A NOTE ON BI-NORMAL CAYLEY GRAPHS

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Abstract. The aim of this paper is to answer a question posed by Li [3] and prove that every bi-normal Cayley graph is not 3-arc-transitive.

Keywords. Cayley graph, bi-Cayley graph, s-arc-transitive graph.

1. Introduction

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We use $\mathsf{Aut}(\Gamma)$ to denote the automorphism group of Γ. The graph Γ is said to be (X,s)-*arc-transitive* for some $X \leq \mathsf{Aut}(\Gamma)$ if it has at least one s-arc and X is transitive on both the vertices and the s-arcs of Γ, where an s-arc means a sequence $v_0, v_1, \cdots, v_s$ of vertices such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. For the case where $X = \mathsf{Aut}(\Gamma)$, an (X,s)-arc-transitive graph is simply called *s-arc-transitive.* A graph is said to be *s-transitive* if it is s-arc-transitive but not $(s+1)$-arc-transitive.

Let G be a finite group and S be a subset of G with $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The Cayley graph $\mathsf{Cay}(G,S)$ of G with respect to S is defined as the graph with vertex set G and edge set $\{\{x,y\} \mid yx^{-1} \in S\}$. Then $\mathsf{Cay}(G,S)$ admits a group $\hat{G} := \{\hat{g} : x \mapsto xg,\ x \in G \mid g \in G\}$ acting regularly on the vertices. The Cayley graph $\mathsf{Cay}(G,S)$ is said to be *normal* if $\hat{G}$ itself is normal in $\mathsf{Aut}(\mathsf{Cay}(G,S))$, or *bi-normal* if $\hat{G}$ has a subgroup of index 2 which is normal in $\mathsf{Aut}(\mathsf{Cay}(G,S))$.

The aim of this paper is to answer a question posed by Li [3]. For $s \geq 2$, Li [3] gave a characterization of s-transitive Cayley graphs, he proved that each connected s-transitive Cayley graph is normal with $s = 2$, or bi-normal (so bipartite) with $s \leq 3$, or a normal cover of one of finite number of graphs. Then the following interesting question was proposed:

Question 1.1. (a) *Do there exists* 3*-transitive bi-normal Cayley graphs?*
(b) *Do there exist s-transitive Cayley graphs for $s = 5$ and $s = 7$?*

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There is a positive answer to part (b) of the above question, one can find such examples in [6] and [1]. But the answer to part (a) is negative. In this paper, we shall prove the following result.

Theorem 1.2. *Each connected bi-normal Cayley graph is not* 3*-arc-transitive.*

2. Bi-Cayley graphs

Let G be a finite group and $S \subseteq G$ which possibly contains the identity element of G. The bi-Cayley graph, denoted by $\mathrm{BCay}(G, S)$, is defined to be the graph with vertex set $G \times \{l, r\}$ and edge set $\{\{(x, l), (y, r)\} \mid x, y \in G, yx^{-1} \in S\}$. Then $\mathrm{BCay}(G, S)$ is a well-defined bipartite graph with two bipartition subsets $G \times \{l\}$ and $G \times \{r\}$. It is easily shown that $\mathrm{BCay}(G, S)$ is connected if and only if $G = \langle SS^{-1}\rangle$ (if and only if $G = \langle S^{-1}S\rangle$), see [2] for example. For each $g \in G$, we define

$$\hat{g} : G \times \{l, r\} \to G \times \{l, r\},\ (x, i) \mapsto (xg, i) \text{ for } i = l, r.$$

It is easy to see that $\hat{g}$ is an automorphism of $\mathrm{BCay}(G, S)$. Set $\hat{G} = \{\hat{g} \mid g \in G\}$. Then $g \mapsto \hat{g}$ gives an isomorphism from G to $\hat{G}$.

Of course, one can define other more automorphisms of bi-Cayley graphs satisfying special conditions such as normalizing $\hat{G}$, see [4, 5] for details. Here, we quote several results which will be used in the following section.

Let $\Gamma = \mathrm{BCay}(G, S)$ be a bi-Cayley graph, let $A = \mathsf{Aut}(\Gamma)$ and $A^+ = A_{G\times\{l\}}$, the set-wise stabilizer of $G\times\{l\}$ in A. Then A^+ is a normal subgroup of A with index no more than 2. Further, denote by N the normalizer $\mathbf{N}_A(\hat{G})$ of $\hat{G}$ in A. Then $N^+ := \mathbf{N}_{A^+}(\hat{G})$ has index no more than 2 in N, and $N_{(1,i)} = N^+_{(1,i)} \le A^+$ for $i = l, r$.

Now we consider the point-wise stabilizers of $(1, l)$ and of $\{(1, l), (1, r)\}$ in N. For $\sigma \in \mathsf{Aut}(G)$ and $h \in G$, we define $\hat{\sigma}$, $\tilde{h}_l$ and $\tilde{h}_r$ as follows:

$$\begin{aligned}
&\hat{\sigma} : G \times \{l, r\} \to G \times \{l, r\};\ (x, l) \mapsto (x^\sigma, l),\ (x, r) \mapsto (x^\sigma, r),\\
&\tilde{h}_l : G \times \{l, r\} \to G \times \{l, r\};\ (x, l) \mapsto (h^{-1}x, l),\ (x, r) \mapsto (x, r),\\
&\tilde{h}_r : G \times \{l, r\} \to G \times \{l, r\};\ (x, l) \mapsto (x, l),\ (x, r) \mapsto (h^{-1}x, r).
\end{aligned}$$

Then $\hat{\sigma}$, $\tilde{h}_l$ and $\tilde{h}_r$ are well-defined permutations on $G\times\{l, r\}$ and fix $G\times\{l\}$ set-wise. Further, we have the following lemma.

Lemma 2.1. *Let $g, h, k \in G$, $\sigma, \tau \in \mathsf{Aut}(G)$ and $\Gamma = \mathrm{BCay}(G, S)$. Then*

(1) $\widehat{\sigma\tau} = \hat{\sigma}\hat{\tau}$, $\hat{\sigma}^{-1}\hat{g}\hat{\sigma} = \widehat{g^\sigma}$, $\tilde{h}_l\hat{g} = \hat{g}\tilde{h}_l$, $\tilde{h}_r\hat{g} = \hat{g}\tilde{h}_r$ *and* $\tilde{h}_l\tilde{k}_r = \tilde{k}_r\tilde{h}_l$*;*
(2) $\hat{\sigma}\tilde{h}_l\tilde{k}_r$ *is an isomorphism from* $\mathrm{BCay}(G, S)$ *to* $\mathrm{BCay}(G, k^{-1}S^\sigma h)$*;*
(3) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N^+$ *if and only if* $S = k^{-1}S^\sigma h$*;*
(4) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,l)}$ *if and only if* $h = 1$ *and* $S = k^{-1}S^\sigma$*;*
(5) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,r)}$ *if and only if* $k = 1$ *and* $S = S^\sigma h$*;*
(6) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,l)(1,r)}$ *if and only if* $h = k = 1$ *and* $S^\sigma = S$*.*

Proof. (1) For any $x \in G$ and $i = l, r$, we have

$$(x,i)^{\widehat{\sigma\tau}} = ((x^\sigma)^\tau, i) = (x^\sigma, i)^{\hat{\tau}} = (x,i)^{\hat{\sigma}\hat{\tau}},$$
$$(x,i)^{\hat{g}\hat{\sigma}} = (xg,i)^{\hat{\sigma}} = ((xg)^\sigma, i) = (x,i)^{\hat{\sigma}\widehat{g^\sigma}}.$$

It follows that the first equations in (1) hold. One may easily check the other three equations.

(2) Set $\omega = \hat{\sigma}\tilde{h}_l\tilde{k}_r$. For $x, y \in G$, we have

$$\begin{aligned} & \{(x,l)^\omega, (y,r)^\omega\} = \{(h^{-1}x^\sigma, l), (k^{-1}y^\sigma, r)\} \in E(\mathrm{BCay}(G, k^{-1}S^\sigma h)) \\ \Leftrightarrow\ & k^{-1}y^\sigma(x^\sigma)^{-1}h \in k^{-1}S^\sigma h \Leftrightarrow (yx^{-1})^\sigma = y^\sigma(x^\sigma)^{-1} \in S^\sigma \Leftrightarrow yx^{-1} \in S \\ \Leftrightarrow\ & \{(x,l),(y,r)\} \in E(\varGamma). \end{aligned}$$

It implies that ω is an isomorphism from $\varGamma$ to $\mathrm{BCay}(G, k^{-1}S^\sigma h)$.

(3) Note that ω fixes $G \times \{l\}$ set-wise and that ω normalizes $\hat{G}$ by (1). Then $\omega \in N^+$ if and only if $\omega \in \mathsf{Aut}(\varGamma)$. If $S = k^{-1}S^\sigma h$ then, by (2), ω is an automorphism of $\varGamma$. Assume $\omega \in \mathsf{Aut}(\varGamma)$. Then ω maps the neighborhood $S \times \{r\}$ of $(1,l)$ onto the neighborhood $Sh^{-1} \times \{r\}$ of $(1,l)^\omega = (h^{-1}, l)$. Noting $(S \times \{r\})^\omega = k^{-1}S^\sigma$, we get $Sh^{-1} = k^{-1}S^\sigma$, and so $S = k^{-1}S^\sigma h$.

Note that $\hat{\sigma}$ fixes both $(1,l)$ and $(1,r)$, $\tilde{h}_l$ fixes $(1,r)$ and $\tilde{k}_r$ fixes $(1,l)$. Then (4), (5) and (6) hold. □

Remark 2.2. *By Lemma* 2.1 (2)*, we get* $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,k^{-1}S) \cong \mathrm{BCay}(G,Sh)$ *for any* $h,k \in G$*. In particular,* $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,s^{-1}S)$ *for any* $s \in S$*. Thus, for a bi-Cayley graph* $\mathrm{BCay}(G,S)$*, one may assume that* S *contains the identity element of* G*.*

Theorem 2.3. (1) $N^+ = \{\hat{\sigma}\tilde{h}_l\tilde{k}_r \mid h,k \in G, \sigma \in \mathsf{Aut}(G), S = k^{-1}S^\sigma h\}$*;*

(2) $N_{(1,l)} = \{\hat{\sigma}\tilde{k}_r \mid \sigma \in \mathsf{Aut}(G),\ k \in G,\ S = k^{-1}S^\sigma\}$*;*
(3) $N_{(1,r)} = \{\hat{\sigma}\tilde{h}_l \mid \sigma \in \mathsf{Aut}(G),\ h \in G,\ S = S^\sigma h\}$*;*
(4) $N_{(1,l)(1,r)} = \{\hat{\sigma} \mid \sigma \in \mathsf{Aut}(G),\ S^\sigma = S\}$.
(5) *If* $\varGamma = \mathrm{BCay}(G,S)$ *is connected, then* $N_{(1,i)}$ *acts faithfully on the neighborhood of* $(1,i)$ *in* $\varGamma$*, where* $i = l, r$*;*

Proof. Let $\omega \in N$. Then ω normalizes $\hat{G}$ and so, for any $x \in G$, we have $\omega^{-1}\hat{x}\omega = \widehat{x'}$ for some $x' \in G$. Define $\sigma : G \to G$; $x \mapsto x'$. It is easily shown that σ is a well-defined bijection on G. For $x, y \in G$, we have

$$\begin{aligned} ((xy)^\sigma, l) &= ((xy)', l) = (1,l)^{\widehat{(xy)'}} = (1,l)^{\omega^{-1}\widehat{xy}\omega} \\ &= (1,l)^{\omega^{-1}\hat{x}\hat{y}\omega} = (1,l)^{\widehat{x'}\widehat{y'}} = (x'y', l) = (x^\sigma y^\sigma, l), \end{aligned}$$

and so $(xy)^\sigma = x^\sigma y^\sigma$. It implies $\sigma \in \mathsf{Aut}(G)$.

Assume $\omega \in N^+$. Then we may set $(1,l)^\omega = (h^{-1}, 1)$ and $(1,r)^\omega = (k^{-1}, r)$ for some $h, k \in G$. Then $\omega = \hat{\sigma}\tilde{h}_l\tilde{k}_r$ follows from

$$(x,l)^{\hat{\sigma}\tilde{h}_l\tilde{k}_r} = (h^{-1}x^\sigma, l) = (h^{-1},l)^{\widehat{x'}} = (h^{-1},l)^{\omega^{-1}\hat{x}\omega} = (1,l)^{\hat{x}\omega} = (x,l)^\omega,$$
$$(x,r)^{\hat{\sigma}\tilde{h}_l\tilde{k}_r} = (k^{-1}x^\sigma, r) = (k^{-1},r)^{\widehat{x'}} = (k^{-1},r)^{\omega^{-1}\hat{x}\omega} = (1,r)^{\hat{x}\omega} = (x,r)^\omega.$$

(1) to (4). By Lemma 2.1 (3), we have $S = k^{-1}S^{\sigma}h$ and (1) holds. Further, (2), (3) and (4) follow from (1) and Lemma 2.1.

(5). By Remark 2.2, we may assume that S contains the identity element of G. Thus $(1, j)$ belongs to the neighborhood $(1, i)$, where $\{i, j\} = \{l, r\}$. Since $1 \in S$ and Γ is connected, we have $G = \langle S^{-1}S\rangle = \langle S\rangle$. Noting that $S \times \{r\}$ is the neighborhood of $(1, l)$ and $S^{-1} \times \{l\}$ is the neighborhood of $(1, r)$, it follows from (4) that the stabilizer $N_{(1,i)(1,j)}$ of $(1, j)$ in $N_{(1,i)}$ acts faithfully on $S \times \{r\}$ for $i = l$ and on $S^{-1} \times \{l\}$ for $i = r$. Thus $N_{(1,i)}$ is faithful on the neighborhood of $(1, i)$. $\square$

Note that $(s, r)^{\hat{\sigma}} = (s, r)$ implies $s^{\sigma} = s$ and $(s^{-1})^{\sigma} = s^{-1}$ for $s \in S$ and $\sigma \in \mathsf{Aut}(G)$ with $S^{\sigma} = S$. We have the following corollary.

Corollary 2.4. *If $1 \neq s \in S$, then $N_{(1,r)(1,l)(s,r)}$ is intransitive on $S^{-1} \times \{l\} \setminus \{(1, l)\}$. In particular, if $1 \in S$, then $N_{(1,r)(1,l)}$ is not transitive on the 3-arcs which contains the arc $((1, r), (1, l))$ of* $\mathrm{BCay}(G, S)$.

3. Proof of Theorem 1.2

Let Γ be a connected bi-normal Cayley graph. Then $\mathsf{Aut}(\Gamma)$ has a normal subgroup, say G, which is semiregular and has exactly two orbits on $V(\Gamma)$. Noting that these two G-orbits give an $\mathsf{Aut}(\Gamma)$-invariant partition of $V(\Gamma)$. It follows that either Γ is not arc-transitive, or Γ is a bipartite graph and those two G-orbits are the bipartition subsets of Γ. Then the following argument completes the proof of Theorem 1.2.

Now let Γ be a connected bipartite graph with $G \leq \mathsf{Aut}(\Gamma)$ acting regularly on both two bipartition subsets U_0 and U_1 of Γ. Then it is easily shown that Γ is a regular graph. Let $\{u_0, u_1\} \in E(\Gamma)$ with $u_0 \in U_0$ and $u_1 \in U_1$. Then each vertex in U_i can be written uniquely as u_i^x for some $x \in G$. Define

$$\phi : V(\Gamma) \to G \times \{l, r\};\ u_i^x \mapsto (x, i),\ i = 0, 2.$$

Then ϕ is a bijection. Set $S = \{s \in G \mid \{u_0, u_1^s\} \in E(\Gamma)\}$. Then, for $x, y \in G$, we have

$$\{u_0^x, u_1^y\} \in E(\Gamma) \Leftrightarrow \{u_0, u_1^{yx^{-1}}\} \in E(\Gamma) \Leftrightarrow yx^{-1} \in S.$$

It follows that ϕ is an isomorphism from Γ to the bi-Cayley graph $\mathrm{BCay}(G, S)$. Further, $\phi^{-1}g\phi = \hat{g} \in \hat{G}$ for all $g \in G$, and $X \leq \mathbf{N}_{\mathsf{Aut}(\Gamma)}(G)$ implies $\phi^{-1}X\phi \leq \mathbf{N}_A(\hat{G})$, where $A = \mathsf{Aut}(\mathrm{BCay}(G, S))$. It follows from Corollary 2.4 that Γ is not $(X, 3)$-arc-transitive for $X \leq \mathsf{Aut}(\Gamma)$ with $G \leq X \leq \mathbf{N}_{\mathsf{Aut}(\Gamma)}(G)$. This completes the proof of Theorem 1.2.

References

[1] M. Conder, On symmetries of Cayley graphs and the graphs underlying regular maps, *J. Algebra*(2008), doi:10.1016.

[2] S.F. Du and M.Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Com. in Algebra* **28**(2000), 2685-2715.

[3] C. H. Li, Finite s-arc transitive Cayley graphsand flag-transitive projective planes, *Proc. Amer. Math. Soc.* **138** (2004), 31-41.

[4] Z.P. Lu, On the automorphism groups of biCayley graphs, *Beijing Daxue Xuebao Ziran Kexue Ban* **39**(2003), 1-5.

[5] Z.P. Lu, Semisymmetric cubic graphs constructed from bi-Cayley graphs of A_n, *Ars Combinatoria* **80**(2006), 117-187.

[6] S.J. Xu, X.G. Fang, J. Wang and M.Y. Xu, 5-arc transitive cubic Cayley graphs on finite simple groups, *European J. Combin.* **28**(2007), 1023-1036.

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