because it can be verified that the square B is a corner of $\alpha(P, b-r)$. Keep in mind that the subshape $\alpha(P, b-r)$ consists of the squares of P occupied by the entries smaller than $b - r$. Note that the entry in the square B of P is $a - r$. Since $a - r < b - r$, the square B must be a square of the subshape $\alpha(P, b-r)$. To verify that B is a corner of $\alpha(P, b-r)$, we need to check that if B' is a square of λ just to the right of B or just below B, then B' does not belong to $\alpha(P, b - r)$, or, equivalently, the entry of P in B' is bigger than or equal to $b - r$. This is obvious owing to the construction of P.

To show that the above construction is reversible, we give a brief description of the reverse procedure. Given a reverse plane partition P in $\mathrm{RPP}(\lambda, k)$ together with an integer $1 \leq i \leq k$ and a corner C of $\alpha(P, i)$, we shall recover a barely set-valued tableau T in $\mathrm{BSSYT}(\lambda, k)$ as follows. Let T' be the tableau obtained from P by joining the entry i into the square C so that the square C has two entries. Increase each entry in T' by t if it is in the t-th row of T'. Let T denote the resulting tableau. It is easily verified that T is a barely set-valued tableau in $\mathrm{BSSYT}(\lambda, k)$. This completes the proof. ■

Figure 3.4 illustrates the construction of the bijection in Theorem 3.2 applying to a barely set-valued tableau T in $\mathrm{BSSYT}(\lambda, k)$ with $\lambda = (4, 4, 2, 1)$ and $k = 2$, where the subshape $\alpha(P, 2)$ is determined by the lattice path in λ drawn with thick line.

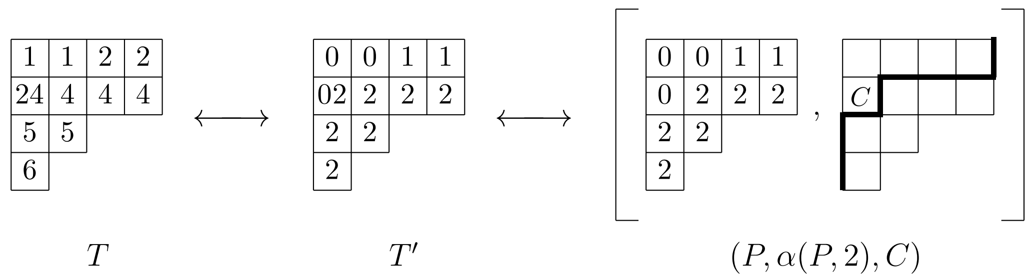

Figure 3.4: An illustration of the bijection in Theorem 3.2.

In the spirit of Theorem 3.2, we have an alternative representation of a barely set-valued tableau involving a designated proper outside corner.

Theorem 3.3. *There is a bijection between the set* $\mathrm{BSSYT}(\lambda, k)$ *and the set of triples* (Q, j, C')*, where* $1 \leq j \leq k$*,* Q *is a reverse plane partition in* $\mathrm{RPP}(\lambda, k)$*, and* C' *is a proper outside corner of the induced subshape* $\alpha(Q, j)$*.*

Proof. The proof is similar to that of Theorem 3.2, and so we only give a description of the construction of the bijection. For a barely set-valued tableau T in $\mathrm{BSSYT}(\lambda, k)$, let T' be the tableau as constructed in the proof of Theorem 3.2. Define Q to be the reverse plane partition in $\mathrm{RPP}(\lambda, k)$ obtained from T' by deleting the entry $a-r$ in B. Set $j = a-r+1$. It can be verified that B is a proper outside corner of $\alpha(Q, a-r+1)$. Then choose C' to be the proper outside corner B. ∎

Figure 3.5 is an illustration of the bijection in Theorem 3.3, where T is a barely set-valued tableau in $\mathrm{BSSYT}(\lambda, k)$ with $\lambda = (4, 4, 2, 1)$ and $k = 2$.

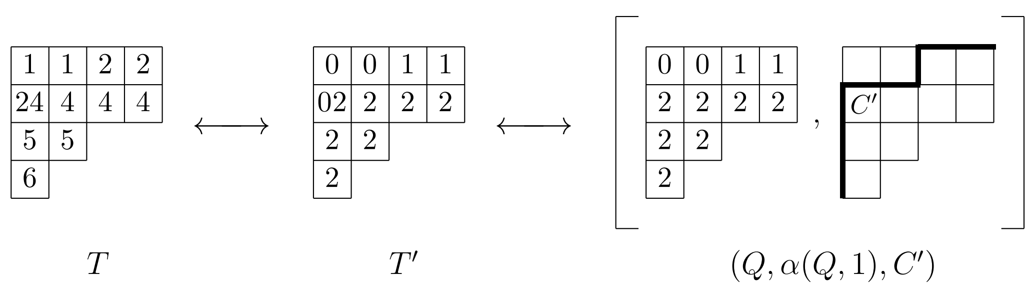

Figure 3.5: An alternative representation.

Proof of Theorem 3.1. As introduced in Section 2, the expected jaggedness of a subshape of λ under the weak distribution equals

$$\frac{\sum_{\mu} |\{(P, i) \in Q(\lambda, k) \mid \alpha(P, i) = \mu\}| \cdot \mathrm{jag}(\mu)}{|Q(\lambda, k)|}, \tag{3.2}$$

where μ ranges over subshapes of λ. To compute the numerator of (3.2), note that

$$\sum_{\mu} |\{(P,i) \in Q(\lambda,k) \mid \alpha(P,i) = \mu\}| \cdot \mathrm{jag}(\mu) = \sum_{(P,i)\in Q(\lambda,k)} \mathrm{jag}(\alpha(P,i)).$$

Let $C(P,i)$ denote the number of corners in the subshape $\alpha(P,i)$, and let $C'(P,i)$ denote the number of proper outside corners of $\alpha(P,i)$. Then we have

$$\mathrm{jag}(\alpha(P,i)) = C(P,i) + C'(P,i).$$

By the definition of the set $Q(\lambda,k)$ in (2.1), we obtain that

$$\begin{aligned}
\sum_{(P,i)\in Q(\lambda,k)} \mathrm{jag}(\alpha(P,i)) &= \sum_{P\in \mathrm{RPP}(\lambda,k)} \sum_{i=1}^{k} \mathrm{jag}(\alpha(P,i)) \\
&= \sum_{P\in \mathrm{RPP}(\lambda,k)} \sum_{i=1}^{k} C(P,i) + \sum_{P\in \mathrm{RPP}(\lambda,k)} \sum_{i=1}^{k} C'(P,i). \qquad (3.3)
\end{aligned}$$

By Theorem 3.2 and Theorem 3.3, both the first double sum and the second double sum in (3.3) are equal to $|\mathrm{BSSYT}(\lambda,k)|$. It follows that

$$\sum_{\mu} |\{(P,i) \in Q(\lambda,k) \mid \alpha(P,i) = \mu\}| \cdot \mathrm{jag}(\mu) = 2|\mathrm{BSSYT}(\lambda,k)|. \qquad (3.4)$$

As to the denominator of (3.2), we notice that $|Q(\lambda,k)| = k|\mathrm{RPP}(\lambda,k)|$. On the other hand, there is an obvious bijection between the set $\mathrm{RPP}(\lambda,k)$ and the set $\mathrm{SYT}(\lambda,k)$. Given $P \in \mathrm{RPP}(\lambda,k)$, one can construct a semistandard Young tableau in $\mathrm{SYT}(\lambda,k)$ from P by increasing each entry in the t-th row of P by t. Therefore,

$$|Q(\lambda,k)| = k|\mathrm{SYT}(\lambda,k)|. \qquad (3.5)$$

Substituting (3.4) and (3.5) into (3.2), the expected jaggedness in (3.2) can be rewritten as

$$\frac{2|\mathrm{BSSYT}(\lambda,k)|}{k|\mathrm{SYT}(\lambda,k)|},$$

which, together with Theorem 2.2, yields

$$\frac{2|\mathrm{BSSYT}(\lambda,k)|}{k|\mathrm{SYT}(\lambda,k)|} = \frac{2rc}{r+c}.$$

This confirms (3.1), and hence the proof is complete. ■

We conclude this paper with a formula on the polynomial $FK(w,\ell)$ with respect to a dominant permutation w corresponding to a balanced shape. Restricting to a dominant permutation corresponding to a rectangular staircase shape, this formula reduces to Conjecture 1.2.

Theorem 3.4. *Let w be a dominant permutation whose Lehmer code is a balanced shape λ with r rows and c columns, and let $\ell = \ell(w)$. Then we have*

$$\frac{FK(w,\ell+1)}{FK(w,\ell)} = \binom{\ell+1}{2}\left(\frac{2xrc}{\ell(r+c)}+1\right). \tag{3.6}$$

Proof. It is sufficient to verify (3.6) for any positive integer x. In the proof of [5, Corollary 6.11], Reiner, Tenner and Yong eatablished the following relation

$$\frac{FK(w,\ell+1)}{FK(w,\ell)} = \binom{\ell+1}{2} + (\ell+1)\frac{|\mathrm{BSSYT}(\lambda,x)|}{|\mathrm{SYT}(\lambda,x)|}. \tag{3.7}$$

Substituting (3.1) into (3.7), we obtain that

$$\frac{FK(w,\ell+1)}{FK(w,\ell)} = \binom{\ell+1}{2}\left(\frac{2xrc}{\ell(r+c)}+1\right),$$

and hence the proof is complete. ■

Notice that by (3.7), it is clear that Theorem 3.4 is equivalent to Theorem 3.1.

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