Inverse Relations and the Products of Bernoulli Polynomials

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Abstract. In this paper, we give two inverse pairs of identities involving products of the Bernoulli polynomials and the Bernoulli polynomials of the second kind.

Key words. Bernoulli polynomial, Bernoulli polynomial of the second kind, inverse relation.

1. Introduction

The Stirling numbers of the first kind $s(n,k)$ and the Stirling numbers of the second kind $S(n,k)$ satisfy the following pair of classical inverse relations:

$$f(n) = \sum_{k=0}^{n} s(n,k)g(k), \;\; g(n) = \sum_{k=0}^{n} S(n,k)f(k). \tag{1}$$

The n-th Bernoulli polynomial $B_n(x)$ and the n-th Bernoulli polynomial of the second kind $b_n(x)$ are defined by the generating functions

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}$$

and

$$\sum_{n=0}^{\infty} b_n(x)t^n = \frac{t(1+t)^x}{\log(1+t)}.$$

Based on (1), we give the following two inverse pairs of identities involving products of the Bernoulli polynomials and the Bernoulli polynomials of the second kind.

Theorem 1. *Let $y = x_1 + \cdots + x_N$. Then for $n \geq N$, we have*

$$\sum_{\substack{k_1+\cdots+k_N=n \\ k_1,\ldots,k_N\geq 0}} \binom{n}{k_1,\ldots,k_N} B_{k_1}(x_1)\cdots B_{k_N}(x_N) = (-1)^{N-1}N\binom{n}{N}$$

$$\times \sum_{j=0}^{N-1}(-1)^j \frac{B_{n-j}(y)}{n-j} \sum_{k=0}^{j} \binom{N+k-j-1}{k} s(N, N-j+k)y^k \tag{2}$$

and for $n \geq N+1$, we have

$$(n-N)\sum_{k=0}^{N}\binom{N}{k}(-y)^k\frac{B_{n-k}(y)}{n-k} = (-1)^N\sum_{k=0}^{N}\binom{n-N+k}{k}^{-1}S(N+1,k+1)$$
$$\times \sum_{\substack{l_0+\cdots+l_k=n-N+k\\ l_0,\ldots,l_k\geq 0}}\binom{n-N+k}{l_0,\ldots,l_k}B_{l_0}\left(y-\sum_{i=1}^{k}x_i\right)\prod_{i=1}^{k}B_{l_i}(x_i). \quad (3)$$

Theorem 2. *Let $y = x_1+\cdots+x_N$. Then for $n \geq N$, we have*

$$\sum_{\substack{k_1+\cdots+k_N=n\\ k_1,\ldots,k_N\geq 0}} b_{k_1}(x_1)\cdots b_{k_N}(x_N) = \frac{(-1)^{N-1}}{(N-1)!}\sum_{j=0}^{N-1}b_{n-j}(y)$$
$$\times\sum_{k=0}^{n-N}S(N-1,N+k-j-1)k!\binom{y+N+k-j-1}{k}\binom{n-j-1}{n-k-N} \quad (4)$$

and for $n \geq N+1$, we have

$$\sum_{k=0}^{n-N-1}b_{n-k}(y)\binom{y+N}{k}\binom{n-1-k}{N} = \frac{1}{N!}\sum_{k=0}^{N}(-1)^k s(N,k)k!$$
$$\times\sum_{\substack{l_0+\cdots+l_k=n-N+k\\ l_0,\ldots,l_k\geq 0}} b_{l_0}\left(y-\sum_{i=1}^{k}x_i\right)\prod_{i=1}^{k}b_{l_i}(x_i) \quad (5)$$

where $\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!$ for complex number x.

Identity (2) was first discovered by Dilcher [3] by induction. Later, Chen [2] proved it by evaluating certain Zeta functions. Follow Dilcher's method, Wu and Pan [5] derived an identity on the sum of products of Bernoulli numbers of the second kind recently, which is the special case of (4) by setting $x_1 = \cdots = x_N = 0$.

2. Proof of Theorem 1

We first give a lemma, in which the second identity is already proved in [1] by induction. Here we offer a different proof.

Lemma 1. *Let N be a positive integer. We have*

$$\frac{1}{(e^t-1)^N} = \frac{1}{(N-1)!}\sum_{k=1}^{N}(-1)^{k-1}s(N,k)\frac{d^{k-1}}{dt^{k-1}}\left(\frac{1}{e^t-1}\right) \quad (6)$$

and

$$(-1)^N\frac{d^N}{dt^N}\left(\frac{1}{e^t-1}\right) = \sum_{k=0}^{N}S(N+1,k+1)\frac{k!}{(e^t-1)^{k+1}}. \quad (7)$$

Proof. By (1), it suffices to prove one of those two formulas, say (6). The right hand side of (6) equals

$$\frac{1}{(N-1)!}\sum_{k=1}^{N}(-1)^k s(N,k)\sum_{j=0}^{\infty} e^{jt} j^{k-1} = \frac{1}{(N-1)!}\sum_{j=0}^{\infty} e^{jt}\sum_{k=1}^{N}(-1)^k s(N,k) j^{k-1}$$
$$= (-1)^N \sum_{j=0}^{\infty}\binom{N-1+j}{j} e^{jt} = \frac{1}{(e^t-1)^N},$$

as desired. □

Lemma 2. *We have the binomial identities,*

$$\sum_{i=0}^{m}\binom{n}{i}(-1)^i = \begin{cases} 0, & m \geq n, \\ (-1)^m\binom{n-1}{m}, & \text{otherwise}, \end{cases} \tag{8}$$

and for complex number x, y,

$$\sum_{i=0}^{n}\binom{x}{i}\binom{y}{n-i} = \binom{x+y}{n}. \tag{9}$$

Lemma 3. *We have*

$$\left[\frac{t^n}{n!}\right]\left(e^{tx}\frac{d^m}{dt^m}\left(\frac{1}{e^t-1}\right)\right) = \sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}\frac{B_{j+n+1}(x)}{j+n+1}, \tag{10}$$

where we denote $[t^n]f(t)$ the coefficient of $[t^n]$ in $f(t)$.

Proof. Since

$$\frac{d^m}{dt^m}\left(\frac{1}{e^t-1}\right) = \frac{d^m}{dt^m}\left(\frac{te^{tx}}{e^t-1}\cdot\frac{e^{-tx}}{t}\right)$$
$$= \sum_{k=0}^{m}\binom{m}{k}\left(\sum_{i=0}^{\infty}\frac{B_i(x)t^{i-k}}{(i-k)!}\right)\left((-1)^{m-k}\frac{(m-k)!}{t^{m-k+1}} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}x^{j+1}t^{j-m+k}}{(j+1)(j-m+k)!}\right)$$
$$= \sum_{i=0}^{\infty} B_i(x)t^{i-m-1}\frac{m!}{i!}\sum_{k=0}^{m}(-1)^{m-k}\binom{i}{k}$$
$$+ \sum_{i,j=0}^{\infty}\frac{B_i(x)(-x)^{j+1}}{j+1}t^{i+j-m}\frac{m!}{i!j!}\sum_{k=0}^{m}\binom{i}{k}\binom{j}{m-k}$$
$$= \sum_{i=0}^{\infty}\frac{B_{i+m+1}(x)t^i}{(i+m+1)i!} + \sum_{i=0}^{\infty}\frac{t^i}{i!}\sum_{j=0}^{i+m}\binom{i+m}{j}\frac{B_j(x)(-x)^{i+m-j+1}}{i+m-j+1}$$

and since

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{i+m} b_j = \sum_{j=0}^{m+n} b_j \sum_{i=0}^{n} a_i - \sum_{j=1}^{n} b_{j+m}\sum_{i=0}^{j-1} a_i,$$

it follows that

$$\begin{aligned}
&\left[\frac{t^n}{n!}\right]\left(e^{tx}\frac{d^m}{dt^m}\left(\frac{1}{e^t-1}\right)\right)\\
&=\sum_{i=0}^{n}\binom{n}{i}\frac{B_{i+m+1}(x)}{i+m+1}x^{n-i}+\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i+m}\binom{i+m}{j}\frac{B_j(x)x^{n+m-j+1}(-1)^{i+m-j+1}}{i+m-j+1}\\
&=\sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}\frac{B_{j+n+1}(x)}{j+n+1}.
\end{aligned}$$

□

Proof of Theorem 1. We first give a proof of (2).

By the generating function of Bernoulli polynomials, it can be checked that the left hand side of (2) is equal to $\left[\frac{t^n}{n!}\right]\left(\frac{t^N e^{ty}}{(e^t-1)^N}\right)$.

From (6), one obtains

$$\sum_{\substack{k_1+\cdots+k_N=n\\ k_1,\ldots,k_N\geq 0}}\binom{n}{k_1,\ldots,k_N}B_{k_1}(x_1)\cdots B_{k_N}(x_N)$$
$$=\frac{1}{(N-1)!}\sum_{k=1}^{N}(-1)^{k-1}s(N,k)\left[\frac{t^{n-N}}{n!}\right]\left(e^{ty}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{1}{e^t-1}\right)\right).$$

From (10), the last summation equals

$$\frac{n!}{(n-N)!}\sum_{k=1}^{N}(-1)^{k-1}s(N,k)\sum_{j=0}^{k-1}\binom{k-1}{j}(-y)^{k-1-j}\frac{B_{j+n-N+1}(y)}{j+n-N+1}.$$

Changing the order of summations and replacing j by $N-1-j$, we complete the proof of (2).

Mutiplying both sides of (7) by $(-1)^N e^{ty}t^{N+1}$, then using Lemma 3 to compare the coefficients of t^n on the both sides of the new formula, we obtain (3). □

3. Proof of Theorem 2

Lemma 4. *Let N be a positive integer. We have*

$$\frac{(-1)^N N!}{\log^{N+1}(1+t)}=\sum_{i=0}^{N}S(N,i)(1+t)^i\frac{d^i}{dt^i}\left(\frac{1}{\log(1+t)}\right), \tag{11}$$

$$\frac{d^N}{dt^N}\left(\frac{1}{\log(1+t)}\right)=\frac{1}{(1+t)^N}\sum_{i=0}^{N}(-1)^i s(N,i)\frac{i!}{\log^{i+1}(1+t)}. \tag{12}$$

Proof. By (1), we only need to prove (11). It is easy to check that

$$\begin{aligned}\frac{(-1)^N N!}{\log^{N+1}(1+t)} &= \left((1+t)\frac{d}{dt}\right)\frac{(-1)^{N-1}(N-1)!}{\log^N(1+t)}\\ &= \left((1+t)\frac{d}{dt}\right)^2\frac{(-1)^{N-2}(N-2)!}{\log^{N-1}(1+t)}\\ &\cdots\\ &= \left((1+t)\frac{d}{dt}\right)^N\frac{1}{\log(1+t)}. \qquad (13)\end{aligned}$$

Note that (see [4])

$$\left((1+t)\frac{d}{dt}\right)^n = \sum_{i=1}^{n} S(n,i)(1+t)^i\frac{d^i}{dt^i}. \qquad (14)$$

Then identity (11) can be deduced from (13) and (14). □

Lemma 5. *Let x be complex number, and m,n,r be integers, for $m,n\geq 0$ we have*

$$[t^n]\left((1+t)^{x+r}\frac{d^m}{dt}\left(\frac{1}{\log(1+t)}\right)\right) = m!\sum_{j=0}^{n} b_{j+m+1}(x)\binom{x+r}{n-j}\binom{m+j}{m}. \qquad (15)$$

Proof. We have

$$\begin{aligned}&(1+t)^{x+r}\frac{d^m}{dt^m}\left(\frac{1}{\log(1+t)}\right) = (1+t)^{x+r}\frac{d^m}{dt^m}\left(\frac{t(1+t)^x}{\log(1+t)}\cdot\frac{1}{t(1+t)^x}\right)\\ &= (1+t)^{x+r}\sum_{k=0}^{m}\binom{m}{k}\sum_{j=k}^{\infty} b_j(x)t^{j-k}\frac{j!}{(j-k)!}\sum_{i=0}^{m-k}\binom{m-k}{i}\\ &\quad\times(-1)^{m-k}\frac{(m-k-i)!}{t^{m-k-i+1}}\binom{x+i-1}{i}\frac{i!}{(1+t)^{x+i}}\\ &= m!\sum_{k=0}^{m}\sum_{j=0}^{\infty} b_j(x)\binom{j}{k}\sum_{i=0}^{m-k}(-1)^{m-k}\binom{x+i-1}{i}\sum_{l=0}^{\infty}\binom{r-i}{l+m+1-i-j}t^l.\end{aligned}$$

Extracting the coefficient of t^n yields

$$m!\sum_{k=0}^{m}\sum_{i=0}^{m-k}(-1)^{m-k}\binom{x+i-1}{i}\binom{r-i}{m+n+1-i-j}\sum_{j=0}^{m+n+1-i} b_j(x)\binom{j}{k}.$$

Now changing the summation order appropriately then applying (8) and (9) and noticing $\binom{-x}{n} = (-1)^n\binom{x+n-1}{n}$, we get the desired result. □

Proof of Theorem 2. By the generating function of Bernoulli polynomials of the second kind, the left side of (4) equals $[t^n]\left(\frac{t^N(1+t)^y}{\log^N(1+t)}\right)$.

From (11) we have

$$\sum_{\substack{k_1+\cdots+k_N=n\\ k_1,\ldots,k_N\geq 0}} b_{k_1}(x_1)\cdots b_{k_N}(x_N)$$
$$= \frac{(-1)^{N-1}}{(N-1)!}\sum_{k=0}^{N-1} S(N-1,k)[t^{n-N}]\left((1+t)^{y+k}\frac{d^k}{dt^k}\left(\frac{1}{\log(1+t)}\right)\right).$$

From (15), the last summation equals

$$\frac{(-1)^{N-1}}{(N-1)!}\sum_{k=0}^{N-1} S(N-1,k)k!\sum_{j=0}^{n-N}\binom{y+k}{n-N-j}\binom{k+j}{k}b_{k+j+1}(y).$$

Now changing the order of summations, and replacing j by $n-k-j-1$, we complete the proof of (4).

Multiplying both sides of (12) by $t^{N+1}(1+t)^{N+y}$, then using Lemma 5 to compare the coefficients of t^n on the both sides of the new formula, we derive (5). □

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