Bijections for 2-Plane Trees and Ternary Trees

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Abstract

According to the Fibonacci number which is studied by Prodinger et al., we introduce the 2-plane tree which is a planted plane tree with each of its vertices colored with one of two colors and -free. The similarity of the enumeration between 2-plane trees and ternary trees leads us to build several bijections. Especially, we found a bijection between the set of 2-plane trees of $n+1$ vertices with a black root and the set of ternary trees with n internal vertices. We also give a combinatorial proof for a relation between the set of 2-plane trees of $n+1$ vertices and the set of ternary trees with n internal vertices.

Keywords: planted plane trees, 2-plane trees, ternary trees, bijection.

AMS Subject Classification: 05A05, 05A15, 05C05

1 Introduction

Trees which were first studied by Cayley [1] play a very important role in combinatorics [10] and appear in a large number of applications in other branches of mathematics. A rooted tree is a tree in which a special vertex is singled out as the root of the tree. The number of rooted trees with n vertices is enumerated by Sloane's $A000081$ [9]. A vertex w is said to be a *child* or *successor* of a vertex v if w is on the next lower level connected to v; the vertex v is then said to be the *parent* of w. The *degree* of v which is usually called the *out-degree* of v is the total number of its children. A *leaf* is a vertex with degree 0, that is a vertex with no child.

A rooted tree in which the children of each vertex are ordered is called a *planted plane tree*. The number of planted plane trees of $n+1$ vertices is enumerated by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1}$$

A *binary tree* is a planted plane tree in which each vertex has at most two children and each child of a vertex is designated as its *left* or *right child* [12].

Prodinger and Tichy in [8] introduced the Fibonacci number $f(G)$ of a (simple) graph G as the total number of all Fibonacci subsets S of the vertex $V(G)$ of G, where a Fibonacci subset S is a (possibly empty) subset of $V(G)$ such that any two vertices of S are not adjacent. In graph theory, a Fibonacci subset is called independent or internally stable set of vertices. The number of Fibonacci subsets is of interest in theoretical chemistry and is called Merrifield-Simmons index. Kirschenhofer et al. in [5] studied the total numbers of the Fibonacci subsets of some kinds of trees, for instance, the binary trees, the t-ary trees, and the planted plane trees.

In [3], Gu et al. introduced the 2-binary tree which is defined as a binary tree with each of its vertices colored with one of two colors, for instance, black or white and the root is colored black. According to the definition, an edge e in a 2-binary tree is of the following eight types: , , , , , , , and . They call a 2-binary tree T *e-free* if and only if there is no edge of type e in T. In that paper, they studied several types of the 2-binary trees and found bijections between those trees and other combinatorial structures. Especially, they built a bijection between the set of -free 2-binary trees with n vertices and the set of ternary trees with n internal vertices. In [7], a simpler bijection between these two sets was presented.

Coloring vertices for plane trees has been studied by several authors [2, 11]. In [2], the authors introduced several families of supper-Catalan numbers by using the idea of giving a restricted bi-color to the existed Catalan structures. In this paper, we introduce a type of planted plane trees, where all the vertices of a planted plane tree are colored with one of two colors, for instance, black or white. Combining trees of this type with the Fibonacci subsets, we focus on the -free type.

Definition 1.1 *A 2-plane tree is a planted plane tree with each of its vertices colored with one of two colors, for instance, black or white and*  *-free.*

In Section 2, we show that there is a bijection between the set of the Fibonacci subsets of planted plane trees of n vertices and the set of 2-plane trees of n vertices.

A *ternary tree* is a planted plane tree in which each vertex has degree 0 or 3, and each child of a vertex is designated as its *left*, *middle*, or *right* child (see [4, 6]). In the literature, this kind of tree is often called complete ternary tree.

The number of ternary trees with n internal vertices which are non-leaves vertices is enumerated by the generalized Catalan number

$$T_n = \frac{1}{2n+1}\binom{3n}{n}. \tag{1.2}$$

Here we give the following definition about ternary trees which is used in this paper.

Definition 1.2 *For a ternary tree T, we define the leftmost path of l_1 as $l_1 l_2 \ldots l_s$ where l_{i+1} is the left child of l_i for $i = 1, 2, \ldots, s-1$, and $l_1, l_2, \ldots, l_s$ are all internal vertices. When l_s is a leaf, we call this path the longest leftmost path of l_1. Likewise, we define the rightmost path of r_1 as $r_1 r_2 \ldots r_t$ where r_{i+1} is the right child of r_i for $i = 1, 2, \ldots, t-1$, and $r_1, r_2, \ldots, r_t$ are all internal vertices. When r_t is a leaf, we call this path the longest rightmost path of r_1.*

In Section 2, we build a bijection between the set of 2-plane trees of $n+1$ vertices with a black root and the set of ternary trees with n internal vertices. In Section 3, we give a combinatorial proof for a relation between the set of 2-plane trees of $n+1$ vertices and the set of ternary trees with n internal vertices. Finally, in Section 4, we study some other relations between 2-plane trees and ternary trees.

2 2-Plane Trees with a Black Root and Ternary Trees

In [8, Corollary 2], the authors studied the average numbers of the Fibonacci subsets of planted plane trees of n vertices, and gave the following results.

Lemma 2.1 *[8, Corollary 2] The average numbers of Fibonacci subsets of planted plane trees of n vertices are given by:*

(a) (not containing the root)

$$a_n := \binom{3n-2}{n-1} \Big/ \binom{2n-2}{n-1}; \tag{2.1}$$

(b) (containing the root)

$$b_n := \frac{n}{n-1}\binom{3n-3}{n-2} \Big/ \binom{2n-2}{n-1}; \tag{2.2}$$

(c) (in total)

$$2\binom{3n-3}{n-1} \Big/ \binom{2n-2}{n-1} \sim \sqrt{3}\cdot\left(\frac{27}{16}\right)^{n-1}, \quad (n\to\infty); \tag{2.3}$$

(d)

$$\frac{a_n}{b_n} = 3 - \frac{2}{n}. \tag{2.4}$$

Combining the Fibonacci subsets of planted plane trees with 2-plane trees, we have the following lemma.

Lemma 2.2 *The number of the Fibonacci subsets of planted plane trees of n vertices is equal to the number of 2-plane trees of n vertices.*

Proof. Given a Fibonacci subset of a planted plane tree, we just color the vertices which belong to the subset with black. Other vertices are colored with white. According to the property of the Fibonacci subset that any two vertices in the Fibonacci subset can not be connected by a edge, we find out that the tree we get is a 2-plane tree. Conversely, for a 2-plane tree, we select all the black vertices to form the Fibonacci subset. It is easy to see that this map is a bijection. ∎

Therefore, multiplying the average numbers in Theorem 2.1 by the Catalan number, we count the numbers of 2-plane trees with a black root or a white root, respectively.

Lemma 2.3 *The numbers of* 2*-plane trees of* n *vertices are given by:*

(a) (with a white root)

$$A_n := \frac{1}{n}\binom{3n-2}{n-1}; \tag{2.5}$$

(b) (with a black root)

$$B_n := \frac{1}{n-1}\binom{3n-3}{n-2}; \tag{2.6}$$

(c) (in total)

$$S_n := \frac{2}{n}\binom{3n-3}{n-1}. \tag{2.7}$$

Due to Lemma 2.1, Lemma 2.2, and Lemma 2.3, we obtain the following theorem by noticing that $B_{n+1} = T_n$.

Theorem 2.4 *There is a bijection between the set of* 2*-plane trees of* $n+1$ *vertices with a black root and the set of ternary trees with* n *internal vertices.*

Proof. We define a map α between these two sets recursively. For a ternary tree with n internal vertices T, we illustrate the bijection by three steps to construct P as a 2-plane tree of $n+1$ vertices with a black root. In each step, we use α_i $(i = 1, 2, 3)$ to denote the map.

Step 1:

We show the bijection α_1 in Figure 2.1. First, we start with an extra black vertex e as the root of P. Then we decompose the ternary tree T into subtrees whose roots are the internal vertices on the longest rightmost path of the root v_1 of T, and map these roots as the white children of the extra black vertex e in turn.

The right picture in Figure 2.1 is the 2-plane tree P corresponding to $\alpha_1(T)$ with a black root e, where v_i' (resp. R_i') corresponds to v_i (resp. R_i) for $i = 1, 2, \ldots, d$ in the ternary tree T.

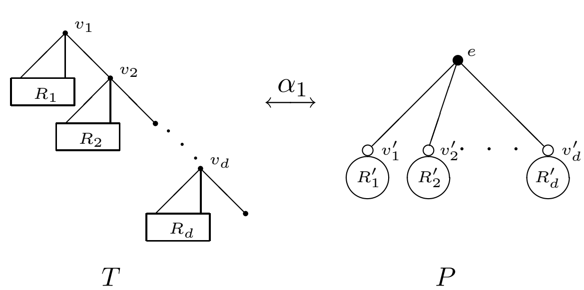

Figure 2.1: Step 1 of the bijection $\alpha = \alpha_1$

Now we map the subtrees with the root v_i for $i = 1, 2, \ldots, d$ by the following two steps. First, we show the map for the left subtree of v_i.

Step 2: In Figure 2.2, l_1 and m_1 are the left and middle children of v_i, and $l_1 l_2 \ldots l_s$ is the rightmost path of l_1. First, we use α_1 to map the subtree with the root l_1, and let these white vertices l'_1, l'_2, …, l'_s corresponding to l_1, l_2, …, l_s be the children of v'_i which corresponds to v_i. Then map the middle child of v_i m_1 to be a black vertex m'_1 as the right brother of l'_s. Here L'_i corresponds to L_i in the ternary tree.

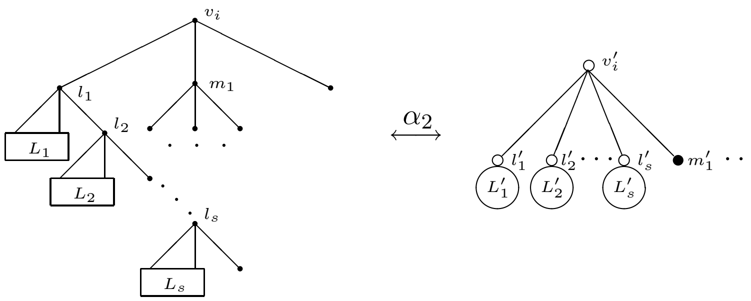

Figure 2.2: Step 2 of the bijection $\alpha = \alpha_2$

In the next step, we show the map for the middle subtree of v_i.

Step 3: In Figure 2.3, the black vertex m'_1 corresponds to m_1 which is the middle child of v_i just as we show in Figure 2.2. First, we use the map α_1 to map the subtrees with the roots k_1 and t_1, respectively. Let the corresponding white vertices $k'_1, k'_2, \ldots, k'_p$ be the children of m'_1 in turn, and let the corresponding white vertices $t'_1, t'_2, \ldots, t'_q$ be the right brothers of m'_1 in turn. Then m_2 which is the middle child of m_1 is mapped to be a black vertex as the right brother of t'_q, and the subtrees of m_2 are mapped by using the map α_3 recursively. That is to say, the left subtree of m_2 are mapped by α_1 to be the subtrees of m'_2, the right subtree of m_2 are mapped by α_1 to be the subtrees of v'_i which are right next to the subtree with the root m'_2, and the middle child of m_2 is mapped to be a black child of v_i right next to the corresponding subtrees of the right subtree of m_2.

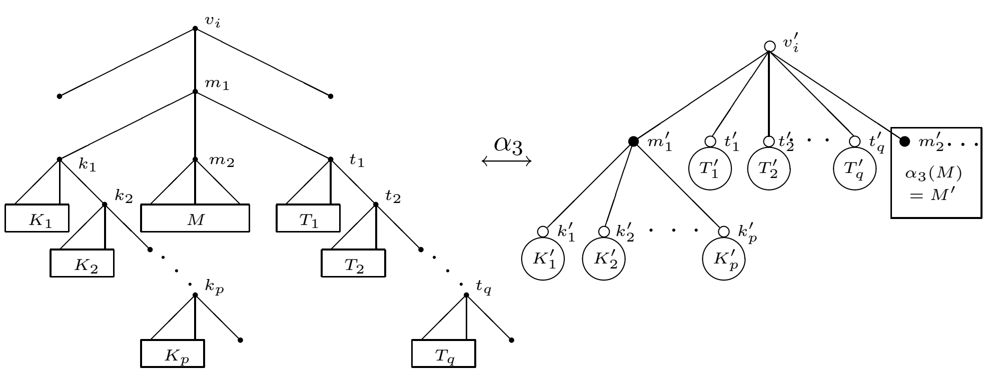

Figure 2.3: Step 3 of the bijection $\alpha = \alpha_3$

According to the three steps, we can find a bijection between the set of 2-plane trees of $n+1$ vertices with a black root and the set of ternary trees with n internal vertices. ∎

Now we give an example in Figure 2.4 to explain the bijection.

According to the bijection α, we have the following corollaries.

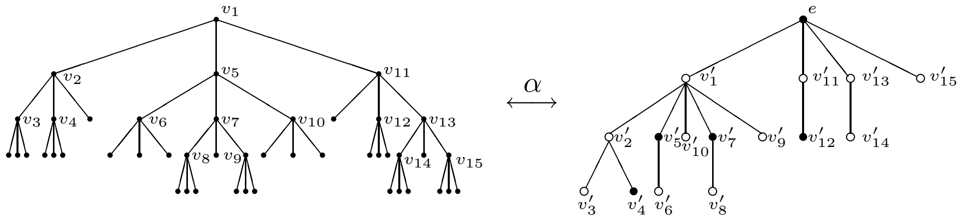

Figure 2.4: An example for the bijection α in Theorem 2.4

Corollary 2.5 *The number of the internal vertices on the longest rightmost path of the root in the set of ternary trees with n internal vertices is equal to the number of the root's children in the set of 2-plane trees of $n+1$ vertices with a black root.*

Corollary 2.6 *The number of the internal vertices as middle children in the set of ternary trees with n internal vertices is equal to the number of the black vertices except for the black root in the set of 2-plane trees of $n+1$ vertices with a black root.*

Corollary 2.7 *The number of the internal vertices as root, left, and right children in the set of ternary trees with n internal vertices is equal to the number of the white vertices in the set of 2-plane trees of $n+1$ vertices with a black root.*

In [3], the authors gave a bijection between the set of $\diagdown$-free 2-binary trees of n vertices and the set of ternary trees with n internal vertices. According to Theorem 2.4 and [3, Theorem 2.1] we have the following result.

Corollary 2.8 *There is a bijection between the set of 2-plane trees of $n+1$ vertices with a black root and the set of $\diagdown$-free 2-binary trees of n vertices.*

3 A Relation between 2-Plane Trees and Ternary Trees

According to Equation (1.2) and Lemma 2.3, we obtain the following theorem.

Theorem 3.1 *We have*

$$2(2n+1)T_n = (n+1)S_{n+1}, \tag{3.1}$$

where T_n enumerates the number of ternary trees with n internal vertices, and S_{n+1} enumerates the number of 2-plane trees with $n+1$ vertices.

Here we describe Equation (3.1) in a bijective combinatorial way. The left-hand side of Equation (3.1) can be interpreted as ternary trees with n internal vertices, where one of the $2n+1$ leaves is colored with one of the two colors blue or red. For convenience, we use b or r to mark the colored leaf. We use E_n to denote this set. Similarly, the right-hand side of Equation (3.1) can be interpreted as 2-plane trees of $n+1$ vertices, where one of the $n+1$ vertices is marked with the label a. We use F_{n+1} to denote the set.

We divide the sets E_n and F_{n+1} into several parts, and then build the bijections between these parts to build a combinatorial proof of Theorem 3.1.

First, we state the divided parts for the set E_n, and give the enumerative formula in each case.

(1) Let E_{b1} (resp. E_{r1}) denote the set of ternary trees with n internal vertices, where one of the middle leaves is marked with b (resp. r). The enumerative formula is $\binom{3n-1}{n-1}$.

(2) Let E_{b2} (resp. E_{r2}) denote the set of ternary trees with n internal vertices, where one of the left leaves on the longest leftmost path of a vertex which is a middle child or the root is marked with b (resp. r). The enumerative formula is $\frac{n+2}{3(2n+1)}\binom{3n}{n}$.

(3) Let E_{b3} denote the set of ternary trees with n internal vertices, where one of the right leaves is marked with b, or one of the left leaves on the longest leftmost path of a right child is marked with b. The enumerative formula is $\frac{n}{2n+1}\binom{3n}{n}$.

(4) Let E_{r0} denote the set of ternary trees with n internal vertices, where v denotes the right child of the root. If v is a right leaf, then v is marked with r; if v is an internal vertex, then the left leaf on the longest leftmost path of v is marked with r. The enumerative formula is $\frac{1}{2n+1}\binom{3n}{n}$.

(5) Let E_{r3} denote the set of ternary trees with n internal vertices, where one of the right leaves is marked with r, or one of the left leaves on the longest leftmost path of a right child is marked with r except for the case in the set E_{r0}. The enumerative formula is $\frac{n-1}{2n+1}\binom{3n}{n}$.

We use F_{bi} (resp. F_{ri}) to denote the corresponding set of E_{bi} (resp. E_{ri}) for $i = 0, 1, 2, 3$.

(1) Let F_{b1} (resp. F_{b2}) denote the set of 2-plane trees of $n+1$ vertices with a black root, where one of the white (resp. black) vertices is marked with a.

(2) Let F_{r1} denote the set of 2-plane trees of $n+1$ vertices with a white root, where all the children of the root are white, and one of the white vertices except for the root is marked with a.

(3) Let F_{r2} denote the set of 2-plane trees of $n+1$ vertices with a white root, where all the children of the root are white, and one of the black vertices or the root is marked with a.

(4) Let F_{b3} denote the set of 2-plane trees of $n+1$ vertices with a white root, where the leftmost child of the root is black, and one of the vertices except for the root is marked with a.

(5) Let F_{r0} denote the set of 2-plane trees of $n+1$ vertices with a white root, where the leftmost child of the root is black, and the root is marked with a.

(6) Let F_{r3} denote the set of 2-plane trees of $n+1$ vertices with a white root, where at least one of the root's children is black, and the leftmost child of the root is white. One of the vertices is marked with a.

Now we derive the enumerative formulas with generating functions of the above subsets. Here we just list a few cases. Other cases are similar.

(1) Enumeration of E_{b1}

Let $T := T(z)$ be the generating function of ternary trees: $T = 1 + zT^3$. Let A be the generating function of ternary trees, where one of the middle leaves is marked with b. We have

$$A = 3zAT^2 + zT^2.$$

To read off coefficients, we use formal residue calculus. Set $z = v/(1+v)^3$, then $T = 1 + v$, $\frac{dz}{dv} = \frac{1-2v}{(1+v)^4}$, and

$$A = \frac{zT^2}{1 - 3zT^2} = \frac{\frac{v}{1+v}}{1 - 3\frac{v}{1+v}} = \frac{v}{1-2v}.$$

Then

$$\begin{aligned}
[z^n]A &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}A \\
&= \frac{1}{2\pi i}\oint \frac{dv(1-2v)(1+v)^{3n+3}}{(1+v)^4 v^{n+1}}\frac{v}{1-2v} \\
&= \frac{1}{2\pi i}\oint \frac{dv(1+v)^{3n-1}}{v^n} \\
&= [v^{n-1}](1+v)^{3n-1} = \binom{3n-1}{n-1}.
\end{aligned}$$

Therefore, the number of ternary trees with n internal vertices, where one of the middle leaves is marked with b is enumerated by $\binom{3n-1}{n-1}$.

(2) Enumeration of F_{b2}

Let

$$B := B(z,u) = \frac{zu}{1-W} \qquad \text{and} \qquad W := W(z,u) = \frac{z}{1-B-W}$$

enumerate the 2-plane trees, according to the vertices and the black vertices. B means that the root is black, and W means that the root is white.

In this case, we are interested in $F = B_u(z,1) = \frac{(v^2+v-1)v}{(2v-1)(1+v)^3}$, with $z = \frac{v}{(1+v)^3}$, $B(z,1) = \frac{v}{(1+v)^2}$, and $W(z,1) = \frac{v}{1+v}$. Then

$$\begin{aligned}
[z^{n+1}]F &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+2}}F \\
&= \frac{1}{2\pi i}\oint \frac{dv(1-2v)(1+v)^{3n+6}}{(1+v)^4 v^{n+2}}\frac{(v^2+v-1)v}{(2v-1)(1+v)^3} \\
&= \frac{1}{2\pi i}\oint \frac{dv(1+v)^{3n-1}(1-v-v^2)}{v^{n+1}} \\
&= [v^n](1+v)^{3n-1}(1-v-v^2) \\
&= [v^n](1+v)^{3n-1} - [v^{n-1}](1+v)^{3n} \\
&= \binom{3n-1}{n} - \binom{3n}{n-1} = \frac{n+2}{3(2n+1)}\binom{3n}{n},
\end{aligned}$$

as desired. Therefore, the enumerative formula for F_{b2} is $\frac{n+2}{3(2n+1)}\binom{3n}{n}$.

(3) Enumeration of F_{r3}

We have $W = \frac{v}{1+v}$, $B = \frac{v}{(1+v)^2}$, and we need

$$F = \frac{zW}{1-B-W} - \frac{zW}{1-W} = \frac{v^3}{(1+v)^3}.$$

Then

$$\begin{aligned}
[z^{n+1}]F &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+2}}F \\
&= \frac{1}{2\pi i}\oint \frac{dv(1-2v)(1+v)^{3n+6}}{(1+v)^4 v^{n+2}}\frac{v^3}{(1+v)^3} \\
&= \frac{1}{2\pi i}\oint \frac{dv(1-2v)(1+v)^{3n-1}}{v^{n-1}} \\
&= [v^{n-2}](1-2v)(1+v)^{3n-1} \\
&= \binom{3n-1}{n-2} - 2\binom{3n-1}{n-3} \\
&= \frac{n-1}{(2n+1)(n+1)}\binom{3n}{n}.
\end{aligned}$$

Now we mark an arbitrary vertex, introducing a factor $n+1$, and obtain, as desired

$$\frac{n-1}{2n+1}\binom{3n}{n},$$

which enumerates the number of 2-plane trees of $n+1$ vertices with a white root, where at least one of the root's children is black, and the leftmost child of the root is white. One of the vertices is marked with a.

In the following subsections, we build the bijections between the two sets in each pair, where the pairs are $\{E_{b1}, F_{b1}\}$, $\{E_{b2}, F_{b2}\}$, $\{E_{b3}, F_{b3}\}$, $\{E_{r0}, F_{r0}\}$, $\{E_{r1}, F_{r1}\}$, $\{E_{r2}, F_{r2}\}$, and $\{E_{r3}, F_{r3}\}$. We also give some combinatorial interpretations for the enumerative formulas. For convenience, we use $|Q|$ to denote the cardinality of a set Q.

3.1 Bijections for $\{E_{b1}, F_{b1}\}$ and $\{E_{r1}, F_{r1}\}$

Lemma 3.2 *There is a bijection between E_{b1} and F_{b1}. Let E_{b1} denote the set of ternary trees with n internal vertices, where one of the middle leaves is marked with b. Similarly, let F_{b1} denote the set of 2-plane trees of $n+1$ vertices with a black root and one of the white vertices is marked with a. The enumerative formula is $\binom{3n-1}{n-1}$.*

Proof. For a ternary tree $T \in E_{b1}$, we use the bijection α in Theorem 2.4 to construct the corresponding 2-plane tree P of $n+1$ vertices with a black root. Then we give an algorithm to mark one of the white vertices with label a in this 2-plane tree with a black root.

The algorithm is described as follows.

For a ternary tree T, we start with the middle leaf which is marked with b. Put a label v on this leaf.

Step 1: If the father of v is a middle child, then move the label v to the father, and repeat Step 1. Otherwise, go to Step 2.

Step 2: Mark the corresponding vertex in P for the father of v with label a, and end the algorithm.

According to the bijection α in Theorem 2.4 and Corollary 2.7 , we notice that the marked vertex in P is white.

It is obvious that the map is a one-to-one correspondence.

In fact, $|E_{b1}|$ enumerates the number of the middle leaves in the set of ternary trees with n internal vertices. We can prove that the number of the middle leaves is one third of the number of all the leaves. For a ternary tree with n internal vertices and one marked middle leaf v, let l (resp. r) denote the left (resp. right) brother of v. After exchanging the marked leaf with the subtree with root l or r, respectively, we obtain two different ternary trees with one marked left or right leaf. According to Equation (1.2), the number of middle leaves in the set of ternary trees with n internal vertices is enumerated by:

$$|E_{b1}| = \frac{2n+1}{3} T_n = \binom{3n-1}{n-1}. \tag{3.2}$$

∎

For example, in Figure 2.4, if the middle child of v_4 is marked with b, then we mark v_2' with label a. If the middle child of v_7 is marked with b, then we mark v_1' with label a.

According to the proof of Lemma 3.2, we can easily get the following corollary.

Corollary 3.3 *The number of the middle leaves in the set of ternary trees with n internal vertices is equal to the number of the white vertices in the set of 2-plane trees of $n+1$ vertices with a black root. The enumerative formula is $\binom{3n-1}{n-1}$.*

Lemma 3.4 *There is a bijection between E_{r1} and F_{r1}. We use E_{r1} to denote the set of ternary trees with n internal vertices, where one of the middle leaves is marked with r. Likewise, we use F_{r1} to denote the set of 2-plane trees of $n+1$ vertices with a white root, where all the children of the root are white, and one of the white vertices except for the root is marked with a. The enumerative formula is $\binom{3n-1}{n-1}$.*

Proof. According to the proof of Lemma 3.2, for a ternary tree in E_{r1}, we only need to change the extra black root to a white root in the corresponding 2-plane tree. The enumerative formula for E_{r1} is the same as that for E_{b1}. ∎

3.2 Bijections for $\{E_{b2}, F_{b2}\}$ and $\{E_{r2}, F_{r2}\}$

Lemma 3.5 *There is a bijection between E_{b2} and F_{b2}. Let E_{b2} denote the set of ternary trees with n internal vertices, where one of the left leaves on the longest leftmost path of a* *vertex which is a middle child or the root is marked with b. Similarly, let F_{b2} denote the set of 2-plane trees of $n+1$ vertices with a black root, where one of the black vertices is marked with a. The enumerative formula is* $\frac{n+2}{3(2n+1)}\binom{3n}{n}$.

Proof. For a ternary tree with n internal vertices $T \in E_{b2}$, we still use the bijection α in Theorem 2.4 to get a 2-plane tree P of $n+1$ vertices with a black root. Now we mark one of the black vertices in the 2-plane tree.

If the marked left leaf is on the longest leftmost path of a middle child, then we put a label a on the corresponding black vertex of this middle child in the corresponding 2-plane tree P; if the marked left leaf is on the longest leftmost path of the root, then we mark the black root with label a in the corresponding 2-plane tree P.

According to the map, we can obviously find the inverse map.

In the set of ternary trees with n internal vertices, according to the proof of Lemma 3.5, we can see that $|E_{b2}|$ enumerates the number of the internal vertices which are a middle child or the root. According to the proof of Lemma 3.2, we also observe that $|E_{b1}|$ enumerates the number of internal vertices which are a left child, a right child, or the root. According to Equation (1.2), we get the following relation:

$$|E_{b2}| = nT_n - |E_{b1}| + T_n = \frac{n+2}{3(2n+1)}\binom{3n}{n}. \tag{3.3}$$

∎

For example, in Figure 2.4, if the left child of v_3 is marked with b, then we mark e with label a. If the left child of v_8 is marked with b, then we mark v_7' with label a.

Corollary 3.6 *The number of the marked left leaves on the longest leftmost paths of the vertices which are the middle children or the root (or the number of the internal vertices which are the middle children or the root) is equal to the number of the black vertices in 2-plane trees of $n+1$ vertices with a black root. The enumerative formula is* $\frac{n+2}{3(2n+1)}\binom{3n}{n}$.

Lemma 3.7 *There is a bijection between E_{r2} and F_{r2}. We use E_{r2} to denote the set of ternary trees with n internal vertices, where one of the left leaves on the longest leftmost path of a vertex which is a middle child or the root is marked with r. Similarly, we use F_{r2} to denote the set of 2-plane trees of $n+1$ vertices with a white root, where all the children of the root are white, and one of the black vertices or the root is marked with a. The enumerative formula is* $\frac{n+2}{3(2n+1)}\binom{3n}{n}$.

Proof. According to the proof of Lemma 3.5, for a ternary tree in E_{r2}, we only need to change the extra black root to a white root in the corresponding 2-plane tree. The enumerative formula for E_{r2} is the same as that for E_{b2}. ∎

3.3 Bijections for $\{\widetilde{E}_{b3}, F_{b3}\}$ and $\{\widetilde{E}_{r0}, F_{r0}\}$

In this section, we first introduce a proposition which can simplify the proofs of the bijections for $\{E_{b3}, F_{b3}\}$, $\{E_{r0}, F_{r0}\}$, and $\{E_{r3}, F_{r3}\}$.

Proposition 3.8 *There is a bijection between the set of ternary trees with n internal vertices, where one of the internal vertices is marked and the set of ternary trees with n internal vertices, where one of the right leaves is marked, or one of the left leaves on the longest leftmost path of a right child is marked.*

Proof. Let M denote the set of ternary trees with n internal vertices, where one of the internal vertices is marked. Let N denote the set of ternary trees with n internal vertices, where one of the right leaves is marked, or one of the left leaves on the longest leftmost path of a right child is marked.

For a marked internal vertex in a ternary tree which belongs to M, let v denote its right child. Now we use the following map to get a ternary tree which belongs to N.

Step 1: Let the father of v be unmarked.

Step 2: If v is a leaf, then we mark the leaf v; if v is a internal vertex, then we find the longest leftmost path of v, and mark the left leaf on this path.

It is easy to see that the map is a one-to-one correspondence. ∎

According to Proposition 3.8, we can build the three new sets $\widetilde{E}_{b3}$, $\widetilde{E}_{r0}$, and $\widetilde{E}_{r3}$ which are in bijection with the sets E_{b3}, E_{r0}, and E_{r3}, respectively.

- Let $\widetilde{E}_{b3}$ denote the set of ternary trees with n internal vertices, where one of the internal vertices is marked with b.
- Let $\widetilde{E}_{r0}$ denote the set of ternary trees with n internal vertices, where the root is marked with r.
- Let $\widetilde{E}_{r3}$ denote the set of ternary trees with n internal vertices, where one of the internal vertices except for the root is marked with r.

In the following, we build the bijections for $\{\widetilde{E}_{b3}, F_{b3}\}$, $\{\widetilde{E}_{r0}, F_{r0}\}$, and $\{\widetilde{E}_{r3}, F_{r3}\}$.

Lemma 3.9 *There is a bijection between $\widetilde{E}_{b3}$ and F_{b3}. Let $\widetilde{E}_{b3}$ denote the set of ternary trees with n internal vertices, where one of the internal vertices is marked with b. Similarly, let F_{b3} denote the set of 2-plane trees of $n+1$ vertices with a white root, where the leftmost child of the root is black, and one of the vertices except for the root is marked with a. The enumerative formula is $\frac{n}{2n+1}\binom{3n}{n}$.*

Proof. For a ternary tree $T \in \widetilde{E}_{b3}$, the map is just like the bijection described in Figure 2.3. But we only consider the tree with the root m_1 as a ternary tree T in the left picture. In the right picture, let v_i' denote an extra white root. Since the root m_1 in T is mapped to be a black vertex m_1', we ensure that the leftmost child of the root is black in the corresponding 2-plane tree. The corresponding vertex of the marked vertex in T is labeled by a. It is easy to see that this map is a bijection.

According to Equation (1.2), the enumerative formula for $\widetilde{E}_{b3}$ is

$$|\widetilde{E}_{b3}| = |E_{b3}| = nT_n = \frac{n}{2n+1}\binom{3n}{n}. \tag{3.4}$$

∎

In Figure 3.5, we use the ternary tree which is given in Figure 2.4 to show the bijection in Lemma 3.9. v_6 is marked with b in the ternary tree, and v_6' is labeled with a.

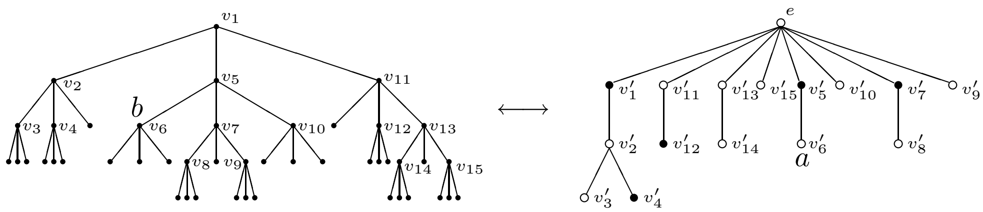

Figure 3.5: Bijection in Lemma 3.9

Lemma 3.10 *There is a bijection between $\widetilde{E}_{r0}$ and F_{r0}. We use $\widetilde{E}_{r0}$ to denote the set of ternary trees with n internal vertices, where the root is marked with r. Likewise, we use F_{r0} to denote the set of 2-plane trees of $n+1$ vertices with a white root, where the leftmost child of the root is black, and the root is marked with a. The enumerative formula is $\frac{1}{2n+1}\binom{3n}{n}$.*

Proof. According to the proof of Lemma 3.9, we only need to change the rule for mapping the marked vertex. When the root of a ternary tree which belongs to $\widetilde{E}_{r0}$ is marked with r, we use the bijection in Lemma 3.9 to map the tree and put a label a on the root of the corresponding 2-plane tree which obviously belongs to F_{r0}.

It is easy to see that the enumerative formula for $\widetilde{E}_{r0}$ is

$$|\widetilde{E}_{r0}| = |E_{r0}| = T_n = \frac{1}{2n+1}\binom{3n}{n}. \tag{3.5}$$

∎

3.4 Bijection for $\{\widetilde{E}_{r3}, F_{r3}\}$

Lemma 3.11 *There is a bijection between $\widetilde{E}_{r3}$ and F_{r3}. Let $\widetilde{E}_{r3}$ denote the set of ternary trees with n internal vertices, where one of the internal vertices except for the root is marked with r. Likewise, let F_{r3} denote the set of 2-plane trees of $n+1$ vertices with a white root, where at least one of the root's children is black, and the leftmost child of the root is white. One of the vertices is marked with a. The enumerative formula is $\frac{n-1}{2n+1}\binom{3n}{n}$.*

Proof. We divide the set $\widetilde{E}_{r3}$ into three parts according to the fact that for a ternary tree $T \in \widetilde{E}_{r3}$, the marked vertex can be in the left, middle, or right subtree of the root. Then we build three bijections to get the set F_{r3}.

Let A_1 (resp. A2 or A3) denote the set of ternary trees with n internal vertices, where the marked vertex is in the left (resp. middle or right) subtree of the root.

We also divide the set F_{r3} into three parts A_i' for $i = 1, 2, 3$. In Figure 3.6, for a 2-plane tree with the root v_0' in F_{r3}, v_s' is the root's first black child from the left side. If the marked vertex is in the area S_i', then this 2-plane tree belongs to the set A_i', for $i = 1, 2, 3$.

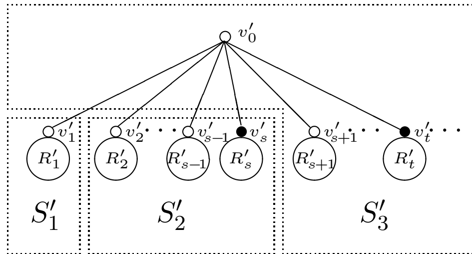

Figure 3.6: 2-plane trees in F_{r3}

Now we build the bijection β_i between A_i and A'_i for $i = 1, 2, 3$. For convenience, we define two kinds of bijective maps which are included in the bijection α in Theorem 2.4. These maps are used in the following proof.

Let α_w denote the bijection which is described in Figure 2.1. Here we consider the tree with the root v_1 in the left picture. We map the tree to a bunch of subtrees with white roots and attach them to a vertex e.

Let α_b denote the bijection which is described in Figure 2.3. Here we only consider the tree with the root m_1 in the left picture. We map the tree to a bunch of subtrees with black or white roots and attach them to a vertex v'_i. Notice that the root m'_1 of the leftmost 2-plane subtree is black.

We should point out that these maps α_w and α_b are defined recursively as in the proof of Theorem 2.4.

1. Bijection between A_1 and A'_1

For a ternary tree $T \in A_1$, the marked internal vertex is in the left subtree of the root. We show the bijection β_1 in Figure 3.7. In the left picture, Let R_i $(i = 1, \ldots, m-1, m+1, \ldots, d)$ denote the left and middle subtrees of v_i, and L_j $(j = 2, 3, 4, 5)$ includes the left, middle, and right subtrees of w_j. The marked vertex with label r is in the area R which includes the subtree with the root v_m where the right subtree of v_m is empty. In the right picture, v'_i (resp. w'_i, R'_i or L'_i) corresponds to v_i (resp. w_i, R_i or L_i).

We observe that for a ternary tree $T \in \widetilde{E}_{r3}$, the vertices v_0 and v_m must exist. By removing these two vertices, we get six subtrees:

(1) Let T_1 denote the subtree with the root v_1 where all the internal vertices on the longest rightmost path of v_1 are $v_1, \ldots, v_{m-1}$.

(2) Let T_2 denote the subtree with the root w_2.

(3) Let T_3 denote the subtree with the root w_3.

(4) Let T_4 denote the subtree with the root w_4.

(5) Let T_5 denote the subtree with the root w_5.

(6) Let T_6 denote the subtree with the root v_{m+1} where all the internal vertices on the longest rightmost path of v_{m+1} are $v_{m+1}, \ldots, v_d$.

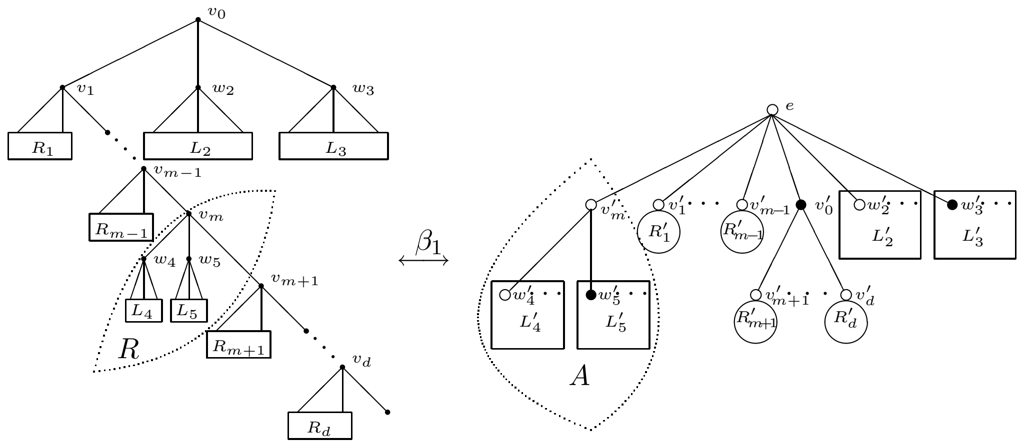

Figure 3.7: The bijection β_1

Now we use the bijections α_w and α_b to build the corresponding 2-plane tree P. See Figure 3.7.

We analyze that for a 2-plane tree $P \in F_{r3}$, the white root, the root's leftmost white child, and the root's leftmost black child must exist:

Step 1: Put an extra white vertex e as the root of P.

Step 2: Let the white vertex v'_m be the first white child of the root e from the left side, and let the black vertex v'_0 be the first black child of the root e from the left side.

Step 3: We apply the bijection α_w to the subtrees T_1, T_2, T_4, and T_6, then apply the bijection α_b to the subtrees T_3 and T_5. The location of the corresponding 2-plane subtrees is shown in Figure 3.7. We can see that there are exactly six positions for the corresponding trees of T_i $(i = 1, 2, \ldots, 6)$.

In order to state the bijection β_1 clearly, we illustrate the positions in Figure 3.8.

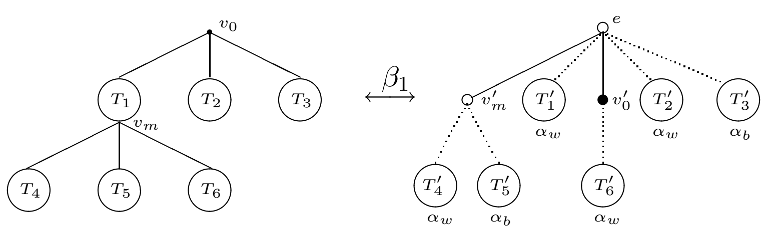

Figure 3.8: The positions for the bijection β_1

We give an example to show the bijection β_1 in Figure 3.9, where v_6 (resp. v'_6) is marked with r (resp. a).

2. Bijection between A_2 and A'_2

For a ternary tree $T \in A_2$, we exchange the left subtree and the middle subtree of the root in the left picture in Figure 3.7. The marked vertex in T is v_m or in T_4 or T_5. Therefore, v_0 and v_m must exist. We map v_0 and v_m to the root's first white child and first black child from

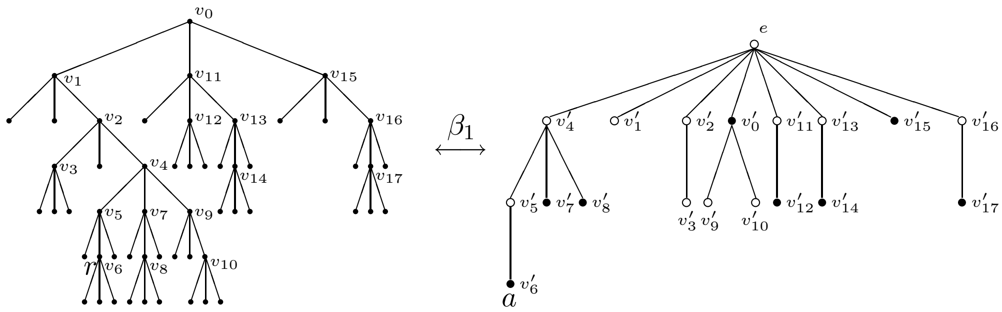

Figure 3.9: An example for the bijection β_1

the left side, respectively. Removing the vertices v_0 and v_m from T, we still get six subtrees. In Figure 3.10, we show the bijection β_2. Here we apply the bijection α_w to the subtrees T_1, T_2, T_4, and T_5, then apply the bijection α_b to the subtrees T_3 and T_6. The marked vertex with label a is v'_m or in T'_4 or in T'_5.

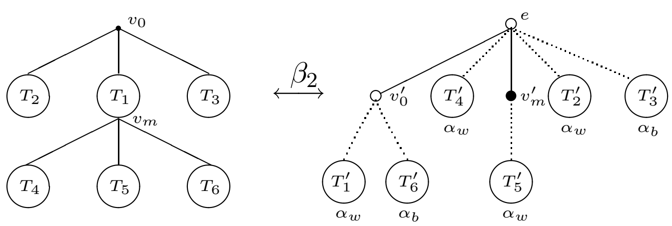

Figure 3.10: The bijection β_2

We give an example to explain the bijection β_2 in Figure 3.11, where v_6 (resp. v'_6) is marked with r (resp. a).

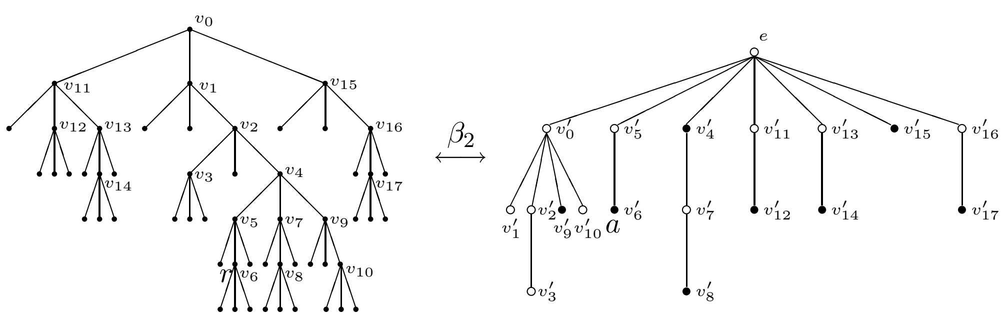

Figure 3.11: An example for the bijection β_2

3. Bijection between A_3 and A'_3

For a ternary tree $T \in A_3$, we exchange the left subtree and the right subtree of the root in the left picture in Figure 3.7. The marked vertex in T is v_m or in T_4 or T_5. Therefore, v_0 and v_m must exist. We map v_0 to be the root's first black child from the left side, and map v_m to be the root in the corresponding 2-plane tree. The extra white vertex is the root's first white child from the left side. Removing the vertices v_0 and v_m from T, we still get the six subtrees. In Figure 3.12, we show the bijection β_3. Here we apply the bijection α_w to the subtrees T_1, T_3, T_4, and T_6, then apply the bijection α_b to the subtrees T_2 and T_5. The marked vertex with label a is v'_m or in T'_4 or in T'_5.

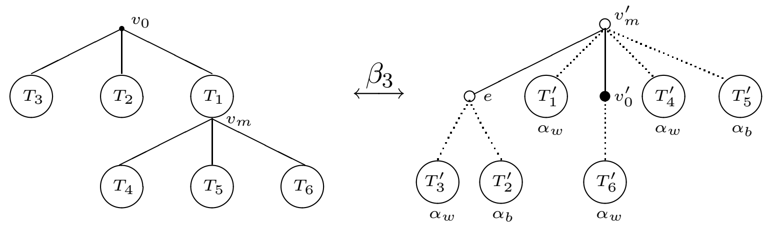

Figure 3.12: The bijection β_3

We give an example to explain the bijection β_3 in Figure 3.13, where v_6 (resp. v'_6) is marked with r (resp. a).

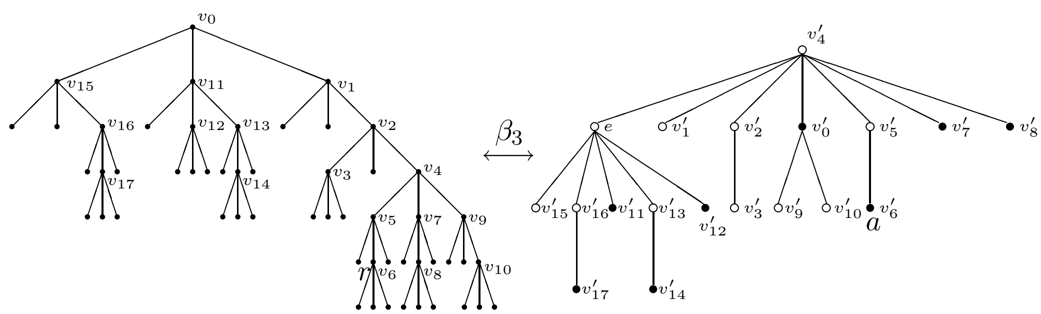

Figure 3.13: An example for the bijection β_3

Since we should put a label on one internal vertex except for the root, we have $n-1$ options for each ternary tree. Therefore, the enumerative formula for $\widetilde{E}_{r3}$ is

$$|\widetilde{E}_{r3}| = |E_{r3}| = (n-1)T_n = \frac{n-1}{2n+1}\binom{3n}{n}. \tag{3.6}$$

∎

After we prove all the seven pairs of the subsets for the sets E_n and F_{n+1}, we finally prove Theorem 3.1. According to the enumerative formula for each case, we have

$$\begin{aligned} |E_n| &= |E_{b1}| + |E_{r1}| + |E_{b2}| + |E_{r2}| + |E_{b3}| + |E_{r0}| + |E_{r3}| \\ &= 2(2n+1)T_n \\ &= 2\binom{3n}{n}. \end{aligned} \tag{3.7}$$

4 Other Relations between 2-Plane Trees and Ternary Trees

Before we prove some other relations between 2-plane trees and ternary trees, we first prove a relation which we find in the computations of the generating functions for the subsets of 2-plane trees in Section 3.

The equation is stated as follows:

$$B^2 = zW, \tag{4.1}$$

where $z = \frac{v}{(1+v)^3}$, $W = \frac{v}{1+v}$, and $B = \frac{v}{(1+v)^2}$.

Now we give a combinatorial proof for Equation (4.1) in the following theorem.

Theorem 4.1 *The number of 2-plane trees of $n-1$ vertices with a white root is equal to the number of ordered pairs of black-rooted 2-plane trees with a total of n vertices.*

Proof. We define a bijection β between these two set. For a 2-plane tree of $n-1$ vertices with a white root, let v_1 denote the root. For the longest rightmost path of v_1, there are two cases:

(1) The longest rightmost path of v_1 has at least one black vertex;

(2) The longest rightmost path of v_1 has no black vertex.

In each case, we use β_i $(i = 1, 2)$ to denote the map.

For the first case, we build the bijection β_1 in Figure 4.14. The left picture is a 2-plane tree of $n-1$ vertices with a white root. Let $v_1v_2 \ldots v_m b_1 \ldots$ denote the longest rightmost path of v_1, where $v_1, v_2, \ldots, v_m$ are all white vertices, and b_1 is the first black vertex on the path from the root. T_i $(i = 1, 2, \ldots, m-1)$ denotes all the subtrees of v_i except for the subtree with the root v_{i+1}. B_1 denotes all the subtrees of b_1, and T_m denotes all the subtrees of v_m except for the subtree with the root b_1.

First, cutting the edge $v_m b_1$ in the left picture in Figure 4.14, we let the subtree with the root b_1 be the first 2-plane tree with a black root. Then we add an extra black vertex e as the root of the second 2-plane tree, and let v_1, v_2, and v_m be the white children of this black vertex. Meanwhile, let T_i $(i = 1, 2, \ldots, m)$ still be the subtrees of v_i in the second 2-plane tree. Now we get the ordered pair of 2-plane trees with n vertices.

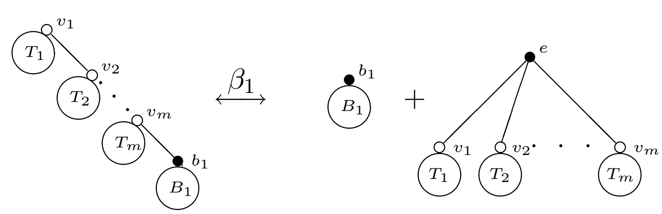

Figure 4.14: The bijection $\beta = \beta_1$ for Case (1)

For the second case, we build the bijection β_2 in Figure 4.15. The left picture is a 2-plane tree of $n-1$ vertices with a white root. Let $v_1v_2 \ldots v_m v_{m+1}$ denote the longest rightmost path of v_1, where $v_1, v_2, \dots, v_m, v_{m+1}$ are all white vertices, and T_i $(i = 1, 2, \dots, m)$ denotes all the subtrees of v_i except for the subtree with the root v_{i+1}.

First, we add an extra black vertex e as the root of the first 2-plane tree, and let v_1, v_2, v_m be the white children of this black vertex. Meanwhile, let T_i $(i = 1, 2, \dots, n)$ still be the subtrees of v_i in the first 2-plane tree. Then we only change the white vertex v_{m+1} to a black vertex, and let this black vertex be the second 2-plane tree.

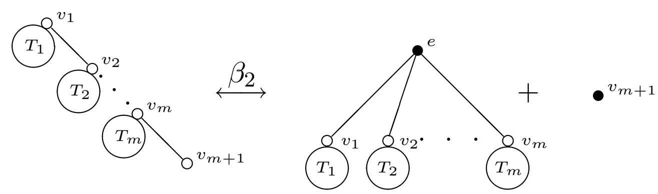

Figure 4.15: The bijection $\beta = \beta_2$ for Case (2)

It is easy to see that the map is a one-to-one correspondence. ∎

Theorem 4.2 *There is a bijection between the set of ternary trees with n internal vertices, where one of the internal vertices is marked and the set of* 2*-plane trees of $n+1$ vertices with a black root, where one of the vertices except for the root is marked.*

Proof. For a ternary tree with n internal vertices, let v denote the marked internal vertex. First we use the bijection α in Theorem 2.4 to map the ternary tree to a 2-plane tree of $n+1$ vertices with a black root. Then we mark the corresponding vertex of v. ∎

Theorem 4.3 *There is a bijection between the set of ternary trees with n internal vertices, where a left/middle/right leaf which does not belong to the right subtree of the root is marked and the set of* 2*-plane trees of n vertices with a white root, where one of the vertices is marked.*

Proof. We first prove the case that a left leaf is marked in the ternary tree. For a ternary tree T with n internal vertices, let v denote the marked left leaf which does not belong to the right subtree of the root. First put a label w on the father of v. Then we let the right subtree of the root replace the leaf v. Finally, we obtain a new ternary tree T_1 with n internal vertices where the right subtree of the root is empty.

Now we map this new ternary tree T_1 to a 2-plane tree of n vertices with a white root by using the bijection α_w and α_b. See Figure 4.16.

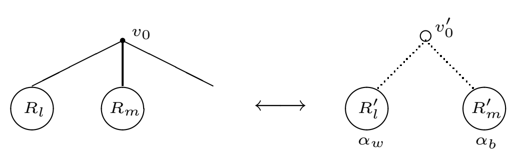

Figure 4.16: The bijection in Theorem 4.3

Step 1: Map the root v_0 of T_1 to be the white root v_0' of the 2-plane tree.

Step 2: Apply the bijection α_w to the left subtree R_l, and apply the bijection α_b to the middle subtree R_m. Then we attach the corresponding R_l' and R_m' to v_0' in turn.

Step 3: Mark the corresponding vertex of w in the 2-plane tree.

It is easy to see that the map is a one-to-one correspondence.

For other cases that in a ternary tree a middle/right leaf which does not belong to the right subtree of the root is marked, the proof is similar to the proof of the above case. ∎

We give an example in Figure 4.17 to explain the bijection in Theorem 4.3, where the left child of v_6 is marked with label b, and v_6' is marked with label a.

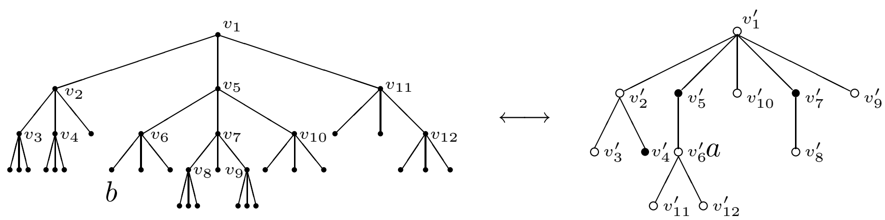

Figure 4.17: An example for Theorem 4.3

For the enumeration of the set of 2-plane trees of n vertices with a white root, where one of the vertices is marked, we use the equation (2.5) in Lemma 2.3.

The number of 2-plane trees of n vertices with a white root is given by

$$A_n := \frac{1}{n}\binom{3n-2}{n-1}. \tag{4.2}$$

Therefore, for the 2-plane trees of n vertices with a white root, where one of the vertices is marked, the number is $\binom{3n-2}{n-1}$.

According to Theorem 4.3, for ternary trees with n internal vertices, where a left/middle/right leaf which does not belong to the right subtree of the root is marked, the number is also $\binom{3n-2}{n-1}$.

Theorem 4.4 *There is a bijection between the set of ternary trees with n internal vertices, where a left/middle/right leaf is marked and the set of 2-plane trees of $n+1$ vertices with a black root, where one of the vertices in the rightmost subtree of the root is marked.*

Proof. We only prove the case that a left leaf of ternary trees with n internal vertices is marked. For a ternary tree T with n internal vertices, let v denote the marked left leaf, and let v_0 denote the root of T. We find the longest rightmost path of v_0 in T denoted by $v_0v_1v_2\ldots v_m$, where $v_0, v_1, \ldots, v_m$ are all internal vertices. Assume that v is in the subtree with the root v_i which belongs to the rightmost path of v_0. Now we construct the corresponding 2-plane tree.

Step 1: Put a label w on the father of the marked leaf v.

Step 2: Let the right subtree of v_i replace the leaf v.

Step 3: Apply the bijection α in Theorem 2.4 to this new ternary tree, and mark the corresponding vertex of w.

It is obvious that the marked vertex in the 2-plane tree is in the rightmost subtree of the root.

It is easy to see that the map is a one-to-one correspondence. ∎

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