A Unification of Two Refinements of Euler's Partition Theorem

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Dedicated to Professor George Andrews on the Occasion of His Seventieth Birthday

Abstract. We obtain a unification of two refinements of Euler's partition theorem respectively due to Bessenrodt and Glaisher. A specialization of Bessenrodt's insertion algorithm for a generalization of the Andrews-Olsson partition identity is used in our combinatorial construction.

Keywords: partition, Euler's partition theorem, refinement, bijection, Bessenrodt's bijection, Andrews-Olsson's theorem.

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1 Introduction

There are several bijective proofs and refinements of the classical partition theorem of Euler. This paper will be concerned with two remarkable bijections obtained by Sylvester [19] and Glaisher [16], see also, [6, pp.8–9]. Glaisher's bijection implies a refinement of Euler's theorem involving the number of odd parts in a partition with distinct parts and the number of parts repeated odd times in a partition with odd parts. On the other hand, as observed by Bessenrodt [9], Sylvester's bijection also leads to a refinement of Euler's theorem. The main result of this paper is a unification of these two refinements that do not directly follow from Sylvester's bijection and Glaisher's bijection.

Let us give an overview of the background and terminology. We will adopt the common notation on partitions used in Andrews [3, Chapter 1]. *A partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

such that $\sum_{i=1}^{r} \lambda_i = n$. The entries λ_i are called the parts of λ, and λ_1 is the largest part. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. The weight of λ is the sum of its parts, denoted by $|\lambda|$. A partition λ can also be represented in the following form

$$\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots),$$

where m_i is the multiplicity of the part i in λ. The conjugate partition of λ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$, where λ'_i is the number of parts of λ that are greater than or equal to i.

Euler's partition theorem reads as follows.

Theorem 1.1 (Euler) *The number of partitions of n with distinct parts is equal to the number of partitions of n with odd parts.*

Let $\mathcal{D}$ denote the set of partitions with distinct parts, and let $\mathcal{D}(n)$ denote the set of partitions of n in $\mathcal{D}$. Similarly, let $\mathcal{O}$ denote the set of partitions with odd parts, and let $\mathcal{O}(n)$ denote the set of partitions of n in $\mathcal{O}$. Sylvester's fish-hook bijection [19], also referred to as Sylvester's bijection, and Glaisher's bijection [6, pp.8-9] have established direct correspondences between $\mathcal{D}(n)$ and $\mathcal{O}(n)$. These two bijections imply refinements of Euler's theorem. There are also several other refinements of Euler's partition theorem, see, for example, [1, 2, 4, 9, 17, 18, 21], [14, pp.51–52], [15, pp.46–47].

Sylvester's refinement [3, p.24] is stated as follows. Recall that a chain in a partition with distinct parts is a maximal sequence of parts consisting of consecutive integers. The number of chains in a partition λ is denoted by $n_c(\lambda)$. The number of different parts in a partition μ is denoted by $n_d(\mu)$. For example, the partition $(8, 7, 5, 3, 2, 1)$ has three chains, and the partition $(8, 6, 6, 5, 4, 4, 2, 1)$ has six different parts.

Theorem 1.2 (Sylvester) *The number of partitions of n into distinct parts with exactly k chains is equal to the number of partitions of n into odd parts (repetitions allowed) with exactly k different parts. In the notation of generating functions, we have*

$$\sum_{\lambda \in \mathcal{D}} z^{n_c(\lambda)} q^{|\lambda|} = \sum_{\mu \in \mathcal{O}} z^{n_d(\mu)} q^{|\mu|}. \tag{1.1}$$

Fine [15, pp.46–47] has derived a refinement of Euler's theorem.

Theorem 1.3 (Fine) *The number of partitions of n into distinct parts with largest part k is equal to the number of partitions of n into odd parts such that the largest part plus twice the number of parts equals $2k+1$. In the notation of generating functions, we have*

$$\sum_{\lambda \in \mathcal{D}} x^{\lambda_1} q^{|\lambda|} = \sum_{\mu \in \mathcal{O}} x^{(\mu_1 - 1)/2 + l(\mu)} q^{|\mu|}. \tag{1.2}$$

Bessenrodt [9] has shown that Sylvester's bijection implies the following refinement, which is a limiting case of the lecture hall theorem due to Bousquet-Mélou and Erikssonin [12, 13]. Let $l_a(\lambda)$ denote the alternating sum of λ, namely,

$$l_a(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots .$$

Theorem 1.4 (Bessenrodt) *The number of partitions of n into distinct parts with alternating sum l is equal to the number of partitions of n with l odd parts. In terms of generating functions, we have*

$$\sum_{\lambda\in\mathcal{D}} y^{l_a(\lambda)} q^{|\lambda|} = \sum_{\mu\in\mathcal{O}} y^{l(\mu)} q^{|\mu|}. \tag{1.3}$$

It has also been shown by Bessenrodt [9] that Sylvester's bijection maps the parameter $n_c(\lambda)$ to the parameter $n_d(\mu)$. Combining the above Theorems 1.2 and 1.3, we arrive at the following equidistribution result.

Theorem 1.5 (Sylvester-Bessenrodt) *The number of partitions of n into distinct parts with largest part k, alternating sum l and m chains is equal to the number of partitions of n into l odd parts with exactly m different parts such that the largest part plus twice the number of parts equals $2k+1$. In terms of generating functions, we have*

$$\sum_{\lambda\in\mathcal{D}} x^{\lambda_1} y^{l_a(\lambda)} z^{n_c(\lambda)} q^{|\lambda|} = \sum_{\mu\in\mathcal{O}} x^{(\mu_1-1)/2+l(\mu)} y^{l(\mu)} z^{n_d(\mu)} q^{|\mu|}. \tag{1.4}$$

Recently, Zeng [21] has found a generating function proof of the above three-parameter refinement (1.4).

From a different angle, Glaisher [16], see also [6, pp.8–9], has given a refinement of Euler's partition theorem. Let $l_o(\lambda)$ denote the number of odd parts in λ, and let $n_o(\mu)$ denote the number of different parts in μ with odd multiplicities.

Theorem 1.6 (Glaisher) *The number of partitions of n into distinct parts with k odd parts is equal to the number of partitions of n with odd parts such that there are exactly k different parts repeated odd times. In terms of generating functions, we have*

$$\sum_{\lambda\in\mathcal{D}} x^{l_o(\lambda)} q^{|\lambda|} = \sum_{\mu\in\mathcal{O}} x^{n_o(\mu)} q^{|\mu|}. \tag{1.5}$$

Given the two bijections of Sylvester and Glaisher, it is natural to ask the question whether the joint distribution of the statistics $(l_o(\lambda), l_a(\lambda))$ of partitions of n with distinct parts coincides with the joint distribution of the statistics $(n_o(\mu), l(\mu))$ of partitions with odd parts. It turns out that this is indeed the case. However, neither Sylvester's bijection nor Glaisher's bijection implies this result. To give a combinatorial proof of this result, we need Bessenrodt's insertion algorithm. Using this algorithm, we can give a new bijection between partitions with distinct parts and partitions with odd parts.

It should be noted that the equidistriubtion of $(l_o(\lambda), l_a(\lambda))$ and $(n_o(\mu), l(\mu))$ can also be deduced from a recent result of Boulet [11] by the manipulation of generating functions.

This paper is organized as follows. In Section 2, we present the main result and some lemmas. Section 3 is devoted to a brief review of Bessenrodt's insert algorithm. In Section 4, we utilize Boulet's formula to give a generating function proof of the two-parameter refinement of Euler's theorem. In Section 5, we give a combinatorial proof of the unification of the refinements of Bessenrodt (1.3) and Glaisher (1.5).

2 The main result

The main result of this paper is the following unification of the refinements of Bessenrodt and Glaisher.

Theorem 2.1 *The number of partitions of n into distinct parts with l odd parts and alternating sum m is equal to the number of partitions of n into exactly m odd parts and l parts repeated odd times. In terms of generating functions, we have*

$$\sum_{\lambda\in\mathcal{D}} x^{l_o(\lambda)} y^{l_a(\lambda)} q^{|\lambda|} = \sum_{\mu\in\mathcal{O}} x^{n_o(\mu)} y^{l(\mu)} q^{|\mu|}. \tag{2.1}$$

For example, Table 2.1 illustrates the case of $n = 7$.

$\lambda \in \mathcal{D}(7)$	$l_o(\lambda)$	$l_a(\lambda)$	$\mu \in \mathcal{O}(7)$	$n_o(\mu)$	$l(\mu)$
(7)	1	7	(1^7)	1	7
$(1,6)$	1	5	$(1^4,3)$	1	5
$(2,5)$	1	3	$(1,3^2)$	1	3
$(3,4)$	1	1	(7)	1	1
$(1,2,4)$	1	3	$(1^2,5)$	1	3

Table 2.1: The case of $n = 7$ for Theorem 2.1.

It is clear that the above theorem reduces to Bessenrodt's refinement (1.3) when $x = 1$ and to Glaisher's refinement (1.5) when $y = 1$.

To prove Theorem 2.1, we proceed to construct a bijection Δ between $\mathcal{D}(n)$ and $\mathcal{O}(n)$ such that for $\lambda \in \mathcal{D}(n)$ and $\mu = \Delta(\lambda) \in \mathcal{O}(n)$, we have

$$l_o(\lambda) = n_o(\mu), \quad l_a(\lambda) = l(\mu).$$

Let $\mathcal{A}_1(n)$ denote the set of partitions of n subject to the following conditions:

1. Only parts divisible by 2 may be repeated.

2. The difference between successive parts is at most 4 and strictly less than 4 if either part is divisible by 2.

3. The smallest part is less than 4.

By considering the conjugate of the 2-modular representation of a partition, it is easy to establish a bijection between $\mathcal{D}(n)$ and $\mathcal{A}_1(n)$.

Lemma 2.2 *There is a bijection φ between $\mathcal{D}(n)$ and $\mathcal{A}_1(n)$. Furthermore, for $\lambda \in \mathcal{D}(n)$ and $\alpha = \varphi(\lambda) \in \mathcal{A}_1(n)$, we have*

$$l_o(\lambda) = l_o(\alpha), \quad l_a(\lambda) = 2r_2(\alpha) + l_o(\alpha), \tag{2.2}$$

where $r_2(\alpha)$ denotes the number of parts congruent to 2 modulo 4 in α.

Let $\mathcal{A}_2(n)$ denote the set of partitions of n subject to the following conditions:

1. No part divisible by 4.

2. Only parts divisible by 2 may be repeated.

We then establish a bijection between $\mathcal{O}(n)$ and $\mathcal{A}_2(n)$ in the spirit of Glaisher's bijection.

Lemma 2.3 *There is a bijection ψ between $\mathcal{O}(n)$ and $\mathcal{A}_2(n)$. Furthermore, for $\mu \in \mathcal{O}(n)$ and $\beta = \psi(\mu) \in \mathcal{A}_2(n)$, we have*

$$n_o(\mu) = l_o(\beta), \quad l(\mu) = 2r_2(\beta) + l_o(\beta). \tag{2.3}$$

In view of the above two lemmas, we see that Theorem 2.1 can be deduced from the following theorem.

Theorem 2.4 *There is a bijection ϕ between $\mathcal{A}_1(n)$ and $\mathcal{A}_2(n)$. Furthermore, for $\alpha \in \mathcal{A}_1(n)$ and $\beta = \phi(\alpha) \in \mathcal{A}_2(n)$, we have*

$$l_o(\alpha) = l_o(\beta), \quad r_2(\alpha) = r_2(\beta). \tag{2.4}$$

We find that Theorem 2.4 can be deduced from Bessenrodt's insertion algorithm which was devised as a combinatorial proof of a generalization of Andrews-Olsson's theorem [5]. Combining the bijection φ for Lemma 2.2, ψ for Lemma 2.3 and ϕ for Theorem 2.4, we are led to a new bijection Δ for Euler's partition theorem which implies the equidistribution of the statistics $(l_o(\lambda), l_a(\lambda))$ of partitions with distinct parts and the statistics $(n_o(\mu), l(\mu))$ of partitions with odd parts.

3 Bessenrodt's insertion algorithm

To provide a purely combinatorial proof of the Andrews-Olsson's theorem [5], Bessenrodt [8] constructs an explicit bijection on the sets of partitions in Andrews-Olsson's theorem, which we call Bessenrodt's insertion algorithm. The original insertion algorithm does not imply to the bijection in Theorem 2.4, we find that the generalized insertion algorithm given by Bessenrodt [10] in 1995 can be used to establish the bijection required by Theorem 2.4.

We give an overview of Bessenrodt's insertion algorithm. Let N be an integer, and let $\mathbb{A}_N = \{a_1, a_2, \ldots, a_r\}$ with $1 \leq a_1 < a_2 < \cdots < a_r < N$. Andrews-Olsson's theorem involves two sets $\mathcal{AO}_1(\mathbb{A}_N; n, N)$ and $\mathcal{AO}_2(\mathbb{A}_N; n, N)$ defined below.

Definition 3.1 *Let $\mathcal{AO}_1(\mathbb{A}_N; n, N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to 0 or some a_i modulo N;*
2. *Only the multiples of N can be repeated;*
3. *The difference between two successive parts is at most N and strictly less than N if either part is divisible by N;*
4. *The smallest part is less than N.*

Definition 3.2 *Let $\mathcal{AO}_2(\mathbb{A}_N; n, N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to some a_i modulo N;*
2. *No part can be repeated.*

The cardinalities of $\mathcal{AO}_1(\mathbb{A}_N; n, N)$ and $\mathcal{AO}_2(\mathbb{A}_N; n, N)$ are denoted by $p_1(\mathbb{A}_N; n, N)$ and $p_2(\mathbb{A}_N; n, N)$ respectively. Andrews-Olsson's theorem is stated as follows.

Theorem 3.3 (Andrews-Olsson) *For any $n \in \mathbb{N}$, we have*

$$p_1(\mathbb{A}_N; n, N) = p_2(\mathbb{A}_N; n, N).$$

By examining the two sets $\mathcal{A}_1(n)$ and $\mathcal{A}_2(n)$ in Theorem 2.4, we find they are somehow analogous to the two sets $\mathcal{AO}_1(\mathbb{A}_N; n, N)$ and $\mathcal{AO}_2(\mathbb{A}_N; n, N)$ in Andrews-Olsson's theorem, but not the special cases. However, we could also apply Bessenrodt's insertion algorithm to establish a bijection between $\mathcal{A}_1(n)$ and $\mathcal{A}_2(n)$. Here we present a more general bijection Φ between two sets $\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$ and $\mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$, and we can restrict the bijection Φ to $\mathcal{A}_1(n)$ and $\mathcal{A}_2(n)$ by setting $N = 2$ and $\mathbb{A}_4 = \{1, 2, 3\}$.

Definition 3.4 *Let $\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to 0 or some a_i modulo $2N$;*
2. *Only the multiples of N can be repeated;*
3. *The difference between two successive parts is at most $2N$ and strictly less than $2N$ if either part is divisible by N;*
4. *The smallest part is less than $2N$.*

Definition 3.5 *Let $\mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to some a_i modulo $2N$;*
2. *Only multiples of N may be repeated;*

The cardinalities of $\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$ and $\mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ are denoted by $c_1(\mathbb{A}_{2N}; n, 2N)$ and $c_2(\mathbb{A}_{2N}; n, 2N)$ respectively. Then we have the following theorem which will be needed to prove Theorem 2.4.

Theorem 3.6 *For any $n \in \mathbb{N}$, we have*

$$c_1(\mathbb{A}_{2N}; n, 2N) = c_2(\mathbb{A}_{2N}; n, 2N).$$

Theorem 3.6 can be proved either by a variant of Bessenrodt's insertion algorithm obtained in 1991, or by specializing a generalization of Bessenrodt's algorithm obtained in 1995.

We outline the first approach by constructing a bijection Φ between $\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$ and $\mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ based on a variant of Bessenrodt's insertion algorithm [8].

For $\lambda \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$, we first extract some parts from λ to form a pair of partitions (α, β), where $\alpha \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N) \cap \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ and β is a partition with parts divisible by $2N$. Then we insert β into α to get a partition $\gamma \in \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$.

The bijection Φ consists of the following two steps.

Step 1: Extract certain parts from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}) \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$.

We now construct a pair of partitions (α, β) based on the partition λ. Let λ_j be a part divisible by $2N$, and λ_t be the smallest part bigger than λ_j. We remove λ_j if λ_t does not exist or the difference between λ_t and λ_{j+1} satisfies the difference condition in $\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$. After removing these parts λ_j, we obtain a partition α^1 in

$\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$, and we can rearrange these parts that have been removed to form partition β^1.

Assume that there are l parts divisible by $2N$ in α^1. Let $t = 1$, and may iterate the following procedure until we get the pair of partitions $(\alpha^{l+1}, \beta^{l+1})$.

- Let α_i^t be the largest part divisible by $2N$ in α^t.
- Subtract $2N$ from $\alpha_1^t, \alpha_2^t, \ldots, \alpha_{i-1}^t$ and remove α_i^t from α^t.
- Rearrange the remaining parts to give a new partition α^{t+1} and add one part of size $(i-1) \cdot 2N + \alpha_i^t$ to β^t to get β^{t+1}.

Then let $\alpha = \alpha^{l+1}$, $\beta = \beta^{l+1}$. It can be seen that $\alpha \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N) \cap \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ and $\beta_1 \leq 2N \cdot l(\alpha)$.

Step 2: Insert β into α to generate a partition $\gamma \in \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$.

For each β_i, we add $2N$ to the first $\beta_i/2N$ parts of α: $\alpha_1, \alpha_2, \ldots, \alpha_{\beta_i/2N}$, then denote by γ the partition obtained by implementing operation. It can be shown that $\gamma \in \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ for $\beta_1 \leq 2N \cdot l(\alpha)$. For the details of the proof, see [8, 20].

The inverse map Φ^{-1} can be described as follows. For $\gamma \in \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$, we extract certain parts from γ to get a pair of partitions (α, β), where $\alpha \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N) \cap \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ and β is a partition with parts divisible by $2N$. Then we insert β to α to form a partition $\lambda \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$.

Formally speaking, the inverse map Φ^{-1} consists of the following two steps.

Step 1: Extraction of parts from γ.

Suppose $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{l(\gamma)}) \in \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$. Let $\alpha = \gamma$, $\beta = \emptyset$ and $t = l(\gamma)$. We can obtain a pair of partitions (α, β) by the following procedures:

- If α_t is divisible by N, then there exists an integer i such that $\alpha_t - \alpha_{t+1} = i \cdot 2N + r_t$, where $0 \leq r_t < 2N$;
- If α_t is not divisible by N, then there exists an integer i such that $\alpha_t - \alpha_{t+1} = i \cdot 2N + r_t$, where $0 < r_t \leq 2N$;
- Subtract $i \cdot 2N$ from the parts $\alpha_1, \alpha_2, \ldots, \alpha_t$; Rearrange these parts to generate a new partition α and add i parts of size $t \cdot 2N$ to β.
- If $t \geq 2$, then replace t by $t-1$ and repeat the above procedure. If $t = 1$, we get a pair of partitions (α, β).

Step 2: Insert β into α.

Assume that (α, β) is a pair of partitions such that $\alpha \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N) \cap \mathcal{C}_2(\mathbb{A}_{2N}; n, 2N)$ and β is a partition with parts divisible by $2N$. We can construct a partition $\lambda \in$

$\mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$. If $\beta_1 \le \alpha_1 + 2N - 1$, we insert all the parts of β into α to generate a new partition λ.

If $\beta_1 > \alpha_1 + 2N - 1$, we set $t = 1$ initially and iterate the following procedure until $\beta_t \le \alpha_1 + 2N - 1$:

- Let i be the largest positive integer such that $\beta_t - i \cdot 2N \ge \alpha_i$, namely for $j > i$ we have $\beta_t - j \cdot 2N < \alpha_j$.
- Add $2N$ to the first i parts $\alpha_1, \alpha_2, \ldots, \alpha_i$, and then insert $\beta_t - i \cdot 2N$ into α in the position before the part α_{i+1}.
- Rearrange the resulted parts to form a new partition α and replace t by $t + 1$.

Finally, we arrive at the condition $\beta_t \le \alpha_1 + 2N - 1$. Then we insert all the remaining parts of β into α to generate a new partition λ. It can be shown that $\lambda \in \mathcal{C}_1(\mathbb{A}_{2N}; n, 2N)$. For the details of the proof, see [8, 20]. ∎

We now turn to the generalization of Bessenrodt's insertion algorithm and we will show how one derives Theorem 3.6 from this generalized algorithm, which implies a generalization of the Andrews-Olsson partition Theorem given in [10]. Let $\mathbb{A}_N = \mathbb{A}'_N \cup \mathbb{A}''_N$ with $\mathbb{A}'_N \cap \mathbb{A}''_N = \emptyset$.

Definition 3.7 *Let $\mathcal{B}_1(\mathbb{A}'_N, \mathbb{A}''_N; n, N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to 0 or some a_i modulo N;*
2. *Only the part congruent to 0 or some a_i belonging to $\mathbb{A}'_N$ modulo N can be repeated;*
3. *The difference between two successive parts is at most N and strictly less than N if either part congruent to 0 or some a_i belonging to $\mathbb{A}'_N$ modulo N;*
4. *The smallest part is less than N.*

Definition 3.8 *Let $\mathcal{B}_2(\mathbb{A}'_N, \mathbb{A}''_N; n, N)$ denote the set of partitions of n satisfying the following conditions:*

1. *Each part is congruent to some a_i modulo N;*
2. *Only part congruent to some a_i belonging to $\mathbb{A}'_N$ modulo N can be repeated.*

The cardinalities of $\mathcal{B}_1(\mathbb{A}'_N, \mathbb{A}''_N; n)$ and $\mathcal{B}_2(\mathbb{A}'_N, \mathbb{A}''_N; n)$ are denoted by $b_1(\mathbb{A}'_N, \mathbb{A}''_N; n)$ and $b_2(\mathbb{A}'_N, \mathbb{A}''_N; n, N)$ respectively.

Bessenrodt's generalization of the Andrews-Olsson theorem is stated as follows.

Theorem 3.9 (Bessenrodt) *For any $n \in \mathbb{N}$, we have*

$$b_1(\mathbb{A}'_N, \mathbb{A}''_N; n, N) = b_2(\mathbb{A}'_N, \mathbb{A}''_N; n, N).$$

Clearly, Andrews-Olsson's Theorem 3.3 can be viewed as the special case $\mathbb{A}'_N = \emptyset$ of Theorem 3.9. Theorem 3.6 is the special case for $2N$ and $\mathbb{A}'_{2N} = \{N\}$. As noted by Bessenrodt [10], the special case $N = 2$, $\mathbb{A}'_2 = \{1\}$ and $\mathbb{A}''_2 = \emptyset$, reduces to Euler's partition Theorem, and Bessenrodt's insertion algorithm for this case coincides with Sylvester's bijection.

4 Connection to Boulet's formula

In this section, we show that our two-parameter refinement (2.1) can be derived from a formula of Boulet. The following four-parameter weight was introduced by Boulet [11] as a generalization of the weight defined by Andrews [7]. Let a, b, c and d be commuting indeterminants. Define the following weight function $\omega(\lambda)$ on the set of all partitions:

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. Boulet obtained the following formula:

$$\sum_{\lambda\in P} \omega(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}, \tag{4.1}$$

where P denotes the set of all integer partitions. It can be easily checked that the generating function of the partitions in which every part appears an even number of times is

$$\prod_{j=1}^{\infty} \frac{1}{(1 - a^j b^j c^j d^j)(1 - a^j b^{j-1} c^j d^{j-1})}.$$

From (4.1), Boulet deduced the generating function for the weight function $\omega(\lambda)$ when λ runs over all partitions with distinct parts ([11, Corollary 2]):

$$\sum_{\lambda\in \mathcal{D}} \omega(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}. \tag{4.2}$$

Making the substitutions $a \mapsto xyq$, $b \mapsto x^{-1}yq$, $c \mapsto xy^{-1}q$, $d \mapsto x^{-1}y^{-1}q$ in (4.1), Boulet derived the following identity due to Andrews [7].

Theorem 4.1 (Andrews) *We have*

$$\sum_{\lambda\in P} x^{l_o(\lambda)} y^{l_o(\lambda')} q^{|\lambda|} = \prod_{j=1}^{\infty} \frac{(1 + xyq^{2j-1})}{(1 - q^{4j})(1 - x^2 q^{4j-2})(1 - y^2 q^{4j-2})}.$$

Using the same substitution in (4.2), we find obtain the following formula for partitions with distinct parts.

Theorem 4.2 *We have*

$$\sum_{\lambda\in\mathcal{D}} x^{l_o(\lambda)}y^{l_o(\lambda')}q^{|\lambda|} = \prod_{j=1}^{\infty}\frac{1+xyq^{2j-1}}{1-y^2q^{4j-2}}. \tag{4.3}$$

On the other hand, it is easy to derive the following generating function formula.

Theorem 4.3 *We have*

$$\sum_{\lambda\in\mathcal{O}} x^{n_o(\lambda)}y^{l(\lambda)}q^{|\lambda|} = \prod_{j=1}^{\infty}\frac{1+xyq^{2j-1}}{1-y^2q^{4j-2}}. \tag{4.4}$$

Proof. We have

$$\begin{aligned}
\prod_{j=1}^{\infty}\frac{1+xyq^{2j-1}}{1-y^2q^{4j-2}} &= \prod_{j=1}^{\infty}\left(\left(1+xyq^{2j-1}\right)\sum_{i=0}^{\infty}y^{2i}q^{(2i)\cdot(2j-1)}\right)\\
&= \prod_{j=1}^{\infty}\sum_{i=0}^{\infty}\left(y^{2i}q^{(2i)\cdot(2j-1)}+xy^{(2i+1)}q^{(2i+1)\cdot(2j-1)}\right)\\
&= \prod_{j=1}^{\infty}\left(1+xyq^{(2j-1)}+y^2q^{2\cdot(2j-1)}+xy^3q^{3\cdot(2j-1)}+\cdots\right)\\
&= \sum_{\lambda\in\mathcal{O}} x^{n_o(\lambda)}y^{l(\lambda)}q^{|\lambda|},
\end{aligned}$$

as desired. ∎

Since $l_a(\lambda) = l_o(\lambda')$ for any partition λ, combining Theorems 4.2 and 4.3 yields Theorem 2.1.

5 A combinatorial proof of the main result

In this section, we give a combinatorial proof of Theorem 2.1. We will use a restricted version of the variant of Bessenrodt's insertion algorithm given in Section 3. We now proceed to give the proofs of Lemma 2.2, Lemma 2.3 and Theorem 2.4.

Proof of Lemma 2.2. For $\lambda\in\mathcal{D}(n)$, define $\alpha=\varphi(\lambda)$ as the 2-modular diagram conjugate of λ. It is necessary to show that $\alpha\in\mathcal{A}_1(n)$. On the one hand, it is easy to see that there is no odd part in α that can be repeated, since there is at most one "1" in each row of the 2-modular diagram of λ. Moreover, the condition that λ is a partition with distinct parts implies that the difference between successive parts in α is at most 4 and strictly less than 4 if either part is divisible by 2, and that the smallest part of α is less than 4.

The reverse map φ^{-1} can be easily constructed. For $\alpha \in \mathcal{A}_1(n)$, we note that its 2-modular diagram conjugate is a partition with distinct parts, namely, $\lambda = \varphi^{-1}(\alpha) \in \mathcal{D}(n)$. Thus φ is a bijection. Furthermore, it is not difficult to check $l_o(\lambda) = l_o(\alpha)$ and $l_a(\lambda) = 2r_2(\alpha) + l_o(\alpha)$. This completes the proof. ∎

Proof of Lemma 2.3. Let $\mu = (1^{m_1}, 3^{m_3}, 5^{m_5}, \ldots, (2t-1)^{m_{2t-1}}) \in \mathcal{O}(n)$. For every multiplicity m_i, we write $m_i = 2h_i + s_i$ ($s_i = 0, 1$). Then we define $\beta = \psi(\mu) = (1^{m'_1}, 2^{m'_2}, 3^{m'_3}, \ldots, k^{m'_k})$, where $m'_{2i+1} = s_{2i+1}$ and $m'_{2i} = h_i$. Clearly, $m'_{4i} = h_{2i} = 0$ and $m'_{2i+1} \leq 1$, and so $\beta \in \mathcal{A}_2(n)$. For example, let $\mu = (1, 3, 7^2, 9, 15)$. Then we have $\beta = (1, 3, 9, 14, 15)$ whose 2-modular diagram is illustrated in Figure 5.1.

15	2	2	2	2	2	2	2	1
7^2	2	2	2	2	2	2	2	
9	2	2	2	2	1			
3	2	1						
1	1							

Figure 5.1: The diagram representation of $\beta = (1, 3, 7^2, 9, 15)$.

The inverse map ψ^{-1} can be easily described. Let $\beta = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots, t^{m_t}) \in \mathcal{B}(n)$. Then we have $m_{4i} = 0$ and $m_{2i-1} = 0$ or 1 for $i \geq 1$. Let

$$\mu = \psi^{-1}(\beta) = (1^{m'_1}, 2^{m'_2}, 3^{m'_3}, \ldots, k^{m'_k}),$$

where $m'_{2i-1} = 2m_{4i-2} + m_{2i-1}$ and $m'_{2i} = 0$ for $i = 1, 2, \ldots$. Obviously, $\psi^{-1}(\beta) \in \mathcal{O}(n)$. It follows that $n_o(\mu) = l_o(\beta)$ and $l(\mu) = 2r_2(\beta) + l_o(\beta)$. This completes the proof. ∎

It is easy to see that Theorem 2.4 follows from Theorem 3.6 by setting $N = 2$ and $\mathbb{A}_4 = \{1, 2, 3\}$, that is, $\mathcal{C}_1(\{1, 2, 3\}; n, 4) = \mathcal{A}_1(n)$ and $\mathcal{C}_2(\{1, 2, 3\}; n, 4) = \mathcal{A}_2(n)$.

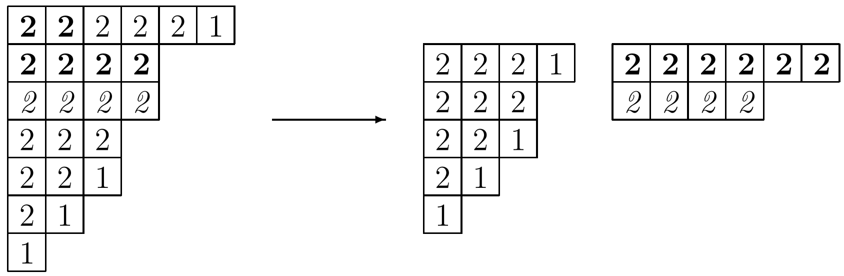
Figures 5.2 and 5.3 illustrate the procedure in Theorem 2.4.

To conclude, we combine the mappings ψ, ϕ and φ to construct the desired map Δ from the set of partitions with distinct parts to the set of partitions with odd parts parts: $\Delta = \psi^{-1} \circ \phi \circ \varphi$. The properties of Δ lead to a proof of Theorem 2.1.

For example, let $\lambda = (17, 16, 14, 10, 7, 4, 2, 1) \in \mathcal{D}(71)$. Then we have

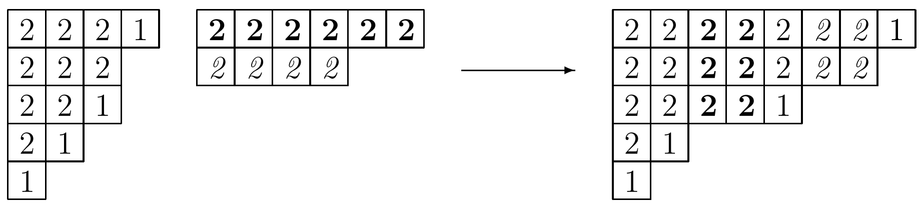
$$\begin{aligned} \alpha &= \varphi(\lambda) = (15, 12, 10, 9, 8, 6, 6, 4, 1), \\ \beta &= \phi(\alpha) = (19, 18, 13, 10, 6, 5), \\ \mu &= \psi^{-1}(\beta) = (3^2, 5^3, 9^2, 13, 19) \in \mathcal{O}(71). \end{aligned}$$

Moreover, $l_o(\lambda) = 3$, $l_a(\lambda) = 9$, $n_o(\mu) = 3$ and $l(\mu) = 9$.



α γ δ

Figure 5.2: Extraction of parts from α.



γ δ β

Figure 5.3: Insertion all parts of δ into γ.

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