A note on vertex-coloring edge-weighting of graphs *

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Abstract

A *k-edge-weighting* w of a graph G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$, to each edge e. An edge-weighting naturally induces a vertex coloring c by defining $c(u) = \Sigma_{e \ni u} w(e)$ for every $u \in V(G)$. A k-edge-weighting of a graph G is *vertex-coloring* if the induced coloring c is proper, i.e., $c(u) \neq c(v)$ for any edge $uv \in E(G)$. When $k \equiv 2 \pmod 4$ and $k \geq 6$, we prove that if G is k-colorable and 2-connected, $\delta(G) \geq k - 1$, then G admits a vertex-coloring k-edge-weighting. We also obtain several sufficient conditions for graphs to be vertex-coloring k-edge-weighting.

Keywords. vertex coloring; edge-weighting;

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1 Introduction

In this paper, we consider only finite, undirected and simple graphs. For a vertex v of a graph $G = (V, E)$, $N(v)$ denotes the set of vertices which are adjacent to v. For a vertex set $S \subseteq V$, $N(S)$ denotes the set of vertices which are adjacent to at least one vertex of S. Let $d(v)$ and $\delta(G)$ denote the degree of a vertex v and the minimum degree of G, respectively. A *k-vertex coloring* c of G is an assignment of k integers, $\{1, 2, \ldots, k\}$, to the vertices of G. The color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two distinct adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. Notations and terminologies that are not defined here may be found in [3]. A *k-edge-weighting* w of

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a graph G is an assignment of an integer weight $w(e) \in \{1, \ldots, k\}$ to each edge e of G. An edge weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \Sigma_{e \ni u} w(e)$ for every $u \in V(G)$. A k-edge-weighting of a graph G is vertex-coloring if for every edge $e = uv$, $c(u) \neq c(v)$ and we call that G admits a *vertex-coloring k-edge-weighting.*

If a graph has an edge as a component, it can not have a vertex-coloring k-edge-weighting. So in this paper, we only consider graphs without K_2 component and refer those graphs as *nice graphs*.

In [5], Karoński, Łuczak and Thomason initiated the study of vertex-coloring k-edge-weighting and they brought forward a conjecture as following.

Conjecture 1.1 *Every nice graph admits a vertex-coloring 3-edge-weighting.*

The above conjecture holds for 3-colorable graphs, which is due to Karoński, Łuczak and Thomason [5]. In fact, they proved a more general result that if k is *odd*, then every k-colorable nice graph admits a vertex-coloring k-edge-weighting. The proof of this result is elegant, by taking advantage of properties of abelian groups. Naturally, we next turn to the cases of k even. Duan, Lu and Yu [4] showed that every k-colorable nice graph admits a vertex-coloring k-edge-weighting for $k \equiv 0 \pmod 4$. In this paper, we continue the study of vertex-coloring k-edge-weighting for $k \equiv 2 \pmod 4$ and $k \geq 6$. We show that a k-colorable 2-connected graph G, where $k \equiv 2 \pmod 4$ and $k \geq 6$, admits a vertex-coloring k-edge-weighting. We can also obtain the same conclusion by eliminating 2-connectivity condition but posing some restriction of degrees.

To conclude this section, we introduce two earlier results as our lemmas for the proofs of main results.

Lemma 1.2 (Karoński, Łuczak and Thomason [5]) *Let G be a connected non-bipartite graph, $\{t_v \mid v \in V(G)\}$ be any given vertex-coloring of G, and k be a positive integer. If $\sum_{v \in V} t_v$ is even, then there exists a k-edge-weighting w of G such that for all $v \in V(G)$, $\sum_{v \in e} w(e) \equiv t_v \pmod k$.*

Lemma 1.3 (Duan, Lu and Yu [4]) *Let G be a k-colorable graph, where $(U_0, U_1, \ldots, U_{k-1})$ denote coloring classes of G. Then G admits a vertex-coloring k-edge-weighting, if any of following conditions holds:*

(i) $k \equiv 0 \pmod 4$*;*

(ii) $\delta(G) \leq k - 2$*;*

(iii) there exists a class U_i with $|U_i| \equiv 0 \pmod 2$ for some $i \in \{0, 1, \ldots, k-1\}$;

(iv) $|V(G)|$ is odd.

2 Connectivity and edge-weighting

In this section, we use connectivity as a sufficient condition to insure a vertex-coloring k-edge-weighting.

Since every nice graph admits a vertex-coloring 13-edge-weighting (see [6]), we need only to consider the cases of $k \equiv 2 \pmod 4$ and $k \leq 12$, i.e., $k \in \{6, 10\}$.

Theorem 2.1 *Let G be a k-colorable graph, where $k \in \{6, 10\}$, and v be a vertex of $V(G)$ with $d(v) = \delta(G)$. Denote $N^{\delta}(v) = \{x \in N(v) \mid d(x) = \delta(G)\}$. If $N^{\delta}(v) = \emptyset$ and $G - v$ is connected, then G admits a vertex-coloring k-edge-weighting.*

Proof. Denote coloring classes of G by $(U_0, U_1, \ldots, U_{k-1})$. If there exists a class U_i with $|U_i| \equiv 0 \pmod 2$, we are done by Lemma 1.3. So we may assume that $|U_i|$ is odd for all $i = 0, 1, \ldots, k-1$.

Without loss of generality, assume $v \in U_0$ and $|N(v) \cap U_i|$ is odd for $i = 1, 2, \ldots, l$, and $|N(v) \cap U_i|$ is even for $i = l+1, \ldots, k-1$ $(0 \leq l \leq k-1)$. Note that $l = 0$ means that $|N(v) \cap U_i|$ is even for $i = 1, \ldots, k-1$, and in this case the proof is similar. Let $W_0 = (U_0 - v) \cup (U_1 \cap N(v))$, $W_i = (U_i - N(v)) \cup (U_{i+1} \cap N(v))$ for $i = 1, 2, \ldots, k-2$, $W_{k-1} = U_{k-1} - N(v)$. Then $|W_i|$ is odd except $i = l$. The number of indices i with $|W_i|$ odd in $\{0, \ldots, k-1\} - \{l\}$ is even, so $\sum_{i=0, i \neq l}^{k-1} |W_i|(i - l + 1)$ is even and thus $\sum_{i=0}^{k-1} |W_i|(i - l + 1)$ is even. Let t_x, where $x \in V(G - v)$, be a given set of vertex-coloring satisfying $t_x \equiv i - l + 1 \pmod k$ for $x \in W_i$. Then $\sum_{x \in V(G-v)} t_x$ is even. So $G - v$ has a vertex-coloring k-edge-weighting such that $c(x) \equiv t_x \equiv i - l + 1 \pmod k$ for all $x \in W_i$ by Lemma 1.2. Assign the edges incident to v with weight 1. Then $c(v) = \delta(G)$ and $N^{\delta}(v) = \emptyset$ implies $c(v) < c(u)$ for $u \in N(v)$. Moreover, $c(x) \equiv i - l + 1 \pmod k$ for all $x \in U_i$ and $x \neq v$. Hence G admits a vertex-coloring k-edge weighting. □

Lemma 2.2 *Let G be a 2-connected graph and v be a vertex of $V(G)$ with $d(v) = \delta(G) \geq 5$. Then there exists $S \subseteq N(v)$ with $|S| = \delta - 3$ such that $G - v$ contains a spanning subgraph M satisfying $d_M(x) \leq d_G(x) - 2$ for all $x \in S$.*

Proof. Suppose, to the contrary, that there exists no such a required connected spanning subgraph in $G - v$. Then we find a connected spanning subgraphs T so that the vertex set $R = \{x \in N(v) \mid d_T(x) \leq d_G(x) - 2\}$ is maximized. Among subgraphs T satisfying the maximality condition of R, we choose a maximum graph with respect to the number of edges, say M. Then $d_M(x) = d_G(x) - 2$ for all $x \in R$.

So $|R| = r \leq \delta - 4$. Let $v_1, v_2, v_3, v_4 \in N(v) - R$. Set $H = M \cup \{vv_1\}$. Then every edge incident with v_1, v_2, v_3, v_4 is a cut-edge of M in $G - v$ since R is maximum and every cut-edge of M is also a cut-edge of H, and vv_1 is a cut-edge of H as well. We observe that $N(v_1) \cup N(v_2) \cup N(v_3) \cup N(v_4) - \{v, v_1, v_2, v_3, v_4\} \geq 4\delta - 10$, that is, there are at least $4\delta - 10$ cut-vertices. Thus we need to add at least $\frac{4\delta-10}{2} + 1 = 2\delta - 4$ edges to H so that the resulting graph is 2-connected, because at least $\frac{4\delta-10}{2}$ edges are required to link the cut-vertices and one more edge incident to v. On the other hand, G is 2-connected, we delete at most $\delta - 1 + r$ edges from G to obtain H, so we have $\delta - 1 + r \geq |E(G) - E(H)| \geq 2\delta - 4$ and $r \geq \delta - 3$, a contradiction. $\square$

Theorem 2.3 *Let G be a k-colorable graph, where $k \equiv 2 \pmod 4$ and $k \geq 6$. If G is 2-connected and $\delta(G) \geq k - 1$, then G admits a vertex-coloring k-edge-weighting.*

Proof. Let $(U_0, \ldots, U_{k-1})$ be coloring classes of G. Let $v \in U_0$ and $d(v) = \delta(G) \geq 5$. Let $\delta(G) \equiv r \pmod k$. Without loss of generality, assume that $|U_0|, \ldots, |U_{k-1}|$ are all odd by Lemma 1.3. If $N(v) \cap U_i = \emptyset$ for some i, we can move v into U_i from U_0, the new classes are also coloring classes of G. Hence $|U_0|$ and $|U_i|$ are both even and so G admits a vertex-coloring k-edge-weighting by Lemma 1.3. So $N(v) \cap U_i \neq \emptyset$ for $i = 1, \ldots, k - 1$. Without loss of generality, suppose $|N(v) \cap U_i|$ is odd for $i = 1, \ldots, l$ and $|N(v) \cap U_i|$ is even for $i = l+1, \ldots, k-1$ (note that $l = 0$ means that there exists no U_i so that $|N(v) \cap U_i|$ is odd). Let $W_0 = (U_0 - v) \cup (N(v) \cap U_1)$, $W_{k-1} = U_{k-1} - N(v)$ and $W_i = (N(v) \cap U_{i+1}) \cup (U_i - N(v))$ for $i = 1, \ldots, k-2$. Then $|W_l|$ is even and $|W_i|$ is odd for any $i \neq l$. By Lemma 2.2, there exists a vertex set $S \subseteq N(v)$ with $|S| = \delta - 3$ such that $G - v$ contains a spanning subgraph M which satisfies $d_M(x) \leq d_G(x) - 2$ for all $x \in S$. (Note that, the spanning subgraph M is obtained by deleting at least one edge incident with each vertex x of S.) Since $|S| = \delta - 3$, there are three vertices in $N(v) - S$, which are in at most three color classes, say U_i, U_j, U_m. Then there are at most three color classes such that $(U_i \cup U_j \cup U_m) \cap N(v) \nsubseteq S$. Thus there are at least two color classes, say U_a and U_b, so that $N(v) \cap U_a \subseteq S$ and $N(v) \cap U_b \subseteq S$.

We consider the following three cases.

Case 1. $|N(v) \cap U_a|$ *is odd and* $|N(v) \cap U_b|$ *is even.*

We may assume $a = l$, $b = l + 1$. We first give a set of target colors t_x for all $x \in V(M)$ so that $\sum_{x \in V(M)} t_x$ is even, as follows.

$$(*) \text{ if } r \text{ is even, } t_x \equiv i - l + r - 1 \pmod k \text{ for } x \in W_i;$$

$$\text{if } r \text{ is odd, } t_x \equiv i - l + r \pmod k \text{ for } x \in W_i.$$

It is not hard to verify that $\sum_{x\in V(M)} t_x$ is even. In fact, if r is even, since $|W_l|$ is even and $|W_i|$ is odd for $i \neq l$, we assign $t_x \equiv r-1 \pmod k$ for $x \in W_l$ and thus t_x is odd. Since $k \equiv 2 \pmod 4$, the number of odd weights in $\{1, 2, \ldots, k\} - \{r-1\}$ is even. And $t_x \equiv i - l + r - 1 \pmod k$ for $x \in W_i$, so $\sum_{x\in V(M)} t_x$ is even. If r is odd, then $t_x \equiv r \pmod k$ for $x \in W_l$, and t_x is also odd. The number of odd weights in $\{1, 2, \ldots, k\} - \{r\}$ is even and thus $\sum_{x\in V(M)} t_x$ is even again.

Then, by Lemma 1.2, we have an edge-weighting of M such that for all $u \in V(M)$, $\sum_{u\in e} w(e) \equiv t_u \pmod k$ and for any two vertices $x, y \in V(M)$, $c(x) \equiv c(y) \pmod k$ if and only if they belong to the same W_i for some i. We assign the edges incident with v with weight 1 and the edges of $E(G-v) - E(M)$ with weight k. Then for any two vertices $x, y \in V(G) - v$, $c(x) \equiv c(y) \pmod k$ if and only if they belong to the same U_i for some i. Now we have $c(v) = \delta(G) \equiv r \pmod k$. For any $x \in N(v)$, $c(x) \equiv r \pmod k$, only if $x \in U_b$ (resp. U_a) when r is even (resp. odd). If $c(x) \equiv c(v) \pmod k$ for $x \in N(v) \cap (U_a \cup U_b)$, then $c(x)$ is greater than $c(v)$ by at least k since $d(x) \geq d(v)$. So we obtain an edge-weighting of G such that the resulting vertex-coloring is proper (see Figure 1).

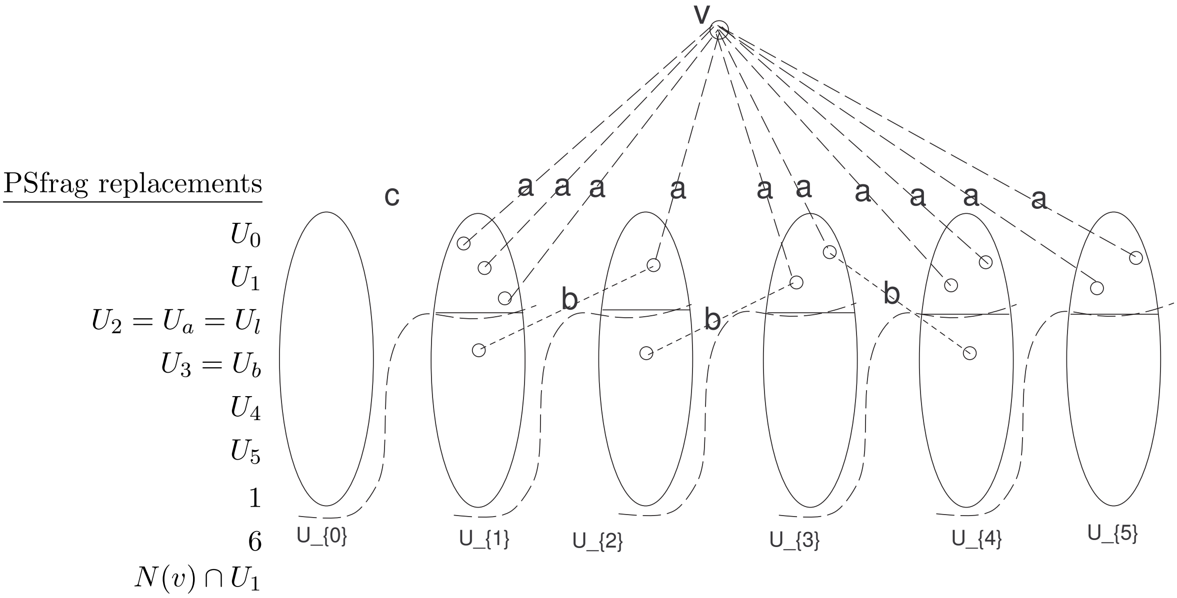

Figure 1: $k = 6$, $l = 2$ and r is odd. The weights of edges in $G - M$ are labeled.

Case 2. Both $|N(v) \cap U_a|$ and $|N(v) \cap U_b|$ are even.

Let $a = l+1$, $b = l+2$. We can give a set of target colors t_x for all $x \in V(M)$. If r is even, we choose $t_x \equiv i-l+r-1 \pmod k$ for $x \in W_i$. If r is odd, we choose $t_x \equiv i-l+r-2 \pmod k$ for $x \in W_i$. It is routine to check that $\sum_{x\in V(M)} t_x$ is even as in Case 1. By Lemma 1.2, we have an edge-weighting of M such that $\sum_{x\in e} w(e) \equiv t_x \pmod k$ for all

$x \in V(M)$. Now we assign all edges incident with v with weight 1, then for any two vertices $x, y \in V(G) - v$, $c(x) \equiv c(y) \pmod{k}$ if and only if they belong to the same U_i for some i. Then $c(v) \equiv r \pmod{k}$. For any $x \in N(v)$, $c(x) \equiv r \pmod{k}$, only if $x \in U_a$ (resp. U_b) when r is even (resp. odd). Then we assign all edges in $E(G - v) - E(M)$ with weight k. For $x \in N(v) \cap (U_a \cup U_b)$ and $c(x) \equiv c(v) \pmod{k}$, we see that $c(x)$ is greater than $c(v)$ by at least k. Still we have $c(x) \equiv c(y) \pmod{k}$ if and only if they belong to the same U_i for some i, for any two vertices $x \neq v \neq y$. So we obtain an edge-weighting of G such that the resulting vertex-coloring is proper.

Case 3. Both $|N(v) \cap U_a|$ and $|N(v) \cap U_b|$ are odd.

Let $a = l - 1$, $b = l$. We give a set of target colors t_x for all $x \in V(M)$. If r is even, we choose $t_x \equiv i - l + r + 1 \pmod{k}$ for $x \in W_i$; if r is odd, $t_x \equiv i - l + r \pmod{k}$ for $x \in W_i$. We can check that $\sum_{x \in V(M)} t_x$ is even. By Lemma 1.2, we have an edge-weighting of M such that $\sum_{x \in e} w(e) \equiv t_x \pmod{k}$ for all $x \in V(M)$. Now we assign all edges incident with v with weight 1 and all edges in $E(G - v) - E(M)$ with weight k. Then for any two vertices $x \neq v \neq y$, $c(x) \equiv c(y) \pmod{k}$ if and only if they belong to the same U_i for some i. For any $x \in N(v)$, $c(x) \equiv r \pmod{k}$, only if $x \in U_a$ (resp. U_b) when r is even (resp. odd). For $x \in N(v) \cap (U_a \cup U_b)$ and $c(x) \equiv c(v) \pmod{k}$, we see that $c(x)$ is greater than $c(v)$ by at least k. So we obtain an edge-weighting of G such that the resulting vertex-coloring is proper. □

Remark: Under the condition of 3-connectivity, the conclusion of Lemma 2.2 can be proved by a constructive method and thus we are able to design an efficient algorithm to find a k-edge-weighting such that the induced vertex-coloring is proper.

3 Vertex-coloring k-edge-weighting with degree conditions

Let a and b be two integers such that $a \leq b$. We denote all integers i with $a \leq i \leq b$ by $[a, b]$. Use degree intervals as sufficient conditions, we have the following theorem.

Theorem 3.1 *Let G be a k-colorable graphs with girth $g(G) \geq 4$, where $k \in \{6, 10\}$. Then*

(i) if $[d(v)+10, 5d(v)-10] \cap [6d(u)-5, 6d(u)-1] = \emptyset$ for any $uv \in E(G)$, then G admits a vertex-coloring 6-edge-weighting.

(ii) if $[d(v) + 36, 9d(v) - 36] \cap [10d(u) - 9, 10d(u) - 1] = \emptyset$ for any $uv \in E(G)$, then G admits a vertex-coloring 10-edge-weighting.

Proof. We only provide a proof for part (i) here, since the proof of part (ii) is very similar and is required only a few minor modifications.

By Lemma 1.3, we may assume that $|U_i|$ is odd for $i = 0, \ldots, 5$ and $\delta(G) \geq 5$. As before, for every $v \in U_j$, $N(v) \cap U_i \neq \emptyset$, for $i \in \{0, \ldots, 5\} - \{j\}$.

Claim 1. There exists a vertex x in U_i, for some i, such that the vertices of $G - N(x)$ in $\bigcup_{j \neq i} U_j$ are all in a same component of $G - N(x)$.

Suppose that Claim 1 is not true. Choose a vertex x such that the size of a maximum component of $G - N(x)$ is largest, say $G_1 = (U_0^1 \cup U_1^1 \cup \ldots \cup U_5^1, E_1)$ is such a component. We may assume $x \in U_0$ and let another component of the graph $G - N(x)$ besides G_1 be $G_2 = (U_0^2 \cup U_1^2 \cup \ldots \cup U_5^2, E_2)$. Without loss of generality, assume that U_i^2 is nonempty for $i = 0, \ldots, l$, where $l \geq 1$. If there exists a vertex $x' \in V(G_2)$ which is not incident with some vertex $u \in N(V(G_1)) \cap N(x)$, then G_1 together with u is in a same component of $G - N(x')$ and the size of the maximum component of $G - N(x')$ is larger than that for x, a contradiction to the choice of x. So every vertex of G_2 is incident with all vertices of $N(V(G_1)) \cap N(x)$ and thus we can find a triangle, a contradiction with $g(G) \geq 4$.

From Claim 1, we see that $G - N(x)$ has a component $G_1 = (U_0^1 \cup U_1^1 \cup \ldots \cup U_5^1, E_1)$ with $U_i^1 = U_i \setminus N(x)$ and all other components are isolated vertices in U_0.

Now we consider two cases.

Case 1. $|N(x) \cap U_i|$ *are odd for $i = 1, \ldots, l$, where $l \geq 1$.*

In this case, $|U_1^1|$ is even. Then it is easy to show that there is a permutation of $U_2^1, U_3^1, U_4^1, U_5^1$, saying W_2, W_3, W_4, W_5 such that $\sum_{i=2}^5 i|W_i|$ is even. Let $W_0 = U_0^1$ and $W_1 = U_1^1$. Then we have a set of target colors t_u for all $u \in V(G_1)$, $t_u = 6$ for $u \in W_0$ and $t_u = i$ for $u \in W_i$, $i \neq 0$. Then $\sum_{u \in V(G_1)} t_u$ is even. By Lemma 1.2, G_1 has a vertex-coloring 6-edge-weighting such that $c(u) \equiv i \pmod 6$ for $u \in W_i$, $i = 0, \ldots, 5$. Next assign the edges xy with weight i if $y \in W_i$ and the other edges in $E(G - v) - E(G_1)$ with 6. Then $c(u) \neq c(v)$ for $u \in W_i$, $v \in W_j$ and $i \neq j$. Note that if $|N(x) \cap U_i| = 1$ for $i = 2, 3, 4, 5$, then $c(x) = d(x) - 4 + 14 = d(x) + 10$, which achieves the lower end of the interval; if $|N(x) \cap U_i| = 1$ for $i = 1, 2, 3, 4$, then $c(x) = 5(d(x) - 4) + 10 = 5d(x) - 10$, which achieves the upper end of the interval. So we have $d(x) + 10 \leq c(x) \leq 5d(x) - 10$. For all $u \in N(x)$, $6d(u) - 5 \leq c(u) \leq 6(d(u) - 1) + 5 = 6d(u) - 1$, which implying $c(x) \neq c(u)$ for all $u \in N(x)$. Therefore we have a vertex-coloring 6-edge-weighting of G.

Case 2. $|N(x) \cap U_i|$ *are even for $i = 1, 2 \ldots, 5$.*

Then we can see $d(x) \geq 10$. In this case, U_i^1 are odd for $i = 1, 2, \ldots, 5$. Note that there is a vertex $u^* \in N(x)$, say $u^* \in U_1$, adjacent to some vertex $v^* \in U_0 \cup U_2 \cup \cdots \cup U_5$. Let G' be the graph obtained from G_1 by adding the vertex u^* and the edge u^*v^*. Let W_2, W_3, W_4, W_5 be a permutation of $U_2^1, U_3^1, U_4^1, U_5^1$ such that $\sum_{i=2}^5 i|W_i|$ is even. Let $W_0 =$

U_0^1 and $W_1 = U_1^1 \cup \{u^*\}$. Then W_1 is even. We assign target colors t_v to $v \in V(G_1)$, where $t_v = 6$ for $v \in W_0$ and $t_v = i$ for $v \in W_i$ $(i \neq 0)$. Then $\sum_{v \in V(G')} t_v$ is even. According to Lemma 1.2, the edges of G' can be assigned weights from $\{1, 2, \ldots, 6\}$ so that $c(u) \equiv i \pmod 6$ for $u \in W_i$, $i = 0, \ldots, 5$. Next assign the edges xy (except xu^*) with weight i if $y \in U_i$ and the remaining edges of $(E(G-v)-E(G_1)) \cup \{xu^*\}$ with 6. As before, $d(x)+15 \leq c(x) \leq 5d(x)-5$. For all $u \in N(x)$, $6d(u)-5 \leq c(u) \leq 6(d(u)-1)+5 = 6d(u)-1$, which implying $c(x) \neq c(u)$ for all $u \in N(x)$. So it is a vertex-coloring 6-edge-weighting of G. □

From the Theorem 3.1, we have the following interesting corollary.

Corollary 3.2 *Let G be a k-colorable $[r, r+1]$-graph with girth $g(G) \geq 4$, where $k \geq 6$ and $r \geq 2$. Then G admits a vertex-coloring k-edge-weighting.*

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