INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY x (200x), #Axx

ON THE NUMBER OF ZERO-SUM SUBSEQUENCES OF RESTRICTED SIZE

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Received: , Revised: , Accepted: , Published:

Abstract

Let $n = 2^\lambda m \geq 526$ with $m \in \{2, 3, 5, 7, 11\}$, and let S be a sequence of elements in $C_n \oplus C_n$ with $|S| = n^2 + 2n - 2$. Let $\mathsf{N}_0^{|G|}(S)$ denote the number of the subsequences with length $n^2(=|G|)$ and with sum zero. Among other results, we prove that either $\mathsf{N}_0^{|G|}(S) = 1$ or $\mathsf{N}_0^{|G|}(S) \geq n^2 + 1$.

1. Introduction and Main Results

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of integers. For $a, b \in \mathbb{Z}$ with $a \leq b$, we define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Let G be an additively written finite abelian group. We denote by $|G|$ the *order* of G, and denote by $\exp(G)$ the *exponent* of G. Let $\mathscr{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathscr{F}(G)$ are called *sequences* over G. If a sequence $S \in \mathscr{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we call $|S| = l$ the *length* of S. For every $g \in G, k \in \mathbb{N}$, let $\mathsf{N}_g^k(S)$ denote the number of subsets $I \subseteq [1, l]$ such that $|I| = k$ and $\sum_{i \in I} g_i = g$. The famous Erdős-Ginzburg-Ziv Theorem asserts that if $|S| \geq 2|G| - 1$ then $\mathsf{N}_0^{|G|}(S) \geq 1$ [5].

When $G = C_n$ is the cyclic group of n elements, $\mathsf{N}_g^n(S)$ has been studied since 1967 by many authors including H.B. Mann, A. Bialostocki and M. Lotspeich, Z. Füredi and D.J. Kleitman, the first author, D.J. Grynkiewicz, and M. Kisin. Let p be a prime and let $S \in \mathscr{F}(C_p)$ with $|S| = 2p-1$. H.B. Mann [19] proved that if no element occurs more than p times in S then $\mathsf{N}_g^p(S) \geq 1$ for every $g \in C_p$. With the same assumption above, the first author [9] proved that $\mathsf{N}_g^p(S) \geq p$ for every $g \in C_p \setminus \{0\}$, and either $\mathsf{N}_0^p(S) = 1$ or $\mathsf{N}_0^p(S) \geq p + 1$. In 1999, the first author [8] showed that for every positive integer n, if $|S| = 2n - 1$ then for every $g \in C_n \setminus \{0\}$ we have $\mathsf{N}_g^n(S) = 0$ or $\mathsf{N}_g^n(S) \geq n$, and either $\mathsf{N}_0^n(S) = 1$ or $\mathsf{N}_0^n(S) \geq n + 1$. In 1992, Bialostocki and Lotspeich [2] formulated the following conjecture.

Conjecture 1.1 *Let $n \geq 2$ be a positive integer, and let $S \in \mathscr{F}(C_n)$. Then*

$$\mathsf{N}_0^n(S) \geq \binom{\lfloor |S|/2 \rfloor}{n} + \binom{\lceil |S|/2 \rceil}{n}.$$

Conjecture 1.1 has been confirmed if one of the following conditions holds:

(i) $n = p^a q^b$ with p, q are primes (M. Kisin, [18]);

(ii) $|S| \geq n^{6n}$ (Füredi and Kleitman, [6]);

(iii) $|S| \leq 6.5n$ (Grynkiewicz, [16]).

However, there is almost no result on $\mathsf{N}_g^{|G|}(S)$ for non-cyclic group G. In this paper we shall obtain some sharp results on $\mathsf{N}_g^{|G|}(S)$ for $G = C_n \oplus C_n$ and $|S| = n^2 + 2n - 2$.

Before we can state our main results (see Corollary 1.4 and 1.6 below) more precisely, let us introduce some notation and terminology first. We write sequence $S \in \mathscr{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

with $\mathsf{v}_g(S) \in \mathbb{N}_0$ for all $g \in G$.

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S. We say that S *contains* g if $\mathsf{v}_g(S) > 0$. The unit element $1 \in \mathscr{F}(G)$ is called the *empty sequence*. A sequence S_1 is called a *subsequence* of S if $S_1 | S$ in $\mathscr{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \leq \mathsf{v}_g(S)$ for all $g \in G$), and it is called a *proper subsequence* of S if it is a subsequence with $1 \neq S_1 \neq S$. Let $S_1, S_2 \in \mathscr{F}(G)$, we denote by $S_1 S_2$ the sequence

$$\prod_{g \in G} g^{\mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)} \in \mathscr{F}(G).$$

If a sequence $S \in \mathscr{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For $g_0 \in G$, we set $g_0 + S = (g_0 + g_1) \cdot \ldots \cdot (g_0 + g_l) \in \mathscr{F}(G)$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathscr{F}(G),$$

we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{the } \textit{length} \text{ of } S,$$

$\mathsf{h}(S) = \max\left\{\mathsf{v}_g(S) | g \in G\right\} \in [0, |S|]$ the *maximum of the multiplicities* of S,

$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S,

$\sum(S) = \left\{\sum_{i \in I} g_i | I \subseteq [1, l] \text{ with} 1 \le |I| \le l\right\}$ the *set of all subsums* of S.

The sequence S is called

- *zero-sumfree* if $0 \notin \sum(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a non-empty zero-sum sequence and every proper subsequence is zero-sumfree,
- a *short zero-sum sequence* if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

We denote by $\mathsf{D}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathscr{F}(G)$ of length $|S| \ge l$ has a nonempty zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant* of G.

Let $n \ge 2$ be a positive integer. We say that n has Property B if every minimal zero-sum sequence in $\mathscr{F}(C_n \oplus C_n)$ of length $2n-1$ contains some element with multiplicity $n-1$. It has been conjectured that

Conjecture 1.2 *Every positive integer $n \ge 2$ has Property B (for e.g., see [11], [12] and [15]).*

Conjecture 1.2 has been confirmed for $n = 2^{\lambda} m$ and $m \in \{2, 3, 5, 7, 11\}$ (See [11], [14]).

Write the elements in $C_n \oplus C_n$ in the form (a, b). Let $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = (0, 1)$. Then every $(a, b) \in C_n \oplus C_n$ can be expressed as $(a, b) = a\mathbf{e_1} + b\mathbf{e_2}$ uniquely. Let $\mathbf{0} = (0, 0)$.

Now we can state our main results precisely.

Theorem 1.3 *Let $G = C_n \oplus C_n$ with $n \ge 2$, and let $S \in \mathscr{F}(G)$ be a sequence of length $|S| = |G| + \mathsf{D}(G) - 1 = n^2 + 2n - 2$. If n has Property B then*

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \ge n$$

for every $g \in G \setminus \{\mathbf{0}\}$.

Corollary 1.4 *Let $n = 2^\lambda m$ with $m \in \{2,3,5,7,11\}$, and let $G = C_n \oplus C_n$. If $S \in \mathscr{F}(G)$ is a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$, then*

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \geq n$$

for every $g \in G \setminus \{\mathbf{0}\}$.

Theorem 1.5 *Let $G = C_n \oplus C_n$ with $n \geq 526$, and let $S \in \mathscr{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$. If n has Property B then*

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq n^2 + 1.$$

Corollary 1.6 *Let $n = 2^\lambda m \geq 526$ with $m \in \{2,3,5,7,11\}$, and let $G = C_n \oplus C_n$. If $S \in \mathscr{F}(G)$ is a sequence of length $|S| = |G| + \mathrm{D}(G) - 1 = n^2 + 2n - 2$, then*

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq n^2 + 1.$$

Now let us give some examples concerning the above results.

Example 1 $G = C_n \oplus C_n$, $S = \mathbf{0}^{n^2+2n-2}$, *then* $\mathsf{N}_g^{|G|}(S) = 0$, *for every* $g \in G \setminus \{\mathbf{0}\}$.

Example 2 $G = C_n \oplus C_n$, $S = \mathbf{0}^{n^2-1}\mathbf{e_1}^n\mathbf{e_2}^{n-1}$, *then* $\mathsf{N}_{\mathbf{e_1}}^{|G|}(S) = n$.

Example 3 $G = C_n \oplus C_n$, $n \geq 3$, $S = \mathbf{0}^{n^2}\mathbf{e_1}^{n-1}\mathbf{e_2}^{n-1}$, *then* $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$.

Example 4 $G = C_n \oplus C_n$, $n \geq 3$, $S = \mathbf{0}^{n^2+1}\mathbf{e_1}^{n-2}\mathbf{e_2}^{n-1}$, *then* $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = n^2 + 1$.

Example 5 $G = C_2 \oplus C_2$, $S = (\mathbf{e_1} + \mathbf{e_2})^2\mathbf{e_1}^2\mathbf{e_2}^2$, *then* $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 3$.

Remarks 1.7 Example 1 and Example 2 show that the bounds in Theorem 1.3 are sharp. Example 3 and Example 4 show that the inequalities in Theorem 1.5 cannot be improved. Example 5 shows that the conclusion of Theorem 1.5 is not true for $G = C_2 \oplus C_2$. Perhaps this is the only exceptional case (see Conjecture 5.3 in Section 5). We believe that the conclusion of Theorem 1.5 is true for all $n \geq 3$, and we have checked it for all $n \leq 10$. It would be interesting to prove Theorem 1.5 for all $n \in [11, 525]$.

2. Preliminaries

To prove Theorem 1.3 and Theorem 1.5 we need some preliminaries begin with the following well known result due to Olson [22].

Lemma 2.1 *$D(C_n \oplus C_n) = 2n - 1$.*

Lemma 2.2 ([15], Theorem 5.8.3) *Every sequence S in $C_n \oplus C_n$ with $|S| = 3n-2$ contains a short zero-sum subsequence.*

Lemma 2.3 ([15], Theorem 5.8.7) *Let $G = C_n \oplus C_n$ with $n \geq 2$, and let $S \in \mathscr{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n-2$. If n has Property B then there is an automorphism ϕ over G such that $\phi(S) = \mathbf{e_2}^{n-1} \prod_{i=1}^{n-1}(\mathbf{e_1} + a_i\mathbf{e_2})$, or $\phi(S) = \mathbf{e_2}^{n-2} \prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})$ with $\sum_{i=1}^{n} a_i \equiv 1 \pmod n$ and $\mathsf{h}(S) = n-2$.*

Lemma 2.4 *Let $n \geq 3$ with n having Property B, and let $G = C_n \oplus C_n$. Let $S_1, S_2 \in \mathscr{F}(G)$ with $|S_1| = |S_2| = 2n-2$. If $\mathsf{h}(S_1) \leq 2n-3$ and $\mathsf{h}(S_2) \leq 2n-3$, then there exist $T_1 | S_1$ and $T_2 | S_2$ such that $\sigma(T_1) = \sigma(T_2)$ and $|T_1| = |T_2| \in [1, 2n-2]$.*

Proof. It is easy to check the lemma for $n = 3$. So, we assume that $n \geq 4$. Let

$$S_1 = \prod_{i=1}^{2n-2}(a_i\mathbf{e_1} + b_i\mathbf{e_2})$$

and

$$S_2 = \prod_{i=1}^{2n-2}(c_i\mathbf{e_1} + d_i\mathbf{e_2}).$$

Let P_{2n-2} denote the symmetric group on $[1, 2n-2]$. Clearly, it suffices to prove that $S_1 - \delta(S_2)$ is not zero-sumfree for some $\delta \in P_{2n-2}$, where $\delta(S_2) = \prod_{i=1}^{2n-2}(c_{\delta(i)}\mathbf{e_1} + d_{\delta(i)}\mathbf{e_2})$.

Assume to the contrary that, $S_1 - \delta(S_2)$ is zero-sumfree for every $\delta \in P_{2n-2}$. By Lemma 2.3, $\mathsf{h}(S_1 - \delta(S_2)) = n-1$ or $n-2$ holds for every $\delta \in P_{2n-2}$.

Case 1: $\mathsf{h}(S_1 - \delta(S_2)) = n-2$ holds for every $\delta \in P_{2n-2}$.

Especially, $\mathsf{h}(S_1 - S_2) = n-2$. Again by Lemma 2.3, there exists an automorphism ϕ over G such that

$$\phi(S_1 - S_2) = \mathbf{e_2}^{n-2} \prod_{i=1}^{n}(\mathbf{e_1} + z_i\mathbf{e_2}).$$

Without loss of generality, we may assume that $\phi = \mathrm{id}$. Furthermore, by rearranging the subscripts, if necessary, we assume that

$$(a_1 - c_1)\mathbf{e_1} + (b_1 - d_1)\mathbf{e_2} = \cdots = (a_{n-2} - c_{n-2})\mathbf{e_1} + (b_{n-2} - d_{n-2})\mathbf{e_2} = \mathbf{e_2}$$

and

$$(a_j - c_j)\mathbf{e_1} + (b_j - d_j)\mathbf{e_2} = \mathbf{e_1} + z_{j-n+2}\mathbf{e_2}$$

for every $j \in [n-1, 2n-2]$.

Since $\mathsf{h}(S_1 - S_2) = n - 2$, we may assume that

$$z_1 \neq z_2.$$

Claim 1. $a_i - c_j \in \{1, 2\}$ holds for any $i, j \in [n+1, 2n-2]$ with $i \neq j$.

Let $i, j \in [n+1, 2n-2]$ with $i \neq j$, and let τ be the transposition $(i, j) \in P_{2n-2}$. Then

$$S_1 - \tau(S_2) = \mathbf{e_2}^{n-2}\Big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\Big)\Big((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\Big)\prod_{k \neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2}).$$

If $a_i - c_j = 0$ then $(a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2} = (b_i - d_j)\mathbf{e_2} \neq \mathbf{e_2}$ follows from $\mathsf{h}(S_1 - \tau(S_2)) = n - 2$. Therefore, $\mathbf{0} \in \sum\Big(\mathbf{e_2}^{n-2}\big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\big)\Big) \subseteq \sum(S_1 - \tau(S_2))$, a contradiction.

Now we assume that $a_i - c_j \in [3, n-1]$. Let $I \subseteq [1, n] \setminus \{1, 2, i-n-2, j-n-2\}$ be a subset with $|I| = n - (a_i - c_j) - 1 \in [0, n-4]$. Then $a_i - c_j + 1 + \sum_{k \in I} 1 = 0$. Therefore

$$\begin{aligned}&\left\{\left(b_i - d_j + z_1 + \sum_{k \in I} z_k\right)\mathbf{e_2}, \left(b_i - d_j + z_2 + \sum_{k \in I} z_k\right)\mathbf{e_2}\right\}\\ &\subseteq \sum\left(\Big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\Big)\prod_{k \neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2})\right).\end{aligned}$$

Since $z_1 \neq z_2$, we have that $b_i - d_j + z_1 + \sum_{k \in I} z_k \neq b_i - d_j + z_2 + \sum_{k \in I} z_k$. Therefore

$$\begin{aligned}\mathbf{0} &\in \sum\left(\mathbf{e_2}^{n-2}\left(b_i - d_j + z_1 + \sum_{k \in I} z_k\right)\mathbf{e_2}\right) \bigcup \sum\left(\mathbf{e_2}^{n-2}\left(b_i - d_j + z_2 + \sum_{k \in I} z_k\right)\mathbf{e_2}\right)\\ &\subseteq \sum\left(\mathbf{e_2}^{n-2}\Big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\Big)\prod_{k \neq i-n+2, j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2})\right)\\ &\subseteq \sum(S_1 - \tau(S_2)),\end{aligned}$$

a contradiction. This proves Claim 1.

Note that $a_i - c_j + a_j - c_i = (a_i - c_i) + (a_j - c_j) = 2$. This forces that $a_i - c_j = 1$ for any pair of $i, j \in [n+1, 2n-2]$ with $i \neq j$. Therefore

$$a_{n+1} = a_{n+2} = \cdots = a_{2n-2} = a \text{ (say)},$$
$$c_{n+1} = c_{n+2} = \cdots = c_{2n-2} = a - 1.$$

Since $\mathsf{h}(S_1 - S_2) = n - 2$, we have that $z_{k-n+2} \neq z_1$ holds for some $k \in [n+1, 2n-2]$. Let $j \in [n+1, 2n-2] \setminus \{k\}$, and let $i = n$. Then repeating the proof above we obtain that

$$a_n = a_{n+1} = \cdots = a_{2n-2} = a,$$
$$c_n = c_{n+1} = \cdots = c_{2n-2} = a - 1.$$

Similarly, we obtain that

$$a_{n-1} = a_{n+1} = \cdots = a_{2n-2} = a,$$
$$c_{n-1} = c_{n+1} = \cdots = c_{2n-2} = a - 1.$$

Hence

$$a_{n-1} = a_n = \cdots = a_{2n-2} = c_{n-1} + 1 = c_n + 1 = \cdots = c_{2n-2} + 1 = a. \tag{1}$$

Claim 2. $a_i - c_j \in \{0, 1\}$ holds for every $i \in [1, n-2]$ and every $j \in [n+1, 2n-2]$.

Let $i \in [1, n-2]$, $j \in [n+1, 2n-2]$, and let θ be the transposition $(i, j) \in P_{2n-2}$. Then

$$S_1 - \theta(S_2) = \mathbf{e_2}^{n-3}\Big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\Big)\Big((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\Big)\prod_{k \neq j-n+2}(\mathbf{e_1} + z_k\mathbf{e_2}).$$

Assume to the contrary that $a_i - c_j \in [2, n-1]$. Let $I \subseteq [1, n] \setminus \{j - n + 2\}$ be any subset with $|I| = n - (a_i - c_j)$. Let $J = [1, n] \setminus \{\{j - n + 2\} \cup I\}$. Then $a_i - c_j + \sum_{k \in I} 1 = 0$ and $a_j - c_i + \sum_{k \in J} 1 = 0$. Therefore

$$\sigma\left(\Big((a_i - c_j)\mathbf{e_1} + (b_i - d_j)\mathbf{e_2}\Big)\prod_{k \in I}(\mathbf{e_1} + z_k\mathbf{e_2})\right) = \left(b_i - d_j + \sum_{k \in I} z_k\right)\mathbf{e_2},$$

and

$$\sigma\left(\Big((a_j - c_i)\mathbf{e_1} + (b_j - d_i)\mathbf{e_2}\Big)\prod_{k \in J}(\mathbf{e_1} + z_k\mathbf{e_2})\right) = \left(b_j - d_i + \sum_{k \in J} z_k\right)\mathbf{e_2}.$$

Since $\mathbf{0} \notin \sum\left(\mathbf{e_2}^{n-3}\left(\left(b_i - d_j + \sum_{k \in I} z_k\right)\mathbf{e_2}\right)\right)$, we infer that

$$b_i - d_j + \sum_{k \in I} z_k \in \{1, 2\}.$$

Similarly

$$b_j - d_i + \sum_{k \in J} z_k \in \{1, 2\}.$$

Note that $a_i - c_j + a_j - c_i + (n-1) = 0$. Similarly to above we have

$$b_i - d_j + b_j - d_i + \sum_{k\in I} z_k + \sum_{k\in J} z_k \in \{1,2\}.$$

These force that $b_i - d_j + \sum_{k\in I} z_k = b_j - d_i + \sum_{k\in J} z_k = 1$ holds for every $I \subseteq [1,n] \setminus \{j-n+2\}$ with $|I| = n - (a_i - c_j)$, which implies $z_1 = z_2$, a contradiction. This proves Claim 2.

Since $a_i - c_j + a_j - c_i = 1$, we have $a_j - c_i \in \{0,1\}$. Therefore

$$a_i - c_j = 0,\ a_j - c_i = 1 \text{ or } \quad a_i - c_j = 1,\ a_j - c_i = 0 \tag{2}$$

holds for every pair of i, j with $i \in [1, n-2]$ and $j \in [n+1, 2n-2]$.

If $a_j - c_i = 0$ then $a_j = a_i$ follows from $a_i - c_i = 0$. By (1), $a_i = a_{n-1} = a_n = \cdots = a_{2n-2}$. Let $t \in [n-1, 2n-2]$. Let γ be the transposition $(i,t) \in P_{2n-2}$. Then

$$S_1 - \gamma(S_2) = \mathbf{e_2}^{n-3}\left((a_i - c_t)\mathbf{e_1} + (b_i - d_t)\mathbf{e_2}\right)\left((a_t - c_i)\mathbf{e_1} + (b_t - d_i)\mathbf{e_2}\right)\prod_{k\neq t-n+2}(\mathbf{e_1} + z_k\mathbf{e_2}).$$

By (1) we have $a_i - c_t = 1$, $a_t - c_i = 0$. Therefore

$$\sigma\left(\left((a_i - c_t)\mathbf{e_1} + (b_i - d_t)\mathbf{e_2}\right)\prod_{k\neq t-n+2}(\mathbf{e_1} + z_k\mathbf{e_2})\right) = \left(b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2}\right)\mathbf{e_2},$$

and

$$(a_t - c_i)\mathbf{e_1} + (b_t - d_i)\mathbf{e_2} = (b_t - d_i)\mathbf{e_2}.$$

Hence

$$\mathbf{0} \notin \sum\left(\mathbf{e_2}^{n-3}((b_t - d_i)\mathbf{e_2})\left(\left(b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2}\right)\mathbf{e_2}\right)\right) \subseteq \sum(S_1 - \gamma(S_2)).$$

This forces that

$$b_t - d_i = b_i - d_t + \left(\sum_{k=1}^{n} z_k\right) - z_{t-n+2} = 1.$$

Since $b_i - d_i = 1$ we have $b_i = b_t$. Therefore, $a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_t\mathbf{e_1} + b_t\mathbf{e_2}$ for every $t \in [n-1, 2n-2]$.

Now we have proved that if $a_j - c_i = 0$ for some $i \in [1, n-2]$ and $j \in [n+1, 2n-2]$, then

$$a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_{n-1}\mathbf{e_1} + b_{n-1}\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2}. \tag{3}$$

Similarly, if $a_i - c_j = 0$ for some $i \in [1, n-2]$ and some $j \in [n+1, 2n-2]$, then

$$c_i\mathbf{e_1} + d_i\mathbf{e_2} = c_{n-1}\mathbf{e_1} + d_{n-1}\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}. \tag{4}$$

From (2), (3) and (4) we infer that there are three possibilities:

(i) $a_1 = a_2 = \cdots = a_{2n-2} = a$, which implies

$$a_1\mathbf{e_1} + b_1\mathbf{e_2} = a_2\mathbf{e_1} + b_2\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2}.$$

(ii) $c_1 = c_2 = \cdots = c_{2n-2} = a - 1$, which implies

$$c_1\mathbf{e_1} + d_1\mathbf{e_2} = c_2\mathbf{e_1} + d_2\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}.$$

(iii) $a_i = a_{n-1} = \cdots = a_{2n-2} = a$ and $c_j = c_{n-1} = \cdots = c_{2n-2} = a - 1$ for some $i, j \in [1, n-2]$ with $i \neq j$, which implies

$$a_i\mathbf{e_1} + b_i\mathbf{e_2} = a_{n-1}\mathbf{e_1} + b_{n-1}\mathbf{e_2} = \cdots = a_{2n-2}\mathbf{e_1} + b_{2n-2}\mathbf{e_2},$$

and

$$c_j\mathbf{e_1} + d_j\mathbf{e_2} = c_{n-1}\mathbf{e_1} + d_{n-1}\mathbf{e_2} = \cdots = c_{2n-2}\mathbf{e_1} + d_{2n-2}\mathbf{e_2}.$$

But we always get a contradiction. This completes the proof of Case 1.

Case 2: $\mathsf{h}(S_1 - \delta(S_2)) = n - 1$ holds for some $\delta \in P_{2n-2}$. Since the proof is similar to and much easier than Case 1, we omit it here. □

Lemma 2.5 *Let $n \geq 3$ with n having Property B, and let $G = C_n \oplus C_n$. Let $S \in \mathscr{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n - 2$. Then for any $g \in G \setminus \{\mathbf{0}\}$, either $\mathrm{v}_g(S) = n - 1$ or there exists a subsequence T of S such that $|T| \geq 2$ and $g = \sigma(T)$.*

Proof. By Lemma 2.1, for any $g \in G \setminus \{\mathbf{0}\}$, $(-g)S$ contains a nonempty zero-sum subsequence S_1. Since S is zero-sumfree, we have $(-g)|S_1$. Let $S_2 = S_1(-g)^{-1}$. Then $g = \sigma(S_2)$. If g is not a term of S then $|S_2| \geq 2$. Let $T = S_2$ and we are done. So we may assume that g is a term of S. Clearly, it suffices to prove that either $\mathrm{v}_g(S) = n - 1$, or there is a subsequence W of S such that g is not a term of W and $g \in \sum(W)$.

By Lemma 2.3 there is an automorphism ϕ over G such that

$$\phi(S) = \mathbf{e_2}^r \prod_{i=1}^{2n-2-r} (\mathbf{e_1} + a_i\mathbf{e_2}),$$

where $r = \mathsf{h}(S) = n - 1$ or $n - 2$. Without loss of generality let $\phi = \mathrm{id}$.

Case 1: $S = \mathbf{e_2}^{n-1} \prod_{i=1}^{n-1} (\mathbf{e_1} + a_i\mathbf{e_2})$.

Subcase 1.1: $a_1 = a_2 = \cdots = a_{n-1}$. Since g is a term of S, $g = \mathbf{e_2}$ or $\mathbf{e_1} + a_1\mathbf{e_2}$. Therefore, $\mathrm{v}_g(S) = n - 1$.

Subcase 1.2: $a_1 = a_2 = \cdots = a_{n-1}$ does not hold. Without loss of generality let $a_1 \neq a_2$. If $g = \mathbf{e_2}$ then $\mathsf{v}_g(S) = n-1$. Now assume $g = \mathbf{e_1} + a_i\mathbf{e_2}$ for some $i \in [1, n-1]$. Note that either $a_i \neq a_1$ and we have $g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-1}(\mathbf{e_1} + a_1\mathbf{e_2})\right)$, or $a_i \neq a_2$ and we have $g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-1}(\mathbf{e_1} + a_2\mathbf{e_2})\right)$.

Case 2: $S = \mathbf{e_2}^{n-2}\prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})$ and $\mathsf{h}(S) = n-2$. By rearranging the subscripts, if necessary, we can assume that $a_1 \neq a_2$. By Lemma 2.3, $\mathbf{e_2} = \sigma\left(\prod_{i=1}^{n}(\mathbf{e_1} + a_i\mathbf{e_2})\right)$. So it remains to check the case that $g = \mathbf{e_1} + a_i\mathbf{e_2}$ for some $i \in [1, n]$.

Subcase 2.1: There are three distinct elements among of $a_1, \ldots, a_n$. Then there are two indices $j, k \in [1, n] \setminus \{i\}$ such that a_i, a_j, a_k are pairwise distinct. Since $[a_j, a_j + n - 2] \cup [a_k, a_k + n - 2] = [0, n-1] \setminus \{a_j + n - 1\} \cup [0, n-1] \setminus \{a_k + n - 1\} = [0, n-1]$, we infer that $\{\mathbf{e_1}, \mathbf{e_1} + \mathbf{e_2}, \ldots, \mathbf{e_1} + (n-1)\mathbf{e_2}\} \subseteq \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right) \cup \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_k\mathbf{e_2})\right)$. Hence

$$g = \mathbf{e_1} + a_i\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right) \cup \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_k\mathbf{e_2})\right).$$

Subcase 2.2: There are exactly two distinct elements among of $a_1, \ldots, a_n$. Let $j \in [1, n]$ with $a_j \neq a_i$. If $a_i \neq a_j + n - 1$ then $g = \mathbf{e_1} + a_1\mathbf{e_2} \in \sum\left(\mathbf{e_2}^{n-2}(\mathbf{e_1} + a_j\mathbf{e_2})\right)$. Otherwise $a_i = a_j + n - 1$. Let r be the number of $k \in \{1, \ldots, n\}$ such that $a_k = a_i$. By Lemma 2.3, $a_1 + a_2 + \cdots + a_n \equiv 1 \pmod{n}$, that is, $ra_i + (n-r)(a_i + 1) \equiv 1 \pmod{n}$. Hence, $r = n-1$ contradicting $\mathsf{h}(S) = n-2$. This completes the proof. □

Lemma 2.6 *Let $n \geq 3$ with n having Property B, and let $G = C_n \oplus C_n$. Let $S \in \mathscr{F}(G)$ be a zero-sumfree sequence of length $|S| = 2n-3$, and let $W \in \mathscr{F}(G)$ be a nonempty zero-sum sequence. If W contains no $\mathbf{0}$ then there exist $W_1|W$ and $S_1|S$ such that $\sigma(W_1) = \sigma(S_1)$ and $1 \leq |W_1| \leq |S_1|$.*

Proof. It is easy to check the lemma for $n \in \{3, 4\}$.

Let $n \geq 5$. We may assume that W is a minimal zero-sum sequence. Let

$$W = g_1 \cdot \ldots \cdot g_w, \text{ where } w = |W| \geq 2.$$

If $(-g_i)S$ contains a nonempty zero-sum subsequence S_1' (say) for some $i \in [1, w]$, then $-g_i|S_1'$ follows from S is zero-sumfree. Let $S_1 = S_1'(-g_i)^{-1}$ and $W_1 = g_i \in \mathscr{F}(G)$. Then $S_1|S$, $g_i = \sigma(S_1)$ and we are done.

Now we may assume that, for any $i \in [1, w]$, $(-g_i)S$ is zero-sum free. By Lemma 2.3, there exists an automorphism ϕ over G such that

$$\phi((-g_1)S) = \mathbf{e_2}^r\prod_{i=1}^{2n-2-r}(\mathbf{e_1} + z_i\mathbf{e_2}),$$

where $\mathsf{h}(\phi((-g_1)S)) = r = n-1$ or $n-2$. Without loss of generality let $\phi = \mathrm{id}$. Then

$$(-g_1)S = \mathbf{e_2}^r \prod_{i=1}^{2n-2-r}(\mathbf{e_1} + z_i\mathbf{e_2}),$$

where $\mathsf{h}((-g_1)S) = r = n-1$ or $n-2$. By rearranging the subscripts, if necessary, we may assume that

$$-g_1 = \mathbf{e_2}, \text{ or } -g_1 = \mathbf{e_1} + z_1\mathbf{e_2}.$$

Case 1: $w = 2$. Then $g_1 + g_2 = \mathbf{0}$.

Subcase 1.1: $-g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. Then $g_2 = -g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. If $r = n-1$, it is easy to see that $g_2 \in \sum\left((\mathbf{e_1} + z_2\mathbf{e_2})\mathbf{e_2}^{n-1}\right) \subseteq \sum(S)$ and we are done. If $r = n-2$ then $\mathsf{h}(z_2z_3 \cdot \ldots \cdot z_n) \le n-2$. By rearranging the subscripts, if necessary, we assume that $z_2 \ne z_3$. Furthermore, we may assume that $z_1 \ne z_2 + (n-1)$. Thus $g_2 \in \sum\left((\mathbf{e_1} + z_2\mathbf{e_2})\mathbf{e_2}^{n-2}\right) \subseteq \sum(S)$ and we are done.

Subcase 1.2: $-g_1 = \mathbf{e_2}$. Then $g_2 = -g_1 = \mathbf{e_2}$. Letting $S_1 = \mathbf{e_2} \in \mathscr{F}(G)$ and $W_1 = g_2 \in \mathscr{F}(G)$ verify the lemma.

Case 2: $w \ge 3$. Let $i, j \in [1, w]$ be an arbitrary pair with $i \ne j$. By Lemma 2.1, $(-g_i)(-g_j)S$ contains a nonempty zero-sum subsequence S_2' (say). Since both $(-g_i)S$ and $(-g_j)S$ are zero-sumfree, we have $(-g_i)(-g_j)|S_2'$. Let $S_2 = S_2'(-g_i)^{-1}(-g_j)^{-1}$. Then $S_2|S$ and $|S_2| \ge 1$. If $|S_2| \ge 2$, setting $S_1 = S_2$ and $W_1 = g_ig_j$ verify the lemma. So, we may assume that $|S_2| = 1$. Therefore, for any $i, j \in [1, w]$ with $i \ne j$,

$$g_i + g_j$$

is a term of S.

Subcase 2.1: $-g_1 = \mathbf{e_1} + z_1\mathbf{e_2}$. Then $g_1 = (n-1)\mathbf{e_1} + (n-z_1)\mathbf{e_2}$. For any $2 \le i \le w$, since $g_1 + g_i$ is a term of S, we infer that $g_i = \mathbf{e_1} + z\mathbf{e_2}$ or $2\mathbf{e_1} + z\mathbf{e_2}$ for some $z \in C_n$. Therefore, for any $i, j \in [2, w]$ with $i \ne j$ we have $g_i + g_j = a\mathbf{e_1} + b\mathbf{e_2}$ for some $a \in \{2, 3, 4\}$, a contradiction of $g_i + g_j$ is a term of S.

Subcase 2.2: $-g_1 = \mathbf{e_2}$. Then $g_1 = (n-1)\mathbf{e_2}$. For any $2 \le i \le w$, since $g_1 + g_i$ is a term of S, we infer that $g_i = 2\mathbf{e_2}$ or $\mathbf{e_1} + z\mathbf{e_2}$ for some $z \in C_n$. If $g_i = 2\mathbf{e_2}$, letting $W_1 = g_i$ and $S_1 = \mathbf{e_2}^2$ verify the lemma. So we may assume that $g_i = \mathbf{e_1} + z\mathbf{e_2}$ for every $2 \le i \le w$. Therefore, for any $i, j \in [2, w]$ with $i \ne j$ we have $g_i + g_j = 2\mathbf{e_1} + z'\mathbf{e_2}$, it is not a term of S, a contradiction. This completes the proof. □

Lemma 2.7 ([7], Theorem 1) *Let G be a finite abelian group, and let $S \in \mathscr{F}(G)$. If $|S| = |G| + \mathsf{D}(G) - 1$ then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge 1$.*

We also need the following technical results.

Lemma 2.8 *Let $n \ge 3, k, p_1, \ldots, p_k$ be positive integers. If $p_1 + p_2 + \cdots + p_k \ge 3n - 2$ and $2 \le p_i \le 2n - 3$ for every $i \in [1, k]$, then $p_1 p_2 \cdots p_k \ge n^2 + 1$.*

Proof. Since $2 \le p_i \le 2n-3$ for every $i \in [1,k]$, we have

$$p_1 p_2 \cdots p_k \ge p_1(p_2 + \cdots + p_k) \ge p_1(3n-2-p_1) \ge (2n-3)(n+1) \ge n^2+1.$$

□

Lemma 2.9 *Let $A_1,\dots,A_l$ be subsets of $[1,k]$ with $|A_1| = \cdots = |A_l| = 2$. If $l \le k$ then there exist a subset $A \subseteq [1,k]$ such that, $|A| \le \frac{k}{2} + \frac{l}{4}$ and $A \cap A_i \ne \emptyset$ holds for every $i \in [1,l]$.*

Proof. By rearranging the subscripts, if necessary, we may assume that $A_1 \cap A_2 \ne \emptyset, A_3 \cap A_4 \ne \emptyset, \dots, A_{2t-1} \cap A_{2t} \ne \emptyset$, and $A_{2t+1},\dots,A_l$ are pairwise disjoint. Put $r = l-2t$. Clearly, $0 \le t \le \frac{l}{2}$ and $r \le \frac{k}{2}$. Now take one element x_i from $A_{2i-1} \cap A_{2i}$ for every $i \in [1,t]$ (Note that $x_1,\dots,x_t$ are not necessarily distinct), and take one element x_{2t+j} from A_{2t+j} for every $j \in [1,r]$. Let

$$A = \{x_1,\dots,x_t,x_{2t+1},\dots,x_l\}.$$

Then, $A \cap A_i \ne \emptyset$ for every $i \in [1,l]$.

It remains to show $|A| \le t + r \le \frac{k}{2} + \frac{l}{4}$. Note that

$$2t + r = l \text{ and } r \le \frac{k}{2}.$$

If $r \le k - \frac{l}{2}$ then $|A| \le t + r = r + \frac{l-r}{2} = \frac{l+r}{2} \le \frac{k}{2} + \frac{l}{4}$. Now assume that $r > k - \frac{l}{2}$. Then, $t = \frac{l-r}{2} < \frac{l-k+\frac{l}{2}}{2} \le \frac{l}{4}$. Therefore, $|A| \le r + t \le \frac{k}{2} + \frac{l}{4}$. This completes the proof. □

3. Proof of Theorem 1.3

Proof. Let $n \ge 3$. Note that $\mathsf{N}_g^{|G|}(S) = \mathsf{N}_g^{|G|}(-x+S)$ holds for every $g \in G$, we may assume that $\mathsf{v}_{\mathbf{0}}(S) = \mathsf{h}(S)$. Let $g \in G \setminus \{\mathbf{0}\}$. Suppose $\mathsf{N}_g^{|G|}(S) \ge 1$, we need to show that $\mathsf{N}_g^{|G|}(S) \ge n$.

By rearranging the subscripts we may assume that

$$S = S_1 S_2,$$

where

$$\begin{aligned} S_1 &= a_1 a_2 \cdot \ldots \cdot a_{n^2-r} \mathbf{0}^r, \\ S_2 &= b_1 b_2 \cdot \ldots \cdot b_{2n-2-\mathsf{h}(S)+r} \mathbf{0}^{\mathsf{h}(S)-r}, \\ g &= \sigma(S_1) = a_1 + a_2 + \cdots + a_{n^2-r}. \end{aligned}$$

We first assume that $\mathsf{h}(S) \le 2n-3$. By Lemma 2.4 there exist $T_1 | a_1 a_2 \cdot \ldots \cdot a_{2n-2}$ and $T_1' | S_2$ such that $\sigma(T_1) = \sigma(T_1')$ and $|T_1| = |T_1'| \ge 1$. By rearranging of the subscripts of S_1 we may assume that

$a_1|T_1$. Again by Lemma 2.4 there exist $T_2|a_2a_3\cdot\ldots\cdot a_{2n-1}$ and $T_2'|S_2$ such that $\sigma(T_2)=\sigma(T_2')$ and $|T_2|=|T_2'|\geq 1$. Clearly, T_1 and T_2 are different. Similarly, we can obtain subsequences $T_3,\ldots,T_n$ of S_1 and subsequences $T_3',\ldots,T_n'$ of S_2 satisfying $|T_i|=|T_i'|$, $\sigma(T_i)=\sigma(T_i')$ for any $i\in[1,n]$, and $T_1,T_2,\ldots,T_n$ are pairwise different. Therefore, $S_1T_1^{-1}T_1', S_1T_2^{-1}T_2',\ldots,S_1T_n^{-1}T_n'$ are pairwise different subsequences of S with sum g and length n^2. So we have $\mathsf{N}_g^{|G|}(S)\geq n$.

Now suppose that $\mathsf{h}(S)\geq 2n-2$. We distinguish four cases.

Case 1: $1\leq r\leq \mathsf{h}(S)-1$. Then $\mathsf{N}_g^{|G|}(S)\geq\binom{\mathsf{h}(S)}{r}\geq\binom{\mathsf{h}(S)}{1}=\mathsf{h}(S)>n$.

Case 2: $r=0$. Then $\mathsf{h}(S)=2n-2$. Since $|S_1|=n^2\geq 3n-2$, by Lemma 2.2, there is a short zero-sum subsequence T of S_1. So $S_1T^{-1}\mathbf{0}^{|T|}$ is a sequence with sum g and length n^2. Replace S_1 by $S_1T^{-1}\mathbf{0}^{|T|}$ and it reduces to Case 1.

Case 3: $r=\mathsf{h}(S)$ and S_2 is not zero-sumfree. Assume that $T|S_2$ and $\sigma(T)=\mathbf{0}$. Replace S_1 by $S_1\mathbf{0}^{-|T|}T$ and it reduces to Case 1 or Case 2.

Case 4: $r=\mathsf{h}(S)$ and S_2 is zero-sumfree. Since $g\neq\mathbf{0}$, there is at least one term of S_1 is not zero. Let $g'|S_1$ and $g'\neq\mathbf{0}$. By Lemma 2.5 we have that either $\mathsf{v}_{g'}(S_2)=n-1$ or there exists a subsequence T of S_2 such that $|T|\geq 2$ and $g'=\sigma(T)$. If $\mathsf{v}_{g'}(S_2)=n-1$ then $\mathsf{N}_g^{|G|}(S)\geq\binom{\mathsf{v}_{g'}(S_1)+\mathsf{v}_{g'}(S_2)}{1}\geq\binom{n}{1}=n$. Now assume that $g'=\sigma(T)$ for some $T|S_2$ with $|T|\geq 2$. Replace S_1 by $S_1g'^{-1}\mathbf{0}^{-|T|+1}T$ and it reduces to Case 1 or Case 2.

It is easy to check the case $n=2$ directly and we omit it here. Now the proof is completed. □

4. Proof of Theorem 1.5

Proof. Let $n\geq 526$. Without loss of generality let $\mathsf{h}(S)=\mathsf{v}_{\mathbf{0}}(S)$. From Lemma 2.7 and Lemma 2.1 we know that $\mathsf{N}_{\mathbf{0}}^{|G|}(S)\geq 1$. Assume that $\mathsf{N}_{\mathbf{0}}^{|G|}(S)\geq 2$. We have to show $\mathsf{N}_{\mathbf{0}}^{|G|}(S)\geq n^2+1$.

By rearranging the subscripts we may assume that

$$S=S_1S_2,$$

where

$$\begin{aligned}S_1&=a_1a_2\cdot\ldots\cdot a_{n^2-r}\mathbf{0}^r,\\ S_2&=b_1b_2\cdot\ldots\cdot b_{2n-2-\mathsf{h}(S)+r}\mathbf{0}^{\mathsf{h}(S)-r},\\ \mathbf{0}&=\sigma(S_1)=a_1+a_2+\cdots+a_{n^2-r}.\end{aligned}$$

We distinguish cases according to the value taken by $\mathsf{h}(S)$.

Case 1. $\mathsf{h}(S)\geq n^2+1$. Since $1\leq r\leq n^2$, $\mathsf{N}_{\mathbf{0}}^{|G|}(S)\geq\binom{\mathsf{h}(S)}{r}\geq\binom{n^2+1}{1}\geq n^2+1$.

Case 2. $\mathsf{h}(S) = n^2$. We have $n^2 - 2n + 2 \le r \le n^2 - 2$ or $r = n^2$. If $n^2 - 2n + 2 \le r \le n^2 - 2$ then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{n^2}{r} \ge \binom{n^2}{2} \ge n^2 + 1$. So we may assume that $r = n^2$. If S_2 is zero-sumfree then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$, a contradiction. If S_2 has a zero-sum subsequence T of length at least 2 then $T\mathbf{0}^{n^2-|T|}$ is a zero-sum sequence of length n^2. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{n^2}{n^2-|T|} \ge \binom{n^2}{2} \ge n^2 + 1$.

Case 3. $2n - 2 \le \mathsf{h}(S) \le n^2 - 1$. We distinguish four subcases according to the value taken by r.

Subcase 3.1: $2 \le r \le \mathsf{h}(S) - 2$. Then $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge \binom{\mathsf{h}(S)}{r} \ge \binom{2n-2}{2} \ge n^2 + 1$.

Subcase 3.2: $0 \le r \le 1$. Then $\mathsf{h}(S) - r \ge n + 2$. Since $n^2 - r \ge n^2 - 1 \ge 3n - 2$, by Lemma 2.2, there is a zero-sum subsequence T of $a_1 a_2 \cdot \ldots \cdot a_{n^2-r}$ with $2 \le |T| \le n$. Now replace S_1 by $S_1 T^{-1}\mathbf{0}^{|T|}$ and it reduces to Subcase 3.1.

Subcase 3.3: $r = \mathsf{h}(S) - 1$. Let $S_1' = S_1\mathbf{0}^{-\mathsf{h}(S)+1}$ and $S_2' = S_2\mathbf{0}^{-1}$. If S_2' contains a nonempty zero-sum subsequence T, then replace S_1 by $S_1 T\mathbf{0}^{-|T|}$ and it reduces to Subcase 3.1 or Subcase 3.2. So we assume that S_2' is zero-sumfree.

If there exist $T|S_1$ and $U|S_2'$ such that $|T| < |U|$ and $\sigma(T) = \sigma(U)$ then replace S_1 by $S_1 U T^{-1}\mathbf{0}^{|T|-|U|}$. Note that $|U| \le 2n - 3$ and it reduces to Subcase 3.1 or Subcase 3.2. So we may assume that $T|S_1'$, $U|S_2'$ and $\sigma(T) = \sigma(U)$ imply

$$|T| \ge |U|. \tag{5}$$

If $\mathsf{h}(S) \ge \frac{n^2+1}{2}$, then by Lemma 2.6 and (5) there exist $T|S_1'$ and $U|S_2'$ such that $|T| = |U|$ and $\sigma(T) = \sigma(U)$. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge 2\binom{\mathsf{h}(S)}{1} \ge n^2 + 1$.

Now we may assume that $\frac{n^2+1}{2} \ge \mathsf{h}(S) \ge 2n - 2$. Since $|S_1'| = n^2 - \mathsf{h}(S) + 1 \ge 2n - 1$, by Lemma 2.6 and (5), there exist $T_1|S_1'$ and $U_1|S_2'$ such that $\sigma(T_1) = \sigma(U_1)$ and $|T_1| = |U_1|$. Without loss of generality let $a_1|T_1$. Since $\left|S_1' a_1^{-1}\right| \ge n^2 - \mathsf{h}(S) + 1 - 1 \ge 2n - 1$, by Lemma 2.1, there is a zero-sum subsequence of $S_1' a_1^{-1}$. Now by Lemma 2.6 and (5), there exist $T_1|S_1' a_1^{-1}$ and $U_1|S_2'$ such that $|T_2| = |U_2|$ and $\sigma(T_2) = \sigma(U_2)$. Clearly, T_1 and T_2 are different. Assume that $a_2|T_2$. Similarly we can obtain subsequences $T_3, \ldots, T_n$ of S_1' and subsequences $U_3, \ldots, U_n$ of S_2' satisfying $|T_i| = |U_i|$ and $\sigma(T_i) = \sigma(U_i)$ for for every $i \in [1, n]$, and $T_1, \ldots, T_n$ are pairwise different. Note that for every $i \in [1, n]$, $S_1' U_i T_i^{-1} 0^{\mathsf{h}(S)-1}$ has sum zero and length n^2, we infer that $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \ge n\binom{\mathsf{h}(S)}{1} \ge n \times (2n - 2) \ge n^2 + 1$.

Subcase 3.4: $r = \mathsf{h}(S)$. If S_2 has a zero-sum subsequence T with $|T| \ge 2$, then replace S_1 by $S_1 T\mathbf{0}^{-|T|}$ and it reduces to Subcase 3.1 or Subcase 3.2.

Now we assume that S_2 is zero-sumfree. Suppose $S_1 = g_1^{\mathsf{v}_{g_1}(S_1)} \cdots g_k^{\mathsf{v}_{g_k}(S_1)}\mathbf{0}^{\mathsf{h}(S)}$, where $g_1, \ldots, g_k, \mathbf{0}$ are distinct elements in G. If there exists a subsequence T of S_2 such that $|T| \ge 2$ and $g_i = \sigma(T)$ for some i, then replace S_1 by $S_1 g_i^{-1}\mathbf{0}^{-|T|+1}T$ and it reduces to Subcase 3.1 or Subcase 3.2 or Subcase 3.3. So by Lemma 2.5 we may suppose that $\mathsf{v}_{g_i}(S_2) = n - 1$ holds for any $i \in [1, k]$. Since $|S_2| = 2n - 2$, we have $k \le 2$.

If $k = 1$ then $\mathrm{v}_{g_1}(S_1) \geq n$. Therefore, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_2)} \geq \binom{n+n-1}{n-1} \geq n^2+1$.

If $k = 2$ then $g_1+g_2 \neq \mathbf{0}$ follows from S_2 is zero-sumfree. Therefore, $\max\{\mathrm{v}_{g_1}(S_1), \mathrm{v}_{g_2}(S_1)\} \geq 2$. Thus, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)}\binom{\mathrm{v}_{g_2}(S_1)+\mathrm{v}_{g_2}(S_2)}{\mathrm{v}_{g_2}(S_1)} \geq \binom{1+n-1}{1}\binom{2+n-1}{2} \geq n \times (n+1) > n^2+1$.

Case 4. $\mathsf{h}(S) \leq 2n-3$. Now rewrite S_1 and S_2 in the form

$$\begin{aligned} S_1 &= g_1^{\mathrm{v}_{g_1}(S_1)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_1)} g_{r_1+1}^{\mathrm{v}_{g_{r_1+1}}(S_1)} \cdots g_{r_1+r_2}^{\mathrm{v}_{g_{r_1+r_2}}(S_1)}, \\ S_2 &= g_1^{\mathrm{v}_{g_1}(S_2)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_2)} g_{r_1+r_2+1}^{\mathrm{v}_{g_{r_1+r_2+1}}(S_2)} \cdots g_{r_1+r_2+r_3}^{\mathrm{v}_{g_{r_1+r_2+r_3}}(S_2)}, \end{aligned}$$

where $g_1, \ldots, g_{r_1+r_2+r_3}$ are distinct elements in G.

Let

$$S_3 = g_{r_1+1}^{\mathrm{v}_{g_{r_1+1}}(S_1)} \cdots g_{r_1+r_2}^{\mathrm{v}_{g_{r_1+r_2}}(S_1)} = S_1\left(g_1^{\mathrm{v}_{g_1}(S_1)} \cdots g_{r_1}^{\mathrm{v}_{g_{r_1}}(S_1)}\right)^{-1}.$$

If $\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1) \geq 3n-3$, then

$$\left(\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)\right) + \cdots + \left(\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)\right) \geq 3n-2.$$

By Lemma 2.8, we have

$$\begin{aligned} \mathsf{N}_{\mathbf{0}}^{|G|}(S) &\geq \binom{\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)} \cdots \binom{\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)}{\mathrm{v}_{g_{r_1}}(S_1)} \\ &\geq \left(\mathrm{v}_{g_1}(S_1) + \mathrm{v}_{g_1}(S_2)\right) \cdots \left(\mathrm{v}_{g_{r_1}}(S_1) + \mathrm{v}_{g_{r_1}}(S_2)\right) \\ &\geq n^2+1. \end{aligned}$$

So we may assume that $\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1) \leq 3n-4$.

Let $N_1 = \binom{\mathrm{v}_{g_1}(S_1)+\mathrm{v}_{g_1}(S_2)}{\mathrm{v}_{g_1}(S_1)} \cdots \binom{\mathrm{v}_{g_{r_1}}(S_1)+\mathrm{v}_{g_{r_1}}(S_2)}{\mathrm{v}_{g_{r_1}}(S_1)}$. Let N_2 denote the number of subsequences T_1 of S_3 satisfying (I) $|T_1| = 2$, and (II) there is a subsequence T_2 of S_2 such that $|T_2| = 2$ and $\sigma(T_1) = \sigma(T_2)$.

Clearly, $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq N_1 + N_2$. So we may assume that

$$N_2 \leq n^2.$$

By Lemma 2.9 there exists a subsequence W of S_3 such that S_3W^{-1} contains no subsequence satisfying both (I) and (II) and such that

$$|W| \leq \frac{|S_3|}{2} + \frac{N_2}{4}.$$

Let $\mathcal{N}_3$ denote the set of nonempty subsequences T_1 of S_3W^{-1} such that $|T_2| = |T_1|$ and $\sigma(T_1) = \sigma(T_2)$ for some $T_2|S_2$. By the definition of $W|S_3$ we know that

$$|T_1| \geq 3 \tag{6}$$

holds for every $T_1 \in \mathcal{N}_3$.

Let $k = |S_3W^{-1}|$. Note that

$$\begin{aligned}|S_3W^{-1}| &= |S_3| - |W| \\ &\geq |S_3| - \frac{|S_3|}{2} - \frac{N_2}{4} = \frac{|S_3|}{2} - \frac{N_2}{4} \\ &\geq \frac{1}{2}\left(n^2 - \left(\mathrm{v}_{g_1}(S_1) + \cdots + \mathrm{v}_{g_{r_1}}(S_1)\right)\right) - \frac{1}{4}n^2 \\ &\geq \frac{1}{4}n^2 - \frac{3}{2}n + 2.\end{aligned}$$

Therefore

$$k \geq \frac{1}{4}n^2 - \frac{3}{2}n + 2. \tag{7}$$

Note that every $T_1 \in \mathcal{N}_3$ is contained by

$$\binom{k - |T_1|}{2n - 2 - |T_1|} = \binom{k - |T_1|}{k - (2n-2)}$$

subsequences of S_3W^{-1} with length $2n-2$. By Lemma 2.4 we have

$$\sum_{T_1 \in \mathcal{N}_3} \binom{k - |T_1|}{k - (2n-2)} \geq \binom{k}{k - (2n-2)}. \tag{8}$$

Let $N_3 = |\mathcal{N}_3|$. Combining (6), (7) and (8) we obtain that

$$\begin{aligned}N_3 &\geq \frac{\binom{k}{k-(2n-2)}}{\binom{k-3}{k-(2n-2)}} = \frac{\binom{k}{2n-2}}{\binom{k-3}{2n-5}} \\ &= \frac{k(k-1)(k-2)}{(2n-2)(2n-3)(2n-4)} \\ &\geq \frac{\left(\frac{1}{4}n^2 - \frac{3}{2}n + 2\right)\left(\frac{1}{4}n^2 - \frac{3}{2}n + 1\right)\left(\frac{1}{4}n^2 - \frac{3}{2}n\right)}{(2n-2)(2n-3)(2n-4)} \\ &\geq n^2 + 1 \text{ (since } n \geq 526).\end{aligned}$$

So $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq N_1 + N_2 + N_3 \geq n^2 + 1$. This completes the proof. □

5. Remarks and Open Problems

Conjecture 1.2 and Theorem 1.3 suggest the following

Conjecture 5.1 *Let $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ be a finite abelian group, where $n_i | n_{i+1}$ for any $i \in [1, r-1]$. Let $S \in \mathscr{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1$. Then*

$$\mathsf{N}_g^{|G|}(S) = 0 \text{ or } \mathsf{N}_g^{|G|}(S) \geq n_1$$

for every $g \in G \setminus \{\mathbf{0}\}$.

It is easy to see that Conjecture 5.1 is true for all elementary abelian groups from the following result.

Proposition 5.2 *Let p be a prime, and let G be a finite abelain p-group. Let $S \in \mathscr{F}(G)$ with $|S| = |G| + D(G) - 1$. Then $\mathsf{N}_g^{|G|}(S) = 0$ or $\mathsf{N}_g^{|G|}(S) \geq p$ for every $g \in G \setminus \{\mathbf{0}\}$, and either $\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 1$ or $\mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq p + 1$.*

Proof. By a result in [10] (or see [13], Theorem 8.3) we know that

$$\mathsf{N}_g^{|G|}(S) \equiv \begin{cases} 1 \pmod{p}, & \text{if } g = \mathbf{0}, \\ 0 \pmod{p}, & \text{otherwise.} \end{cases}$$

Now the proposition follows. □

Conjecture 5.3 *Let G be a finite abelian group. Let $S \in \mathscr{F}(G)$ be a sequence of length $|S| = |G| + \mathrm{D}(G) - 1$. If $G \neq C_2 \oplus C_2$, then*

$$\mathsf{N}_{\mathbf{0}}^{|G|}(S) = 0 \text{ or } \mathsf{N}_{\mathbf{0}}^{|G|}(S) \geq |G| + 1.$$

ACKNOWLEDGMENTS

The authors would like to thank the referee and Professor Landman for their very useful suggestion that improved the presentation of this paper. This research was supported in part by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, the National Science Foundation of China.

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