New improvements on connectivity of cages*

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Abstract

A (δ, g)-cage is a δ-regular graph with girth g and with the least possible number of vertices. In this paper, we show that all (δ, g)-cages with odd girth $g \geq 9$ are r-connected, where $(r-1)^2 \leq \delta + \sqrt{\delta} - 2 < r^2$ and all (δ, g)-cages with even girth $g \geq 10$ are r-connected, where r is the largest integer satisfying $\frac{r(r-1)^2}{4} + 1 + 2r(r-1) \leq \delta$. Those results support a conjecture of Fu, Huang and Rodger that all (δ, g)-cages are δ-connected.

Key words: Cage; Girth; Superconnectivity

1 Introduction

In this paper, we only consider simple graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and $N_G(v)$ denotes the neighborhood of a vertex v in G. If $S \subseteq V$, then the subgraph of G induced by S is denoted by $G[S]$. For $u, v \in V$, $d_G(u, v)$ denotes the distance between u and v in G. For $S, W \subseteq V$, define $d_G(S, W) = min\{d_G(s, w) \mid s \in S, w \in W\}$. By *deleting* a vertex u from a graph G, we mean to delete the vertex u from G together with all the edges incident with u. By *connecting* two vertices we mean to join the two vertices by an edge.

A k-connected graph G is called *k-superconnected* if every k-vertex cutset $S \subseteq V(G)$ is a trivial cut set. The k-edge-superconnectivity is defined similarly.

The *girth* $g = g(G)$ is the length of a shortest cycle in G. A (δ, g)-*graph* is a regular graph of degree δ and girth g. Let $f(\delta, g)$ denote the smallest integer ν such that there exists a (δ, g)-graph having ν vertices. A (δ, g)-cage is a (δ, g)-graph with $f(\delta, g)$ vertices.

Cages were introduced by Tutte in 1947, and have been extensively studied (see survey [17] for more information). Fu, Huang and Rodger [6] proved that all cages are 2-connected, and then subsequently showed that all cubic cages are 3-connected. They then conjectured that (δ, g)-cages

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are δ-connected. Daven and Rodger [2], and independently Jiang and Mubayi [7], proved that all (δ, g)-cages are 3-connected for $\delta \geq 3$. In [13, 18], it has been shown that every $(4, g)$-cage is 4-connected.

Recently, the following two results were obtained.

Theorem 1. (Lin, Miller and Balbuena [8]) *Let G be a (δ, g)-cage with $\delta \geq 3$ and odd girth $g \geq 7$. Then G is r-connected with $r \geq \sqrt{\delta+1}$.*

Theorem 2. (Lin et al. [11]) *Let G be a (δ, g)-cage with $\delta \geq 4$ and even girth $g \geq 10$. Then G is $(r+1)$-connected, where r is the largest integer satisfying $r^3 + 2r^2 \leq \delta$.*

In this paper, we improve the bound of r in Theorem 1 to $\sqrt{\delta + \sqrt{\delta} - 2}$ when g is odd. For even girth g, we show that (δ, g)-cage is $(r+1)$-connected, where r is the largest integer satisfying $\frac{1}{4}r^3 + \frac{3}{2}r^2 - \frac{7}{4}r + 1 \leq \delta$.

2 Known Results

We often use the following theorems and lemmas.

Monotonicity Theorem. ([3, 6]) If $\delta \geq 3$ and $3 \leq g_1 < g_2$, then $f(\delta; g_1) < f(\delta; g_2)$.

Lemma 1. ([1, 4, 5, 14, 15]) *Let G be a graph with girth g, and minimum degree δ. Assume that S is a cutset with cardinality $|S| \leq \delta - 1$. Then, for any connected component C in $G - S$, there exists a vertex $x \in V(C)$ such that $d(x, S) \geq \lfloor (g-1)/2 \rfloor$.*

Theorem 3. ([10, 16]) *Every (δ, g)-cage is δ-edge-connected.*

Theorem 4. ([9, 12]) *Every (δ, g)-cage is edge-superconnected.*

Lemma 2. *Let H be a bipartite graph with bipartition (U, W), where $|U| = |W| = m$. Suppose that $d(v) \leq 1$ for each $v \in W$ and the maximum degree of H is at most $m-1$. Denote $H^* = (V^*, W^*)$ as a copy of H. Then there exist two one-to-one mappings $f : W \mapsto U^*$ and $f^* : W^* \mapsto U$ such that no 4-cycle created in the graph $H \cup H^* \cup E(f) \cup E(f^*)$.*

Proof. It is clear that H contains at most m edges. Using Hall's theorem, it is easy to show that, between the vertices of U and W, there exists one-to-one mapping f, which satisfies $f(u) = w$, $d(u, w) > 1$, where $u \in U$ and $w \in W$.

Connecting the vertices of W and U^* based on the mapping $f^{'} : w \rightarrow u^*$, if $f(u) = w$. After these new edges are added, we have a new graph G, we claim that there is a way to connect vertices of W^* and U in G such that there are no new 4-cycles created.

Considering the graph G, it is easy to verify that for every vertex in U, there is a vertex in W^* at distance at least four apart. Otherwise, suppose $t \in U$ is connected to $w_1, \ldots, w_x \in W$ in H, then $w_1, \ldots, w_x$ are connected to $u_1^*, \ldots, u_x^*$. Furthermore, $u_1^*, \ldots, u_x^*$ are connected to all the vertices in W^*. Since H^* is a copy of H, the edges $t^*w_1^*, \ldots, t^*w_x^*$ also exist in H^*. From the mapping $f^{'}$, it follows that the vertices in W connected to t^* must have distance more than one to t in H, this implies that $w_1, \ldots, w_x$ are not connected to t^*, which also implies that t^* must be different from $u_1^*, \ldots, u_x^*$. Therefore, H^* contains edges $t^*w_1^*, \ldots, t^*w_x^*$ and edges between $\{u_1^*, \ldots, u_x^*\}$ and all vertices of W^*, so there are more than m edges in total in H^*, a contradiction.

Follow the same reasoning, it is also easy to show that, in G, for any arbitrary number of vertices in U, there are at least the same number of vertices in W^* which are at distance 4 apart. From Hall's theorem, it follows that there exists a mapping between U and W^* such that there are no new 4-cycles created. □

3 Main Results

To prove the main results, we adopt the same approaches as in [8, 11] and refine the techniques to get the improvements on connectivity. The idea is to use two copies of a suitable subgraph from a (δ, g)-cage to construct a (δ, g')-graph with $g' \geq g$ but having less vertices than the original graph and thus contradicting to the definition of cages. Throughout the paper, notion x^* denotes the copy of x, where x could be a single vertex, a set of vertices or a subgraph.

Theorem 5. *Let G be a (δ, g)-cage with $\delta \geq 3$ and odd girth $g \geq 9$. Then G is r-connected, where $(r-1)^2 \leq \delta + \sqrt{\delta} - 2 < r^2$.*

Proof. Since every (δ, g)-cage with $\delta \geq 3$ is 3-connected, the theorem holds for $\delta \leq 8$. So we may assume $\delta \geq 9$. We reason by contradiction. Assume that there exists a cutset of order less than r. Let $S = \{s_1, \ldots, s_k\}$ be a cutset with $|S| = k < r$ and C be one of the smallest components of $G - S$. Without loss of generality, assume that $|C|$ is minimized among all cutsets of cardinality at most $r - 1$. Now we partition S into two subsets S_1 and S_2, where $S_1 = \{s \mid d_C(s) \geq k + 1\}$ and $S_2 = \{s \mid d_C(s) \leq k\}$. Let $|S_1| = m$ and $|S_2| = k - m$. By the choice of S, we may assume $|N_C(S_2)| > |S_2|$ and $|N_C(v)| \geq 2$ for all $v \in S_2$.

We consider two cases according to the cardinality of S_1.

Case 1. $|S_1| = m \geq 1$

Without loss of generality, assume that $S_1 = \{s_1, \ldots s_m\}$ and $S_2 = \{s_{m+1}, \ldots, s_k\}$. Let $S' = S_1 \cup N_C(S_2)$. Then

$$\begin{aligned}
|S'| &\leq |S_1| + |N_C(S_2)| \\
&\leq |S_1| + |E(N_C(S_2), S_2)| \\
&\leq m + (k-m)k \\
&\leq m + (r-1-m)(r-1) \\
&\leq m + \delta + \sqrt{\delta} - 2 - m(\sqrt{\delta + \sqrt{\delta} - 2} - 1) \\
&= (\delta - 1) + (2m-1) + \sqrt{\delta} - m\sqrt{\delta + \sqrt{\delta} - 2}
\end{aligned}$$

If $m = 1$, then $|S'| \leq (\delta - 1) + 1 + \sqrt{\delta} - \sqrt{\delta + \sqrt{\delta} - 2} < \delta$ as $\delta \geq 9$. Furthermore, $|S'|$ is an integer, so $|S'| \leq \delta - 1$. If $m \geq 2$, then $|S'| \leq (\delta-1) + (2m-1) - (m-1)\sqrt{\delta} \leq (\delta-1) + (2m-1) - 3(m-1) = (\delta - 1) + (2 - m) \leq (\delta - 1)$. Thus $|S'|$ is smaller than δ and note that S' is also a cutset.

By Lemma 1, there exists a vertex $v \in C$ such that $d(v, S') \geq (g-1)/2$. Let $N(v) = \{v_1 \ldots, v_\delta\}$. Note that there are at most m paths of length $(g-3)/2$ from $N(v)$ to S_1. Otherwise, by Pigeonhole

Principle, there are two vertices v_i, v_j from $N(v)$ at distance $(g-3)/2$ to a vertex $s \in S$, a cycle of length less than g is formed by two paths from v to s via v_i and v_j, respectively, which is a contradiction. Thus we may assume that $d(\{v_{m+1}, \ldots, v_\delta\}, S_1) \geq (g-1)/2$. Following the same arguments, we see that there are at most k paths of length less than $(g-3)/2$ from $N(v_i) - v$ to S for each $i = 1, \ldots, m$. Hence there are at least $\delta - k - 1$ vertices, denoted by T_i, in $N(v_i) - v$ such that $d(T_i, S) \geq (g-1)/2$, for $i = 1, \ldots, m$.

Moreover $|N_C(S_2)| \leq \delta - m - 1$, and there are at most $|N_C(S_2)|$ paths of length $(g-3)/2$ from $N(v)$ to $N_C(S_2)$. Now we may choose a subset $L \subseteq \{v_{m+1}, \ldots, v_\delta\}$ such that $|L| = |N_C(S_2)|$. For each $v_j \in L$, there are at most $|L| - 1$ paths of length at most $(g-3)/2$ from v_j to $N_C(S_2)$, otherwise a cycle of length less than g is formed since $N_C(v) \geq 2$ for all $v \in S_2$. Moreover, the set $\{v_{m+1}, \ldots, v_\delta\} - L$ is at distance at least $(g-1)/2$ to $N_C(S_2)$.

Since $d(L, N_C(S_2)) \geq (g-3)/2$, we construct a bipartite graph B with bipartition $(L, N_C(S_2))$ such that an edge $st \in E(B)$ if and only if $d_C(s,t) = (g-3)/2$, where $s \in L$ and $t \in N_C(S_2)$. Clearly, B satisfies the conditions in Lemma 2. Thus there exist two one-to-one mappings $f : N_C(S_2) \mapsto L^*$ and $f^* : N_C(S_2^*) \mapsto L$ such that no 4-cycle appears in graph $B \cup B^* \cup E(f) \cup E(f^*)$.

Consider the subgraph $G_1 = G[(C - \{v, v_1, \ldots, v_m\}) \cup S_1] - E[S_1]$ and take another copy of the subgraph G_1, denote it by G_1^*. The corresponding sets of interest are denoted by S_1^*, S_2^* and T_i^*, $i = 1, \ldots, m$. We construct a δ-regular graph G' with girth at least g by using G_1 and G_1^*. The order of the new graph G' is $2|V(G_1)| = 2(|V(C)| - 1) < |V(G)|$. Thus we construct a (δ, g')-graph with $g' \geq g$ and $|V(G')| < |V(G)|$. By Monotonicity Theorem, this contradicts to the assumption that G is a (δ, g)-cage. The construction is given below.

(a) For $i = 1, \ldots, m$, each vertex $s_i \in S_1$ is of degree at least $k+1$ and T_i in G_1 contains at least $\delta - k - 1$ vertices at distance at least $(g-1)/2$ to S_1, thus we connect s_i with $d_{G[G-C]}(s_i)$ distinct vertices in T_i^*. Similarly, we make the corresponding connections between the vertices in S_1^* and T_i.

After the operation is carried out in (a), for each $i = 1, \ldots, m$, $d_{G[G-C]}(s_i)$ vertices of $N(v_i) - v$ (respectively, $N(v_i^*) - v^*$) are of degree δ and the remaining are of degree $\delta - 1$.

(b) Connect each vertex in $N_C(S_2)$ to a vertex in L^*, and connect each vertex in $N_C(S_2^*)$ to a vertex in L according to the two one-to-one mappings f and f^* given in the graph $B \cup B^*$. After this operation, all the vertices in $L \cup L^*$ are of degree δ, but some vertices in $N_C(S_2) \cup N_C(S_2)^*$ might be of degree less than δ. Since $|N_C(S_2)| \leq |E(S_2, N_C(S_2))| \leq \delta - m - 1$, so we can connect some vertices in $N_C(S_2)$ to the vertices in $\{v_{m+1}^*, \ldots, v_\delta^*\} - L^*$ such that all vertices in $N_C(S_2)$ are of degree δ. Similarly, we make the corresponding connections between the vertices in $N_C(S_2^*)$ and $\{v_{m+1}, \ldots, v_\delta\} - L$.

(c) After the operations are carried out in (a) and (b), all the vertices are of degree δ or $\delta - 1$. To obtain a δ-regular graph, we connect the vertices of degree $\delta - 1$ in G_1 with the corresponding vertices in G_1^*, and connect each pair of matched vertices by an edge.

Thus we have constructed a new graph G' that is δ-regular (see Figure 1). Next, taking $g \geq 7$ into account, we show that the girth of G' is at least g.

Clearly, we only need to show this for any new cycle, say $\mathcal{C}$, which is introduced in the construction. All new cycles have to use at least two new edges, so we consider the following six cases.

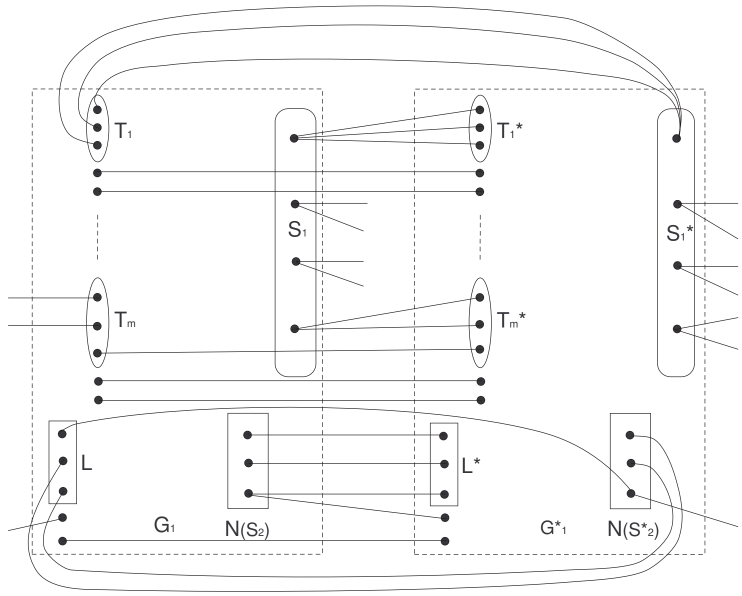

Figure 1: Illustration of the construction

- If $\mathcal{C}$ goes through two edges in (a), then the cycle $\mathcal{C}$ is of length at least $(g-2)+2 \geq g$ or $(g-1)/2+(g-1)/2+2 > g$.
- If $\mathcal{C}$ goes through two edges in (b), and the two edges in G' are correspond to $E(f)$ or $E(f^*)$, then $\mathcal{C}$ is of length at least $2+(g-1)/2+(g-3)/2 = g$. Otherwise $\mathcal{C}$ is also of length at least g, since $d(\{v^*_{m+1}, \ldots, v^*_\delta\} - L^*, N_C(S_2)) \geq (g-1)/2$ and no 4-cycle created in $B \cup B^* \cup E(f) \cup E(f^*)$.
- If $\mathcal{C}$ goes through two edges in (c), then $\mathcal{C}$ is of length at least $2(g-4)+2 \geq g$.
- If $\mathcal{C}$ goes through one edge in (a) and one edge in (b), then $\mathcal{C}$ is of length at least $(g-3)+2+1 = g$ or $(g-1)/2+2+(g-3)/2 \geq g$.
- If $\mathcal{C}$ goes through one edge in (a) and one edge in (c), then $\mathcal{C}$ is of length at least $(g-4)+2+(g-5)/2 \geq g$.
- If $\mathcal{C}$ goes through one edge in (b) and one edge in (c), then $\mathcal{C}$ is of length at least $(g-3)+2+(g-3)/2 \geq g$.

Case 2. $|S_1| = m = 0$.

Then $d_C(s_i) \leq k$ for $1 \leq i \leq k$. Now we partition S_2 into two subsets S_3 and S_4, where $S_3 = \{s \mid d_C(s) = k\}$ and $S_4 = \{s \mid d_C(s) \leq k-1\}$. Then $|S_3| \geq 2$. Otherwise, since $\delta \geq 9$, we have

$$
\begin{aligned}
|E(S, N_C(S))| &\leq k + (k-1)(k-1) \\
&\leq (r-1) + (r-2)^2 \\
&= (r-1)^2 - r + 2 \\
&\leq \delta + \sqrt{\delta} - 2 - \sqrt{\delta + \sqrt{\delta} - 2} + 2 \\
&= \delta + \sqrt{\delta} - \sqrt{\delta + \sqrt{\delta} - 2} \\
&< \delta
\end{aligned}
$$

But we know $E(S, N_C(S))$ is an edge-cut of G, which is a contradiction to Lemma 4. Now let $R_1 = \{s_1, s_2\} \subseteq S_3$ and $R_2 = S - R_1$. Note that

$$
\begin{aligned}
|R_1 \cup N_C(R_2)| &= |R_1| + |N_C(R_2)| \\
&\leq |R_1| + |E(R_2, N_C(R_2))| \\
&\leq 2 + k(k-2) \\
&\leq 2 + (r-1)(r-3) \\
&= (r-1)^2 - 2r + 4 \\
&\leq \delta + \sqrt{\delta} - 2 - 2\sqrt{\delta + \sqrt{\delta} - 2} + 4 \\
&< \delta + 2 - \sqrt{\delta + \sqrt{\delta} - 2} \\
&< \delta - 1.
\end{aligned}
$$

The last inequality is due to the assumption $\delta \geq 9$ and the condition that $|R_1 \cup N_C(R_2)|$ is an integer. Thus by Lemma 1, there exists a vertex $v \in V(C)$ such that $d(v, R_1 \cup N_C(R_2)) \geq (g-1)/2$. From $N(v)$, we can find two vertices v_1 and v_2 such that there are at most $k-1$ paths of length less than $(g-1)/2$ from $\{N(v_i) - v \mid 1 \leq i \leq \delta\}$ to S. If two such vertices do not exist, it implies that there are at least $(\delta - 1)k$ paths of length less than $(g-1)/2$ from $\cup_{i=1}^{\delta}(N(v_i) - v)$ to S. Note that $(\delta - 1)k > (r-1)k \geq k^2$ and $|E(S, N_C(S))| \leq k^2$, which implies a cycle of length less than g.

From $N_C(v_1)$ and $N_C(v_2)$, we can find two sets $T_1 \subseteq N_C(v_1)$ and $T_2 \subseteq N_C(v_2)$ such that $d(T_i, S) \geq (g-1)/2$, where $|T_i| = \delta - k$ and $i = 1, 2$. Also there are at most two paths of length less than $(g-1)/2$ from $N(v)$ to R_1. We may assume $d(v_i, R_1) \geq (g-1)/2$ for $5 \leq i \leq \delta$. Moreover, $|R_2| < |N_C(R_2)| \leq |E(N_C(R_2), R_2)| \leq \delta - 4$. We may choose a subset $L \subseteq \{v_5, \ldots, v_\delta\}$ such that $|L| = |N_{G_1}(R_2)|$ and for each $v_j \in L$, there are at most $|L| - 1$ paths of length at most $(g-3)/2$ from v_j to $N_C(R_2)$.

We construct a bipartite graph B with bipartition $(L, N_C(R_2))$ such that $st \in E(B)$ if and only if $d_C(s, t) = (g-3)/2$, where $s \in L$ and $t \in N_C(R_2)$. Clearly, B satisfies the conditions in Lemma 2. Thus there exist two one-to-one mappings $f : N_C(R_2) \mapsto L^*$ and $f^* : N_C(R_2^*) \mapsto L$ such that no 4-cycle created in graph $B \cup B^* \cup E(f) \cup E(f^*)$.

Consider the subgraph $G_1 = G[(C - \{v, v_1, v_2\}) \cup R_1] - E(G[R_1])$ and take another copy of the subgraph G_1, denote it by G_1^*. The corresponding sets of interest are denoted by R_1^*, R_2^* and T_i^*,

$i = 1, 2$. Similar to the proof in Case 1, we construct a δ-regular graph G' with girth at least g by using G_1 and G_1^*.

(a) For $i = 1, 2$, each vertex $s_i \in R_1$ connects to $d_{G-C}(s_i)$ distinct vertices in T_i^*. Similarly, we make the corresponding connections between the vertices in R_1^* and T_i.

After the operation is carried out in (a), for each $i = 1, 2$, $d_{G-C}(s_i)$ vertices of $N(v_i) - v$ (respectively, $N(v_i^*) - v^*$) are of degree δ and the remaining are of degree $\delta - 1$.

(b) Connect each vertex in $N_C(R_2)$ to a vertex in L^*, and connect each vertex in $N_C(R_2^*)$ to a vertex in L according to the two one-to-one mappings f and f^* given in the graph $B \cup B^*$. After this operation, all vertices in $L \cup L^*$ are of degree δ, but some vertices in $N_C(R_2) \cup N_C(R_2)^*$ might be of degree less than δ. Since $|N_C(R_2)| \leq |E(R_2, N_C(R_2))| \leq \delta - 4$, so we can connect a vertex in $N_C(R_2)$ to the vertices in $\{v_4^*, \ldots, v_\delta^*\} - L^*$ such that all vertices in $N_C(R_2)$ are of degree δ. Similarly, we make the corresponding connections between the vertices in $N_C(R_2^*)$ and $\{v_4, \ldots, v_\delta\} - L$.

(c) After the operations are carried out in (a) and (b), all vertices are of degree δ or $\delta - 1$. To obtain a δ-regular graph, we connect the vertices of degree $\delta - 1$ in G_1 with the corresponding vertices in G_1^*, and connect each pair of matched vertices by an edge.

Thus we have constructed a new δ-regular graph G'. Verifying the girth of G' can be done in the same fashion as in Case 1. □

Theorem 6. *Let G be a (δ, g)-cage with $\delta \geq 4$ and even girth $g \geq 10$. Then G is $(r+1)$-connected, where r is the largest integer such that $\frac{r(r-1)^2}{4} + 1 + 2r(r-1) \leq \delta$.*

Proof. In [13], (δ, g)-cages with $g \geq 10$ are showed to be 4-connected. Thus if $\delta \leq 16$, the theorem holds. So assume $r \geq 4$ and $\delta \geq 17$. Suppose, to the contrary, $\kappa(G) < r + 1$. Then G has a cutset $S = \{s_1, \ldots, s_k\}$ with $k \leq r$. Let C be a smallest component of $G - S$ and let $G_1 = G[V(C) \cup S] - E(G[S])$.

Now we partition the set S into three subsets (see Figure 2).

$X = \{u \mid d_{G_1}(u) \leq r, u \in S\}$,

$Y = \{u \mid r + 1 \leq d_{G_1}(u) \leq rx + r - x, u \in S\}$,

$Z = \{u \mid d_{G_1}(u) \geq rx + r - x + 1, u \in S\}$.

where $|X| = x$, $|Y| = y$ and $|Z| = z$. Thus $r \geq k = |X| + |Y| + |Z| = x + y + z$. By Lemma 4, it follows $|Z| \geq 1$, otherwise, $E(N(S), S)$ is an edge-cut and $|E(N(S), S)| \leq rx + (r-x)(rx+r-x) = (r^2 - r)x + (1 - r)x^2 + r^2 < \delta$, a contradiction to Theorem 4.

Based on this partition, we conclude:

$|N(X) \cap V(C)| \leq rx$,

$|N(Y) \cap V(C)| \leq y(xr + r - x)$,

$|N(X) \cap V(C) \cup Y \cup Z| \leq xr + r - x < r^2$.

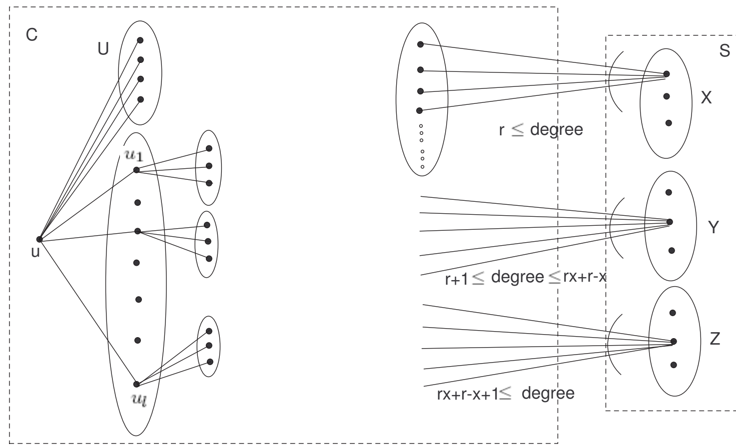

Figure 2: The structure of X, Y and Z

Let $F = (N(X) \cap V(C)) \cup Y \cup Z$. Obviously, the set F is also a vertex-cut whose cardinality is less than δ. Instead of considering the vertex-cut S, we focus on this new vertex-cut F in the rest of the proof.

By Lemma 1, there exists a vertex $u \in C$ such that the distance from u to F is at least $g/2-1$. Let W stand for the set of edges of the subgraph induced by $Y \cup Z$, that is, $W = E(G[Y \cup Z])$. It is easy to see that there are at most $y(xr + r - x)$ vertices in $N(u)$ which are at distance $g/2 - 1$ or $g/2 - 2$ to Y in the graph $G_1 - W$, because all shortest paths in $G_1 - W$ of length $g/2 - 1$ or $g/2 - 2$ from vertices in $N(u)$ to vertices in Y must go via the vertices in $N(Y) \cap V(C)$. As $|N(Y) \cap V(C)| \leq y(xr + r - x)$, there are at most $y(xr + r - x)$ disjoint paths of length $g/2 - 1$ or $g/2 - 2$ from $N(u)$ to Y. Otherwise, by the Pigeonhole Principle, there exists a cycle of length less than g in the graph, which goes through u, two distinct vertices in $N(u)$, and a vertex in $N(Y) \cap V(C)$, a contradiction.

Since $|F - Y| = |N(X) \cap V(C) \cup Z| \leq rx + z = rx + r - x - y$, using the arguments as in the previous cases, we see that among the vertices left, at least $\delta - y(xr + r - x)$ vertices are in $N(u)$, and there are at most $rx + z$ vertices which have distance $g/2 - 2$ in G to $(N(X) \cap V(C)) \cup Z$. Moreover, because of

$$\begin{aligned} yrx + y(r - x) + z + 2rx &= rxy + yr - xy + r - x - y + 2rx \\ &\leq \frac{r(r-1)^2}{4} + yr - xy + r - x - y + 2rx \\ &= \frac{r(r-1)^2}{4} + 1 + 2r(r-1), \end{aligned}$$

we have

$$\delta - y(xr + r - x) - z - 2rx \geq \delta - \frac{r(r-1)^2}{4} - 1 - 2r(r-1) \geq 0.$$

Therefore, there are at least

$$\delta - y(xr + r - x) - z - rx \geq rx$$

vertices in $N(u)$ which have distance at least $g/2$ to Y and at least $g/2-1$ to $(N(X) \cap V(C)) \cup Z$ in $G - W$. Thus, we have

$$d(v, F) \geq g/2 - 2 \text{ for all } v \in N(u)$$

and there exists a set $U = \{u_1, \ldots, u_t\} \subseteq N(u)$, where $t \geq rx$, such that

$$d(U, Y) \geq g/2 \text{ and } d(U, F \backslash Y) \geq g/2 - 1.$$

For each vertex u_i in $N(u)$, denote by U_i the vertices in $N(u_i) - u$ which have distance at least $g/2-1$ to F in $G_1 - W$. It is clear that $|U_i|$ is at least $\delta - 1 - rx - z - y$, since $|F| \leq rx + y + z$. Denote by $\widehat{U}_i$ the set of vertices in $N(u_i) - u$ which have distance at least $g/2-1$ to $X \cup Y \cup Z$ in $G_1 - W$. So $U_i \subseteq \widehat{U}_i$. It is easy to see that $|\widehat{U}_i| \geq \delta - r - 1$ as $|X \cup Y \cup Z| \leq r$. To summarize, there exist two sets $U_i \subset \widehat{U}_i \subset N(u_i) - u$, $i = 1, 2$, with $|U_i| \geq \delta - 1 - rx - z - y$ and $|\widehat{U}_i| \geq \delta - r - 1$ such that

$$d(U_i, F) \geq g/2 - 1,$$
$$d(\widehat{U}_i, X \cup Y \cup Z) \geq g/2 - 1,$$
$$d(\widehat{U}_i, F) \geq g/2 - 2.$$

Using the similar approach as before, we construct a (δ, g')-graph with smaller size. Taking the subgraph of $G - W$ induced by $V(C) \cup Y \cup Z - \{u\}$ and deleting some vertices (which are described in the proof later), we denote the resulting graph by H. Take another copy of H and denote it by H^*, the corresponding sets of interests in H^* are $U^* = \{u_1^*, u_2^*, \ldots, u_t^*\}$, Y^* and Z^*. We join the vertices of H and H^* by some edges, which are described below, to construct a new graph G'. The new graph G' is δ-regular and its girth is at least g but with fewer vertices than G. By Monotonicity Theorem, we arrive at a contradiction and thus the theorem is proved.

The connections are described below (see Figure 3 for an illustration).

(a) The degrees of vertices in $N(X) \cap V(C)$ are unknown at this point, however we know that the number of new edges that should be added in order to achieve degree δ for all the vertices in $N(X) \cap V(C)$ is at most rx. Therefore, every vertex, say x_i, in $N(X) \cap V(C)$ is connected to $|N(x_i) \cap V(C)|$ vertices in U^*. Note that $|U| \geq rx$, thus this operation is well defined. We make the same connections between $N(X^*) \cap V(C^*)$ and U. It is obvious that now the vertices in $N(X) \cap V(C)$ and $N(X^*) \cap V(C^*)$ have degree δ.

(b) There are at least $|Y| + |Z|$ vertices left in $N(u)$ with degree $\delta - 1$ which are at distance at least $g/2 - 1$ to F. The the same statement applies to $N(u^*)$. Every vertex y_i in Y is arbitrarily connected with one of these remaining vertices in $N(u^*)$, say u_i^*. We remove u_i^* from the graph and connect y_i to some vertices in $\widehat{U_i^*}$ such that y_i has degree δ. Note that $|\widehat{U_i^*}| \geq \delta - r - 1$ and $|N(y_i) \cap V(C)| \leq \delta - r - 1$. Therefore, we guarantee that the degree of y_i equals to δ by connecting it to vertices in $\widehat{U_i^*}$. We make the similar connections between Y^* and $N(u)$.

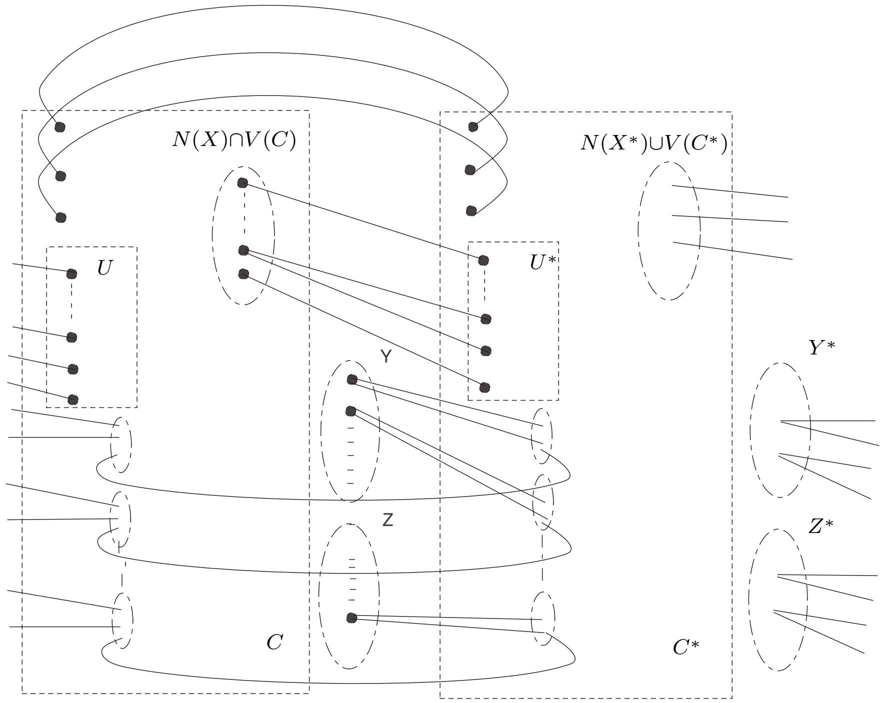

Figure 3: Illustration of G'

(c) At this stage, there are at least $|Z|$ vertices left in $N(u)$ with degree $\delta - 1$ and these vertices are at distance at least $g/2 - 1$ to F. Each vertex z_j of Z is arbitrarily connected with a vertex in $N(u^*)$, say u_j^*. We remove u_j from the graph and connect z_j to some vertices in U_j^* such that z_j has degree δ. Note that $|U_j^*| \geq \delta - (1 + rx + z + y)$ and $\delta - d_{G_1}(z_j) \leq \delta - (rx + z + y) - 1$. Therefore, we can connect z_j to some vertices of U_j^* to insure that degree of z_j is δ. We make the similar connections between Z^* and $N(u)$.

(d) The rest of the vertices in the graph have degree δ or $\delta - 1$. We connect each vertex $x \in V(H)$ with degree $\delta - 1$ to its copy $x^* \in V(H^*)$.

The graph G' is a δ-regular graph. It is not hard to verify that this graph has girth at least g in the same way as what we did in the proof of the previous theorem. Now we have constructed a (δ, g')-graph G' with girth $g' \geq g$ but $|V(G')| < |V(G)|$, arriving at a contradiction by Monotonicity Theorem. □

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