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A Note on Totally Free Matroids

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Abstract It is known matroids obtained from a totally free uniform matroid $U_{2,n}$ by a sequence of segment–cosegment and cosegment–segment exchanges are totally free (Geelen et al., in J Combin Theory Ser B 92:55–67, 2004). In this paper, we prove matroids obtained from any totally free matroid by a sequence of segment–cosegment and cosegment–segment exchanges are also totally free.

Keywords Totally free matroids · Segment–cosegment and cosegment–segment exchanges

1 Introduction

Unique representability is of great importance in matroid representation theory. In fact, it is no coincidence that finite fields $GF(q)$ for which the sets of excluded minors have been completely determined are those over which every 3-connected $GF(q)$-representable matroid is uniquely representable [2,8,9,12,13]. Recall binary matroids and ternary matroids over $GF(3)$ have a unique representation property, and 3-connected quaternary matroids are also uniquely representable over $GF(4)$. Hence, the presence of inequivalent representations of matroids over fields is the major barrier to progress in matroid representation theory; and more techniques are needed to develop the theory

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[5,6,14]. The notion, totally free matroid, is defined to understand the behavior of inequivalent representation of 3-connected matroids. It turns out that the number of inequivalent representation of a 3-connected matroid is bounded above by the number of inequivalent representation of a totally free minor. Then we need to study the property of totally free matroids in a well-closed class (namely, a class of matroids closed under isomorphisms, minors, dualities). Surprisingly, all totally free matroids in a well-closed class can be found by an inductive search [5].

In [11], Oxley et al. defined generalized $\Delta - Y$ exchange or segment–cosegment exchange, and studied the class of matroids that can be obtained from an totally free uniform matroid $U_{2,n}$ by a sequence of segment–cosegment and cosegment–segment exchanges via a vertex-labeled tree, which is also called *quasi-lines* by [7]. Recall the following two key results from [7]:

Lemma 1.1 *Quasi-lines are totally free.*

Lemma 1.2 *Totally free matroids without the uniform matroid $U_{3,6}$ as a minor are quasi-lines.*

Then by Lemmas 1.1 and 1.2, and the fact that quasi-lines have no $U_{3,6}$-minor [11, Lemma 6.1], Geelen et al. [7] proved that Kahn's conjecture holds for all 3-connected matroids without $U_{3,6}$ as a minor.

Now our main result, which extends Lemma 1.1 and is of its own interests as a property of matroids, can be stated as follows.

Theorem 1.3 *Matroids obtained from a totally free matroid M by a sequence of segment–cosegment and cosegment–segment exchanges are totally free.*

For this paper, the matroid terminologies will follow Oxley [10] except that the simplification and cosimplification of a matroid M are denoted by $\mathrm{si}(M)$ and $\mathrm{co}(M)$ respectively. The orthogonal property that a circuit and a cocircuit of a matroid can not contain exactly one common element will be used repeatedly in our proofs. In Sects. 2 and 3, some necessary preliminaries on fixed elements and totally free matroids, and generalized $\Delta - Y$ exchange are presented respectively. In Sect. 4, proof of Theorem 1.3 is given.

2 Fixed Elements and Totally Free Matroids

Let M be a matroid with the ground set $E(M)$. Elements x and x' of M are *clones* if the function exchanging x with x' and fixing other points in $E(M)$ is an automorphism of M. *A clonal class* of M is a maximal subset $X \subseteq E(M)$ such that any two points of X are clones. Parallel class and series class, the set of loops and the set of coloops are called *trivial* clonal classes; and other clonal classes are called *nontrivial* ones. *A clonal set* of M is a subset of a nontrivial clonal class containing at least two elements. Clearly, the clone sets of M and its dual matroid M^* are coincide.

For any $x \in E(M)$, call the matroid M' obtained from M by cloning x with x' (a point not in $E(M)$) if M' is a single element extension of M by x' satisfying x and x' clones in M'. Note such a matroid M' always exists because x' can be parallel with x in M'. If $\{x, x'\}$ is an independent set in M', then we call M' is obtained from M by independently cloning x with x', and call x is *not fixed* in M. Otherwise, we call x is *fixed* in M. Dually, we call M' is obtained from M by cocloning x with x' if M' is a single element coextension of M by x' such that x and x' are clones in M'. Similarly, if x and x' are coindependent in M', then we say M' is obtained from M by coindependently cloning x with x', and say x is *not cofixed* in M. Otherwise, we say x is *cofixed* in M.

Let F_1 and F_2 be flats of M. Then (F_1, F_2) is a *modular pair of flats* if

$$r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2),$$

where $r(F)$ denotes the rank of flat F. A *modular cut* $\mathcal{F}$ of a matroid M is a collection of flats of M with the following properties:

(i) *If* $F_1, F_2 \in \mathcal{F}$ *and* (F_1, F_2) *is a modular pair, then* $F_1 \cap F_2 \in \mathcal{F}$;
(ii) *for any* $F \in \mathcal{F}$, *any flat of* M *containing* F *is also in* $\mathcal{F}$.

A flat of a matroid is *cyclic* if it is a union of circuits. For a set $\mathcal{F}$ of flats of a matroid, the unique minimal modular cut containing $\mathcal{F}$ is called the modular cut generated by $\mathcal{F}$ and is denoted by $\langle \mathcal{F} \rangle$.

Proposition 2.1 [4, Corollary 3.5] *Let e be an element of a matroid M. Then e is fixed in M if and only if* $\mathrm{cl}(e)$, *the flat of M generated by e, is in the modular cut generated by the cyclic flats of M containing e.*

Obviously, if x and x' are independent clones in M, then x is not fixed in $M \backslash x'$. The next proposition extends the observation.

Proposition 2.2 [6, Proposition 4.9] *If x and x' are independent clones in M, then x is fixed in neither M nor $M \backslash x'$. Dually, if x and x' are coindependent clones in M, then x is cofixed in neither M nor M/x'.*

By definitions, if x and x' are both independent clones and coindependent clones, then x is neither fixed nor cofixed in M. However, it is possible for x to be fixed in M/x' and for x to be cofixed in $M \backslash x'$.

Proposition 2.3 [5, Proposition 4.9] *Elements x and x' are clones in a matroid M if and only if the set of cyclic flats of M containing x is equal to the set of cyclic flats containing x'.*

By Proposition 2.3, we obtain

Corollary 2.4 *Let F be a cyclic flat of a matroid M, and A a clonal set of M. Then either $F \cap A = \emptyset$ or $F \cap A = A$.*

A matroid M is *totally free* if the following conditions hold:

(i) M is 3-connected with $|E(M)| \geq 4$; and
(ii) if e is fixed in M, then $\mathrm{co}(M \backslash e)$ is not 3-connected, and if e is cofixed in M, then $\mathrm{si}(M/e)$ is not 3-connected.

Note M is totally free if and only if M^* is totally free. A *clonal triple* or a *clonal pair* means a clonal set of size 3 or 2, respectively.

Lemma 2.5 [5, Lemma 8.8] *If* $\{a, b, c\}$ *is a triad or a triangle of a totally free matroid* M, *then* $\{a, b, c\}$ *is a clonal triple.*

Lemma 2.6 [5, Lemma 8.7] *If* $\{a, b, c\}$ *is a triangle of a totally free matroid* M *with at least 5 elements, then no triad of* M *meets* $\{a, b, c\}$.

Lemma 2.7 [7, Corollary 2.10] *Let* M *be a totally free matroid with at least 5 elements, and* e *an element of* $E(M)$. *Then* $M\backslash e$ *is totally free if* e *is an element of an triangle of* M; *and* M/e *is totally free provided* e *is an element of a triad of* M.

Now we arrive at the following corollary which will be frequently used in the sequel.

Corollary 2.8 *Let* A *be a coindependent set of a totally free matroid* M *with at least 3-element. If* $M|A \cong U_{2,|A|}$, *then*

(i) *A is a clonal set of M, and every element in A is neither fixed nor cofixed; and*
(ii) *$M\backslash(A - a)$ is connected for any element a in A.*

Proof Since M is totally free, M is 3-connected, that is, it has neither parallel classes nor series classes. Then (i) follows from Proposition 2.2 and Lemma 2.5. To prove (ii), let b be another element of A disjoint from a. Since A is a coindependent set of M, $|E(M) - A| \geq 2$. By repeatedly using Lemma 2.7, $M\backslash(A - (a \cup b))$ is totally free. Hence, $M\backslash(A - (a \cup b))$ is 3-connected which implies $M\backslash(A - a)$ is connected, namely, (ii) holds.

By duality, the following corollary holds.

Corollary 2.9 *Let* A *be an independent set of a totally free matroid* M *with at least 3-element. If* $M^*|A \cong U_{2,|A|}$, *then*

(i) *A is a clonal set of M, and every element in A is neither fixed nor cofixed; and*
(ii) *$M/(A - a)$ is connected for any element a in A.*

3 Generalized $\Delta - Y$ Exchange

The generalized $\Delta - Y$ exchange was first studied by Oxley et al. [11]. The operation of $\Delta - Y$ and $Y - \Delta$ exchanges are of basic importance in graph theory. For matroids, these operations are defined in terms of the generalized parallel connection [3]. Let M_1 and M_2 be two matroids satisfying $M_1|T = M_2|T$, where $T = E(M_1) \cap E(M_2)$. Suppose T is a modular flat of M_1. Here a flat F of a matroid M is *modular* if

$$r(F) + r(F') = r(F \cap F') + r(F \cup F') \text{ for all flats } F' \text{ of } M.$$

Put $N = M_1|T$. The *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_i)$ is a flat of M_i, $i = 1, 2$. When $M_1 \cong M(K_4)$ and N

is a triangle of this matroid, [1] defined a $\Delta - Y$ *exchange on* M *across* T to be the matroid obtained by $P_N(M(K_4), M)$ by deleting T. Oxley et al. [11] generalized this operation as follows.

Firstly, a matroid Θ_k is introduced to generalize the role played by $M(K_4)$ in the $\Delta - Y$ exchange. On one hand, Θ_k can be obtained from a free matroid $U_{k,k}$ by adding a point to each hyperplane of the latter so that each of these hyperplanes becomes a circuit in the resulting matroid and so that the restriction of Θ_k to the added points is a k-point line. On the other hand, we can describe Θ_k as follows: The ground set of Θ_k consists of a k-element line and a k-element coline with the property that each $(k-1)$-element subset of the coline forms a circuit with an element of the line. Denote the line of Θ_k by A and the coline by B, where

$$A = \{a_1, a_2, \dots, a_k\}, \quad B = \{b_1, b_2, \dots, b_k\}.$$

Obviously, A is a modular flat of Θ_k. For $k > 2$ the non-spanning circuits of Θ_k are

(i) *all subsets* $(B - \{b_i\}) \cup \{a_i\}$ *for all* $i \in \{1, 2, \dots, k\}$, *and*
(ii) *all* 3 *-elements of* A.

If X is a subset of $E(M)$ with $|X| \geq 2$ and $M|X = U_{2,|X|}$, then X is a *segment* of M. A *cosegment* of M is a segment of M^*. Since we would like an operation whose inverse is the dual of the original operation, in defining this operation we shall impose the additional condition A is coindependent in M. In this case, A is a *strict segment* of M. By duality, a *strict cosegment* of M is an independent cosegment of M. Let A be a strict segment of M and define $\Delta_A(M)$ as the matroid obtained from $P_A(\Theta_k, M)\backslash A$ by relabeling the element b_i by $a_i (1 \leq i \leq k)$. We call this operation a Δ_A*-exchange* or a *segment–cosegment exchange on* A.

Let M be a matroid for which M^* has a $U_{2,k}$-restriction on the set A. If A is independent in M, then $\nabla_A(M)$ is defined as $(\Delta_A(M^*))^*$, that is, $[P_A(\Theta_k, M^*)\backslash A]^*$. This operation will also be referred to as a ∇_A*-exchange* or a *cosegment–segment exchange on* A. By Corollary 2.12 in [11], these operations are inverse mutually, i.e., $\Delta_A(\nabla_A(M)) \cong M$.

Notice Θ_2 is isomorphic to the matroid obtained form $U_{2,2}$ by adding exactly one element in parallel with each element of the ground set, and Θ_3 is isomorphic to $M(K_4)$. In addition, $\Delta_A(M) \cong M$ for any strict segment A with $|A| = 2$, and by duality $\nabla_A(M) \cong M$ for any strict cosegment with $|A| = 2$; and in both cases, the isomorphism is simply the function exchanging the two members of A and fixing other elements.

In the rest of the paper, fix

$$A = \{a_1, a_2, \dots, a_k\}, \quad k = |A|. \tag{3.1}$$

Lemma 3.1 [11, Lemma 2.6] *Let* A *be a coindependent set in a matroid* M *with* $M|A \cong U_{2,|A|}$. *Then*

$$r(\Delta_A(M)) = r(M) + k - 2.$$

By the definition of $Y - \Delta$ *exchange and Lemma* 3.1*, we have*

Corollary 3.2 *Assume $\nabla_A(M)$ is well defined. Then $r(\nabla_A(M)) = r(M) - k + 2$.*

Lemma 3.3 [11, Lemma 2.9] *Let $\Delta_A(M)$ be the matroid with ground set $E(M)$ that is obtained from M by a Δ_A-exchange. Then a subset of $E(M)$ is a basis of $\Delta_A(M)$ if and only if it is a member of one of the following sets:*

(i) $\{A \cup B' : B' \text{ is a basis of } M/A\}$;
(ii) $\{(A - a_i) \cup B'' : 1 \leq i \leq k \text{ and } B'' \text{ is a basis of } M/a_i\backslash(A - a_i)\}$; *and*
(iii) $\{(A - \{a_i, a_j\}) \cup B''' : 1 \leq i < j \leq k \text{ and } B''' \text{ is a basis of } M\backslash A\}$.

By Lemma 3.3, it is easy to obtain

Corollary 3.4 *Suppose $\Delta_A(M)$ is well defined. Then*

$$r_{\Delta_A(M)}(X) = \begin{cases} |X \cap A| + r_M(X - A), & if\ |X \cap A| < k - 1, \\ k - 1 + r_{M/a}(X - A), & if\ X \cap A = A - a, where\ a \in A, \\ k + r_{M/A}(X - A), & if\ X \cap A = A. \end{cases}$$

By the dual of Corollary 3.4, we obtain

Corollary 3.5 *Suppose $\nabla_A(M)$ is well defined. Then*

$$r_{\nabla_A(M)}(X) = \begin{cases} r_M(X), & if\ X \cap A = \emptyset, \\ 1 + r_{M/(A-a)}(X - a), & if\ X \cap A = a, \\ 2 + r_{M/A}(X - A), & if\ |X \cap A| \geq 2. \end{cases}$$

By Corollaries 3.4 and 3.5, *if $C \cap A = \emptyset$, then C is a circuit of M if and only if C is a circuit of $\Delta_A(M)$ or $\nabla_A(M)$*. In Sect. 4, this result will be used directly without explanation.

Lemma 3.6 [11, Corollary 2.16] *Suppose that $\Delta_A(M)$ is well defined. Then*

(i) *If $x \in E(M) - A$ and A is a coindependent in $M\backslash x$, then $\Delta_A(M\backslash x)$ is defined and $\Delta_A(M)\backslash x = \Delta_A(M\backslash x)$.*
(ii) *If $x \in E(M) - cl(A)$, then $\Delta_A(M/x)$ is defined and $\Delta_A(M)/x = \Delta_A(M/x)$.*

Lemma 3.7 [11, Corollary 2.17] *Let M be a matroid and $A \subseteq E(M)$. Suppose*

$$x \in E(M) - A,\ |E(M) - A| \geq 3,\ and\ k \geq 3.$$

Then

(i) *suppose $\Delta_A(M)$ is defined,*
  (a) *if $M\backslash x$ is 3-connected, then $\Delta_A(M\backslash x)$ is defined and $\Delta_A(M)\backslash x = \Delta_A(M\backslash x)$,*
  (b) *if M/x is 3-connected, then $\Delta_A(M/x)$ is defined and $\Delta_A(M)/x = \Delta_A(M/x)$;*
(ii) *assume $\nabla_A(M)$ is defined,*

(a) *if* $M\backslash x$ *is* 3-*connected, then* $\nabla_A(M\backslash x)$ *is defined and* $\nabla_A(M)\backslash x = \nabla_A(M\backslash x)$,
(b) *if* M/x *is* 3-*connected, then* $\Delta_A(M/x)$ *is defined and* $\nabla_A(M)/x = \nabla_A(M/x)$.

Lemma 3.8 [11, Lemma 2.20] *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\Delta_A(M)$. Moreover, if $\{x, x'\}$ is independent in M, it is independent in $\Delta_A(M)$, and if $\{x, x'\}$ is coindependent in M, it is coindependent in $\Delta_A(M)$.*

Lemma 3.9 [11, Corollary 2.21] *Let x and x' be clones in a matroid M. If $A \cap \{x, x'\}$ is empty or $A \supseteq \{x, x'\}$, then x and x' are clones in $\nabla_A(M)$. Moreover, if $\{x, x'\}$ is independent in M, it is independent in $\nabla_A(M)$, and if $\{x, x'\}$ is coindependent in M, it is coindependent in $\nabla_A(M)$.*

Lemma 3.10 [11, Lemma 2.10] *Let A be a coindependent set in a matroid M with $M|A \cong U_{2,|A|}$.*

(i) *If X is a subset of $E(M)$ avoiding A, then e is in the closure of X in M if and only if e is in the closure of X in $\Delta_A(M)$.*
(ii) *If $\{e, f\}$ is a cocircuit of M, then $\{e, f\}$ is a cocircuit of $\Delta_A(M)$. Conversely, if $\{e, f\}$ is a cocircuit of $\Delta_A(M)$ avoiding A, then $\{e, f\}$ is a cocircuit of M.*

4 Proof of Theorem 1.3

To prove Theorem 1.3, it suffices to prove that any matroid obtained from M by a *single* segment–cosegment exchange or cosegment–segment exchange is totally free. Further, by definitions of segment–cosegment exchange and cosegment–segment one, it suffices to verify that any matroid obtained from totally free matroid M by a single segment–cosegment exchange is totally free. This result is known when $|E(M)| = 4$ by Lemma 1.1; we prove it is true by a series of lemmas and corollaries when $|E(M)| \geq 5$.

To begin, we introduce the well-known connectivity function. Let M be a matroid with ground set $E = E(M)$ and rank function r_M. The connectivity function λ_M of M is defined on all subsets X of E by

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r_M(E).$$

Clearly, $\lambda_M(X) \geq 0$. We also denote $\lambda_M(X)$ by $\lambda_M(X, Y)$, where (X, Y) is a partition of E. In the rest of the paper, since the matroids considered are totally free, for convenience, we assume M is totally free with at least 5 elements here and hereafter.

Note that, in general, 3-connectivity is not preserved under a Δ_A-exchange or dually under a ∇_A-exchange. For example, the matroid obtained from Q_6 by performing a Δ_3-exchange on one of its triangle is not 3-connected [11]. Q_6 is the matroid obtained by placing a point on the intersection of two lines of $U_{3,5}$.

Lemma 4.1 *Let A be a coindependent set of a 3-connected matroid N with $N|A \cong U_{2,|A|}$. If $N\backslash(A - a)$ does not contain any coloops for any $a \in A$, then $\Delta_A(N)$ is 3-connected.*

Proof When $|A| = 2$, $\Delta_A(N) \cong N$. So $\Delta_A(N)$ is 3-connected. Hence, we can assume $|A| \geq 3$.

It is an immediate consequence of the definition of generalized $\Delta - Y$ exchange that $\Delta_A(N)$ has neither loops nor nontrivial parallel classes. Suppose $\Delta_A(N)$ has a coloop $\{c\}$. Then, by Lemma 3.3, $\{c\} \cap A = \emptyset$ and $\{c\}$ is a coloop of N/A. Therefore, $\{c\}$ is a coloop of N. A contradiction, since N is 3-connected. Thus $\Delta_A(N)$ has no coloops. Since N has no nontrivial series classes, by Lemma 3.10(ii), $\Delta_A(N)$ has no nontrivial series classes avoiding A. It is a straightforward consequence of Lemma 3.3 that no nontrivial series class is contained in A. Hence, we can assume there exists some nontrivial series class $C^* = \{a, b\}$ satisfying $a \in A$ and $b \notin A$. Then by Lemma 3.3, b must be a coloop of $N/a\backslash(A - a)$, in particular, b is a coloop of $N\backslash(A - a)$, which is a contradiction. Thus, $\Delta_A(N)$ has no nontrivial series classes. Hence, to prove $\Delta_A(N)$ is 3-connected, it suffices to prove $\lambda_{\Delta_A(N)}(X, Y) \geq 2$ for any partition (X, Y) of E with $|X| \geq 3$ and $|Y| \geq 3$. Clearly $\lambda_N(X, Y) \geq 2$.

Case 1 $A \subseteq X$ or $A \subseteq Y$.

Without loss of generality suppose $A \subseteq X$, then by Corollary 3.4,

$$\begin{aligned}\lambda_{\Delta_A(N)}(X, Y) &= r_{\Delta_A(N)}(X) + r_{\Delta_A(N)}(Y) - r_{\Delta_A(N)}(E)\\ &= |A| + r_{N/A}(X - A) + r_N(Y) - (|A| + r_N(E) - 2)\\ &= r_N(X) + r_N(Y) - r_N(E) + 2 - r_N(A)\\ &= \lambda_N(X, Y)\\ &\geq 2.\end{aligned}$$

Case 2 $1 < |A \cap X| < |A| - 1$ and $1 < |A \cap Y| < |A| - 1$.

By Corollary 3.4, we have

$$\begin{aligned}\lambda_{\Delta_A(N)}(X, Y) &= r_{\Delta_A(N)}(X) + r_{\Delta_A(N)}(Y) - r_{\Delta_A(N)}(E)\\ &= |A \cap X| + r_N(X - A) + |A \cap Y| + r_N(Y - A)\\ &\quad -(|A| + r_N(E) - 2)\\ &= r_N(X - A) + r_N(Y - A) - r_N(E) + 2\\ &= \lambda_{N\backslash A}(X - A, Y - A) + 2\\ &\geq 2.\end{aligned}$$

Case 3 $|A \cap X| = |A| - 1$ or $|A \cap Y| = |A| - 1$.

Assume $|A \cap X| = |A| - 1$ and $A - X = \{a\}$. First, we show $N\backslash(A - a)$ is connected. Assume to the contrary that $N\backslash(A - a)$ is not connected. Then there exists a partition (S, T) of $E(N\backslash(A - a))$ such that $\lambda_{N\backslash(A-a)}(S, T) = 0$, where $a \in S$. Obviously, $|S| \geq 2$ and $|T| \geq 2$ since N is 3-connected and $N\backslash(A - a)$ has no coloops. On the other hand, note that, $\lambda_N(S \cup A, T) \leq 1$. A contradiction since N is 3-connected. So $N \backslash (A - a)$ is connected, and consequently, $\lambda_{N\backslash(A-a)}(X - A + a, Y - a) \geq 1$. Therefore, by Corollary 3.4,

$$\begin{aligned}\lambda_{\Delta_A(N)}(X, Y) &= r_{\Delta_A(N)}(X) + r_{\Delta_A(N)}(Y) - r_{\Delta_A(N)}(E)\\ &= |A| - 1 + r_{N/a}(X - A) + 1 + r_N(Y - a) - (|A| + r_N(E) - 2)\\ &= r_N(X - A + a) + r_N(Y - a) - r_N(E) + 2 - 1\\ &= \lambda_{N\backslash(A-a)}(X - A + a, Y - a) + 1\\ &\geq 2.\end{aligned}$$

So far we have proven $\lambda_{\Delta_A(N)}(X, Y) \geq 2$. Thus $\Delta_A(N)$ is 3-connected. □

Note that, by the proof of Case 3 in Lemma 4.1, we know, in fact, if A is a coindependent set of a 3-connected matroid N satisfying $N|A \cong U_{2,|A|}$, then $N\backslash(A - a)$ without any coloops for any $a \in A$ is equal to $N\backslash(A - a)$ connected for any $a \in A$.

Lemma 4.2 *Let A be a coindependent set of M with $M|A \cong U_{2,|A|}$. Then $\Delta_A(M)$ is 3-connected.*

Proof When $|A| = 2$, the result is trivial. Hence assume $|A| \geq 3$. By Corollary 2.8 (ii), $M\backslash(A - a)$ is connected for any element $a \in A$, in particular, $M\backslash(A - a)$ does not contain any coloops. Hence, the lemma holds according to Lemma 4.1. □

The dual of Lemma 4.2 is as follows.

Corollary 4.3 *Let A be an independent set of M with $M^*|A \cong U_{2,|A|}$. Then $\nabla_A(M)$ is 3-connected.*

Lemma 4.4 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. Then A is a clonal set of $\Delta_A(M)$.*

Proof It suffices to prove that for any two elements $a_i, a_j \in A$, a_i and a_j are clones in $\Delta_A(M)$ (A is given by (3.1)). Thus we need to prove that for any $B \in \mathcal{B}(\Delta_A(M))$ if $a_i \in B$ but $a_j \notin B$, then $B - a_i + a_j \in \mathcal{B}(\Delta_A(M))$. By Corollary 2.8(i), $M/a_i\backslash a_j \cong M/a_j\backslash a_i$. Then

$$M/a_i\backslash(A - a_i) \cong M/a_j\backslash(A - a_j).$$

From Lemma 3.3, we see that $B - a_i + a_j \in \mathcal{B}(\Delta_A(M))$. □

By duality, we have

Corollary 4.5 *Let A be an independent set of M with at least three elements and $M^*|A \cong U_{2,|A|}$. Then A is a clonal set of $\nabla_A(M)$.*

Combining Lemmas 4.2, 4.4 and Corollary 4.5 with Proposition 2.2, we obtain

Corollary 4.6 (i) *If A is a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$, then every element in A is neither fixed nor cofixed in $\Delta_A(M)$.*
(ii) *If A is an independent set M with at least three elements and $M^*|A \cong U_{2,|A|}$, then every element in A is neither fixed nor cofixed in $\nabla_A(M)$.*

Lemma 4.7 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. If $x \in E - A$ and C^* is a cocircuit of M satisfying $x \in C^* \subseteq A \cup x$, then $C^* = A \cup x$.*

Proof By orthogonality, we obtain $|A| - 1 \leq |C^* \cap A| \leq |A|$. Suppose $C^* \cap A = A - a$, where $a \in A$. Then x is coloop of $M\backslash(A - a)$. However, by Corollary 2.8(ii), $M\backslash(A - a)$ is connected, which is a contradiction. Hence, $|C^* \cap A| = |A|$, and consequently, $C^* = A \cup x$. □

Lemma 4.8 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \in E - A$, if C^* is a triad of $\Delta_A(M)$ satisfying $x \in C^*$ and $C^* \cap A$ is nonempty, then $A \cup x \in \mathcal{C}^*(M)$.*

Proof Suppose $a \in C^* \cap A$ and $C^* = \{a, x, y\}$.

Case 1 y is in A.

Since C^* is a triad of $\Delta_A(M)$, C^* must meet every basis of $\Delta_A(M)$. According to Lemma 3.3, x must be a coloop of $M\backslash A$. Thus there exists some cocircuit ${C_1}^*$ of M satisfying ${C_1}^* \subseteq A \cup x$ and $x \in {C_1}^*$. It is a consequence of Lemma 4.7 that $A \cup x \in \mathcal{C}^*(M)$.

Case 2 y is not in A.

By Lemma 3.3, every basis of $M/a\backslash(A - a)$ must meet at least one of x and y. Using the fact that M is 3-connected, $M|A \cong U_{2,|A|}$ and A is a clonal set of M, easily we can deduce that $M/a\backslash(A - a)$ is connected. Thus $\{x, y\}$ is a cocircuit of $M/a\backslash(A - a)$, that is, $(A - a) \cup x \cup y$ contains some cocircuit ${C_1}^*$ of M. Obviously, both x and y are in ${C_1}^*$. By orthogonality, $|{C_1}^* \cap A| \geq |A| - 1$. Therefore, ${C_1}^* = (A - a) \cup x \cup y$. Let a' be an arbitrary element in A disjoint from a. By Corollary 2.8(i), a and a' are clones in M. Then ${C_2}^* = (A - a') \cup x \cup y$ is also a cocircuit of M. Hence, $A \cup x = {C_1}^* \cup {C_2}^* - y$ contains a cocircuit ${C_3}^*$ of M, that is to say, $A \cup x$ is codependent in M. Since A is coindependent in M, $x \in {C_3}^*$. So by Lemma 4.7, $A \cup x \in \mathcal{C}^*(M)$. □

Lemma 4.9 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \in E - A$, if $co(M\backslash x)$ is not 3-connected, then either x is not fixed in M or $co(\Delta_A(M)\backslash x)$ is also not 3-connected.*

Proof If x is in some triangle or triad of M, then by Proposition 2.2 and Lemma 2.5, x is not fixed in M. Hence assume x is neither in any triangle nor in any triad, and consequently, $\mathrm{co}(M\backslash x) = M\backslash x$. Depending on whether $A \cup x$ is a cocircuit of M, there are two cases to consider.

Case 1 $A \cup x \notin \mathcal{C}^*(M)$.

Since $A \cup x \notin \mathcal{C}^*(M)$, by Lemma 4.8, $\Delta_A(M)$ contains no triad C^* such that $x \in C^*$ and $C^* \cap A \neq \emptyset$. Hence, if exist some triad C^* of $\Delta_A(M)$ with $x \in C^*$, then $C^* \cap A = \emptyset$. By Corollary 3.4, we have

$$r_{\Delta_A(M)}(E - C^*) = |A| + r_{M/A}(E - C^* - A) = |A| + r_M(E - C^*) - 2,$$

Hence,

$$r_M(E - C^*) = r_{\Delta_A(M)}(E - C^*) + 2 - |A|.$$

So

$$\begin{aligned} r_{M^*}(C^*) &= |C^*| + r_M(E - C^*) - r_M(E) \\ &= |C^*| + r_{\Delta_A(M)}(E - C^*) - |A| - r_M(E) + 2 \\ &= |C^*| + r_{\Delta_A(M)}(E - C^*) - r_{\Delta_A(M)}(E) \\ &= r_{(\Delta_A(M))^*}(C^*) \\ &= 2. \end{aligned}$$

Since M is 3-connected, M has no nontrivial series classes. So C^* is also a triad of M containing x, which is a contradiction. Hence, if $A \cup x \notin \mathcal{C}^*(M)$, then x is not in any triad of $\Delta_A(M)$. Therefore, $\text{co}(\Delta_A(M)\backslash x) = \Delta_A(M)\backslash x$.

Since $\text{co}(M\backslash x) = M\backslash x$ is connected but not 3-connected, M is not a uniform matroid of rank-2. Furthermore, since A is coindependent in M, $|E(M\backslash x) - A| \geq 2$. If $E(M\backslash x) - A = \{x_1, x_2\}$, then $\{x_1, x_2\}$ is a cocircuit of $M\backslash x$. Hence $\{x, x_1, x_2\}$ is a triad of M, which is a contradiction. So $|E(M\backslash x) - A| \geq 3$.

Since x is not in any triad of M and M is 3-connected, there exists some 2-separation (X, Y) of $M\backslash x$ such that $|X| \geq 3$ and $|Y| \geq 3$. If A is a subset of X or Y, say $A \subseteq X$, then

$$\lambda_{\Delta_A(M)\backslash x}(X, Y) = \lambda_{M\backslash x}(X, Y) = 1.$$

Hence, $\Delta_A(M)\backslash x$ is not 3-connected. So assume both $X \cap A$ and $Y \cap A$ are nonempty and $|X \cap A| \geq |Y \cap A| \geq 1$. If there is some 2-separation (X', Y') of $M\backslash x$ corresponding to (X, Y) such that A is a subset of X' or Y', then

$$\lambda_{\Delta_A(M)\backslash x}(X', Y') = \lambda_{M\backslash x}(X', Y') = 1,$$

which implies $\Delta_A(M)\backslash x$ is not 3-connected. We are in the position to prove the existence of such 2-separation (X', Y') of $M\backslash x$.

If $X \subset A$ or $Y \subset A$, say $X \subset A$, then let $X' = A$, $Y' = Y - A$. Obviously, $(A, Y - A)$ is the needed 2-separation. Therefore, suppose neither X nor Y is a proper subset of A. We prove it by two subcases.

Subcase 1. $|Y \cap A| \geq 2$.

Since $|E(M\backslash x) - A| \geq 3$, at least one of $|X - A|$ and $|Y - A|$ is larger than one. Suppose $|X - A| > 1$. Let $X' = X - A$, $Y' = Y \cup A$. Then $(X - A, Y \cup A)$ is the needed 2-separation of $M\backslash x$. The case $|Y - A| > 1$ can be treated similarly to the case $|X - A| > 1$.

Subcase 2. $|Y \cap A| = 1$.

Since $|Y| \geq 3$, $|Y - A| \geq 2$. Let $X' = X \cup A$, $Y' = Y - A$. Then (X', Y') is the needed 2-separation of $M\backslash x$.

Case 2 $A \cup x \in \mathcal{C}^*(M)$.

Let F be an arbitrary cyclic flat of M containing x. Using orthogonality, $A \cap F \neq \emptyset$. Since A is a clonal set of M, by Corollary 2.4, $A \subseteq F$. Let $\mathcal{F}$ denote the collection of all cyclic flats of M containing x. Evidently, for every element F' in $\langle \mathcal{F} \rangle$, we have $A \cup x \subseteq F'$. Hence, $\text{cl}_M(x) = \{x\}$ is not in $\langle \mathcal{F} \rangle$. By Proposition 2.1, x is not fixed in M. □

The dual of Lemma 4.9 is as follows.

Corollary 4.10 *Let A be an independent set of M with at least three elements and $M^*|A \cong U_{2,|A|}$. For any $x \in E - A$, if si(M/x) is not 3-connected, then either x is not cofixed in M or si$(\nabla_A(M)/x)$ is also not 3-connected.*

Corollary 4.11 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \in E - A$, if co$(\Delta_A(M)\backslash x)$ is 3-connected, then x is not fixed in M.*

Proof Suppose x is fixed in M. Then co$(M\backslash x)$ is not 3-connected. Hence, by Lemma 4.9, co$(\Delta_A(M)\backslash x)$ is also not 3-connected, which is a contradiction. □

Lemma 4.12 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \notin cl_M(A)$, if si(M/x) is not 3-connected, then either x is not cofixed in M or si$(\Delta_A(M)/x)$ is also not 3-connected.*

Proof If x is in some triangle or a triad of M, then by Proposition 2.2 and Lemma 2.5, x is not cofixed in M. Thus assume x is neither in any triangle nor in any triad of M. Then si$(M/x) = M/x$. Assume x is in some triangle C of $\Delta_A(M)$. Since M is 3-connected and x is not in any triangle of M, $r_M(C) = 3$. If $C \cap A = \emptyset$, then by Corollary 3.4, $r_{\Delta_A(M)}(C) = r_M(C) = 3$; which contradicts to $C \in \mathcal{C}(\Delta_A(M))$. Therefore $C \cap A \neq \emptyset$ and $|C \cap A| = 1, 2$. Let $C = \{x, y, z\}$ and suppose $y \in A$. If $|C \cap A| = 1$, then following from Lemma 3.3, there exists some basis B of $\Delta_A(M)$ such that $C \subseteq B$. This contradicts to C is a circuit of $\Delta_A(M)$. Thus $|C \cap A| = 2$, that is, $\{y, z\} \subseteq A$. Similarly, if $|A| \geq 4$, then there is some basis B of $\Delta_A(M)$ such that $C \subseteq B$; which contradicts to C is a circuit of $\Delta_A(M)$. Hence $|A| = 3$. Then by the definition of $\Delta_A(M)$, there exists some element a in A such that x is parallel with a in M. This contradicts to M is 3-connected. Hence, x is also not in any triangle of $\Delta_A(M)$. Therefore si$(\Delta_A(M)/x) = \Delta_A(M)/x$. By Lemma 3.6(ii),

$$\text{si}(\Delta_A(M)/x) = \Delta_A(M)/x = \Delta_A(M/x).$$

Since M/x is connected but not 3-connected, $|E(M/x) - A| \geq 2$. Assume

$$E(M/x) - A = \{x_1, x_2\}.$$

Then $r(M/x) = 3$ and $\{x_1, x_2\}$ is a cocircuit of M/x. Hence $\{x_1, x_2\}$ is also a cocircuit of M. This contradicts to M is 3-connected. So $|E(M/x) - A| \geq 3$.

Let $M_1 = M/x$. Since x is not in any triangle of M and M is 3-connected, there is some 2-separation (X, Y) of M_1 such that $|X| \geq 3$ and $|Y| \geq 3$. Similarly to proving Case 1 of Lemma 4.9, we can prove $\Delta_A(M)/x$ is not 3-connected. □

The dual of Lemma 4.12 is as follows.

Corollary 4.13 *Let A be an independent set of M with at least three elements and $M^*|A \cong U_{2,|A|}$. For any $x \notin cl_{M^*}(A)$, if $co(M\backslash x)$ is not 3-connected, then either x is not fixed in M or $co(\nabla_A(M)\backslash x)$ is not 3-connected.*

With Lemma 4.12 in mind, following the same line as the proof of Corollary 4.11, we can obtain

Corollary 4.14 *Let A be a coindependent set of M with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \notin cl_M(A)$, if $si(\triangle_A(M)/x)$ is 3-connected, then x is not cofixed in M.*

Lemma 4.15 *Suppose $\nabla_A(M)$ is well defined. Let x be an element in $E - A$. If there exists some circuit C of $\nabla_A(M)$ satisfying $x \in C \in \mathcal{C}(\nabla_A(M))$, then there is some circuit C_1 of M such that $x \in C_1 \subseteq A \cup C$.*

Proof Since every three-element of A is a triangle of $\nabla_A(M)$, $|C \cap A| \leq 2$. We prove the lemma in three cases: $|C \cap A| = 0, 1$, or 2.

Case 1 $C \cap A = \emptyset$.

Clearly, $C_1 = C$ is a circuit of M.

Case 2 $C \cap A = \{a\}$.

By Corollary 3.5, we have

$$r_{\nabla_A(M)}(C) = 2 + r_M((C - a) \cup (A - a)) - |A| = |C| - 1,$$
$$r_{\nabla_A(M)}(C - x) = 2 + r_M((C - \{x \cup a\}) \cup (A - a)) - |A| = |C| - 1.$$

Then

$$r_M((C - a) \cup (A - a)) = |A| + |C| - 3,$$
$$r_M((C - \{x \cup a\}) \cup (A - a)) = |A| + |C| - 3.$$

Hence, $x \in \mathrm{cl}_M((C - \{x \cup a\}) \cup (A - a)) \subseteq \mathrm{cl}_M(A \cup C - x)$; which implies that there exists some circuit C_1 of M such that $x \in C_1 \subseteq A \cup C$.

Case 3 $|C \cap A| = 2$.

Similarly to Case 2, we can show that $x \in C_1 \subseteq A \cup C$ for some circuit C_1 of M. □

Lemma 4.16 *Let A be an independent set of M with at least three elements and $M^*|A \cong U_{2,|A|}$. For any element $x \in E - A$, if F is a cyclic flat of $\nabla_A(M)$ containing x, then F is also a cyclic flat of M containing x.*

Proof By Corollary 2.4 and Corollary 4.5, $F \cap A = \emptyset$ or $F \cap A = A$. Note every subset C of E disjoint from A is a circuit of M if and only if C is a circuit of $\nabla_A(M)$. Hence, if we can prove F is a flat of M, then by Lemma 4.15, F is also a cyclic flat of M. Therefore, it suffices to prove F is a flat of M.

Assume F is not a flat of M. Then there exists some element e in $E - F$ such that $e \in \mathrm{cl}_M(F)$, namely, there is some circuit C of M satisfying $e \in C \subseteq F \cup e$. If $C \cap A$ is empty, then $C \in \mathcal{C}(\nabla_A(M))$ and $e \in \mathrm{cl}_{\nabla_A(M)}(F)$. This contradicts to F is a flat of $\nabla_A(M)$. So $C \cap A$ is nonempty. According to whether $F \cap A = \emptyset$ or $F \cap A = A$, there are two cases to consider.

Case 1 $F \cap A = \emptyset$.

Since $C \cap A \neq \emptyset$ and $F \cap A = \emptyset$, $C \cap A = e \in A$. By Corollary 2.9(i),

$$r_M((C - e) \cup (A - e)) = r_M(C) = r_M(C - e) = |C| - 1.$$

By Corollary 3.5,

$$\begin{aligned} r_{\nabla_A(M)}(C - e) &= r_M(C - e) = |C| - 1, \\ r_{\nabla_A(M)}(C) &= 2 + r_M((C - e) \cup (A - e)) - |A| \\ &= |C| - |A| + 1 < |C| - 1. \end{aligned}$$

This contradicts to $r_{\nabla_A(M)}(C) \geq r_{\nabla_A(M)}(C - e)$. Hence F is a flat of M.

Case 2 $F \cap A = A$.

Since $e \in E - F$ and $F \cap A = A$, $e \notin A$.
Subcase 1. $|A \cap C| \geq 2$.
Since A is a clonal set of M,

$$r_M(C \cup A) = r_M((C - e) \cup A) = r_M(C - e) = |C| - 1.$$

Then by Corollary 3.5, we have

$$\begin{aligned} r_{\nabla_A(M)}(C) &= 2 + r_M(C \cup A) - |A| = 2 + |C| - 1 - |A| = |C| + 1 - |A|, \\ r_{\nabla_A(M)}(C - e) &= 2 + r_M((C - e) \cup A) - |A| = 2 + |C| - 1 - |A| = |C| + 1 - |A|. \end{aligned}$$

Hence, $e \in \mathrm{cl}_{\nabla_A(M)}(C - e) \subseteq \mathrm{cl}_{\nabla_A(M)}(F) = F$. This is a contradiction.

Subcase 2. $|A \cap C| = 1$.

Similarly to *Subcase 1*, we can verify F is a flat of M.

Hence, F is a flat of M. □

Lemma 4.17 *Let A be an independent set of M with at least three elements and $M^*|A \cong U_{2,|A|}$ and $x \in cl_{M^*}(A) - A$. Then x is not fixed in $\nabla_A(M)$.*

Proof Let F be an arbitrary cyclic flat of $\nabla_A(M)$ containing x. By Lemma 4.16, F is also a cyclic flat of M. Since $x \in \mathrm{cl}_{M^*}(A) - A$ and M is 3-connected, $M^*|(A \cup x) \cong U_{2,|A|+1}$. It follows from Corollary 2.9(i) that $A \cup x$ is a clonal set of M. By Corollary 2.4, $A \cup x \subseteq F$. Hence, $A \cup x$ is contained in every element of $\langle \mathcal{F} \rangle$, where $\mathcal{F}$ is the collection of all cyclic flats of $\nabla_A(M)$ containing x. Therefore, $\mathrm{cl}_{\nabla_A(M)}(x) = \{x\}$ is not in $\langle \mathcal{F} \rangle$. By Proposition 2.1, x is not fixed in M. □

Lemma 4.18 *Let A be an independent subset of E with at least three elements and $M^*|A \cong U_{2,|A|}$. For any $x \in E - A$, if x is not fixed in M, then x is not fixed in $\nabla_A(M)$.*

Proof If $x \in \mathrm{cl}_{M^*}(A) - A$, then by Lemma 4.17, x is not fixed in $\nabla_A(M)$. Now we assume $x \notin \mathrm{cl}_{M^*}(A) - A$. Then x is not a coloop of $M \backslash A$. Let M' be the matroid obtained from M by independently cloning x with x'. Clearly, A is also independent in M' and $r_{M'}(E') = r_M(E)$. Let $E' = E \cup x'$. Since x is not a coloop of $M \backslash A$, $r_{M'}(E' - A) = r_M(E - A)$. Hence, by Corollary 3.5,

$$\begin{aligned} r_{(M')^*}(A) &= |A| + r_{M'}(E' - A) - r_{M'}(E') \\ &= |A| + r_M(E - A) - r_M(E) \\ &= r_{M^*}(A) = 2. \end{aligned}$$

Since M has no nontrivial series classes, M' contains no nontrivial series classes. Hence, $(M')^*|A \cong U_{2,|A|}$. Then $\nabla_A(M')$ is well defined. Since $M' \backslash x' \cong M$ is 3-connected, then by Lemma 3.7(ii), $\nabla_A(M' \backslash x')$ is defined and $\nabla_A(M') \backslash x' \cong \nabla_A(M' \backslash x') \cong \nabla_A(M)$. By Lemma 3.9, $\{x, x'\}$ is an independent clone of $\nabla_A(M')$. Hence x is not fixed in $\nabla_A(M)$ due to Proposition 2.2. □

By duality, we obtain the following corollary.

Corollary 4.19 *Let A be a coindependent subset of E with at least three elements and $M|A \cong U_{2,|A|}$. For any $x \in E - A$, if x is not cofixed in M, then x is not cofixed in $\Delta_A(M)$.*

Lemma 4.20 *Let A be a coindependent subset of E with $M|A \cong U_{2,|A|}$. Then $\Delta_A(M)$ is a totally free matroid.*

Proof Firstly, by Lemma 4.2, $\Delta_A(M)$ is 3-connected. If $|A| = 2$, then clearly $\Delta_A(M) \cong M$. Hence $\Delta_A(M)$ is totally free. Thus assume $|A| \geq 3$.

Let x be an arbitrary element in E. First, assume $\mathrm{co}(\Delta_A(M) \backslash x)$ is 3-connected, we shall prove x is not fixed in $\Delta_A(M)$. If $x \in A$, then by Corollary 4.6(i), x is not fixed in $\Delta_A(M)$. Hence, assume $x \notin A$. Then by Corollary 4.11, x is not fixed in M. Independently cloning x with x' in M, we obtain a new matroid M'. Note A is also a coindependent set of M' and every three-element subset of A is a triangle of M'. Therefore $\triangle_A(M')$ is well defined. Since $M'\backslash x' \cong M$ is 3-connected, by Lemma 3.7(i), $\triangle_A(M')\backslash x' = \triangle_A(M'\backslash x') = \triangle_A(M)$. By Lemma 3.8, $\{x, x'\}$ is a clonal set of $\triangle_A(M')$. Hence x is not fixed in $\triangle_A(M)$ according to Proposition 2.2.

Secondly, suppose $\mathrm{si}(\triangle_A(M)/x)$ is 3-connected. If $x \in A$, then it follows from Corollary 4.6(i) that x is not cofixed in $\triangle_A(M)$. Now suppose $x \notin A$.

Case 1 $x \in \mathrm{cl}_M(A) - A$.

Note $x \in \mathrm{cl}_M(A) - A$ implies that x is in some triangle of M. Hence x is not cofixed in M according to Corollary 2.8(i). By Corollary 4.19, x is also not cofixed in $\triangle_A(M)$.

Case 2 $x \notin \mathrm{cl}_M(A)$.

From Corollary 4.14, x is not cofixed in M. Then by Corollary 4.19, x is also not cofixed in $\triangle_A(M)$.

Hence, $\triangle_A(M)$ is a totally free matroid. □

The duality of Lemma 4.20 is as follows:

Corollary 4.21 *Let A be an independent set of M with $M^*|A \cong U_{2,|A|}$. Then $\nabla_A(M)$ is a totally free matroid.*

Proof of Theorem 1.3 Note the only totally free matroid M with $|E(M)| < 5$ is $U_{2,4}$. Following from Lemma 4.20, Corollary 4.21 and Lemma 1.1, we obtain Theorem 1.3 immediately. □

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