On k-tuple domination of random graphs*

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Abstract. In graph $G = (V, E)$, a vertex set $D \subseteq V$ is called a domination set if any vertex $u \in V \backslash D$ is connected to at least one vertex in D. Generally, for any natural number k, the k-tuple domination set D is a vertex set such that any vertex $u \in V \backslash D$ is connected to at least k vertices in D. The k-tuple domination number is the minimum size of k-tuple domination sets. It is known the 1-tuple domination number (i.e. domination number) of classical random graphs $G(n, p)$ with fixed $p \in (0, 1)$ asymptotically almost surely (*a.a.s.*) has a two point concentration (Wieland and Godbole [11]). In this paper, we prove the 2-tuple domination number of $G(n, p)$ with fixed $p \in (0, 1)$ *a.a.s.* has a two-point concentration.

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1. Introduction and main result

In a graph $G = (V, E)$, a vertex is said to dominate itself and its neighbors. A dominating set of G is a subset $D \subseteq V$ such that any vertex in $V \backslash D$ is dominated by at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of dominating sets of G. In general, for any natural number k, Harary and Haynes [5] introduced the following k-tuple dominating set: a k-tuple dominating set of G is a subset D of V such that any vertex outside D is connected to at least k vertices in D (in this case, we say every vertex in $V \backslash D$ is dominated by at least k vertices of D). Define k-tuple domination number $\gamma_k(G)$ as the minimum cardinality of k-tuple dominating sets of G. Obviously, $\gamma_k(G) \geq \gamma(G) = \gamma_1(G)$.

Liao and Chang [8] studied k-tuple domination in graphs from an algorithmic point of view. They obtained a linear-time algorithm on the k-tuple domination problem for some graphs and proved the mentioned problem is NP-complete for others. Note the following type upper bounds on k-tuple domination for graphs $G = (V, E)$ were obtained in [2], [3], [4], [8] and [12] (under a certain condition):

$$\gamma_k(G) \leq c|V| \text{ for some constant } c \in (0, 1).$$

As mentioned before, k-tuple domination problem is NP-complete for some graphs, it is interesting to study it in random graphs. Wieland and Godbole [11] studied the domination

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number of classical random graphs $G(n,p)$, and verified that $\gamma_1(G(n,p))$ is concentrated at two-point asymptotically almost surely (*a.a.s.*) if $p \in (0,1)$ is fixed or $p = p(n)$ approaches zero sufficiently slowly, namely, *a.a.s.*,

$$\lfloor \log_b n - \log_b(\log_b n \cdot \log n) \rfloor + 1 \le \gamma_1(G(n,p)) \le \lfloor \log_b n - \log_b(\log_b n \cdot \log n) \rfloor + 2,$$

where $b = 1/(1-p)$, and $\lfloor x \rfloor$ is the largest integer which is no more than x for any $x \in \mathbb{R}$.

In this paper we study the 2-tuple domination number (namely double domination number in [6]) of $G(n,p)$. Our main result is stated as follows.

Theorem 1.1. Let $p \in (0,1)$ be a constant and $b = 1/(1-p)$. In $G(n,p)$, *a.a.s.*,

$$\left\lfloor \log_b n - \log_b \log n + \log_b \frac{p}{1-p} \right\rfloor + 1 \le \gamma_2(G(n,p)) \le \left\lfloor \log_b n - \log_b \log n + \log_b \frac{p}{1-p} \right\rfloor + 2.$$

Write $\mathbf{P}(\cdot)$, $\mathbf{E}(\cdot)$ and $\mathbf{V}(\cdot)$ for the probability, expected value and variance of a random variable respectively. Denote by $\mathbf{COV}(\cdot,\cdot)$ the covariance of two random variables. For two functions $f(n)$, $g(n)$ of a natural valued parameter n, $f(n) = o(g(n))$ means $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = \Omega(g(n))$ means that for some constant $c > 0$, $f(n) \ge cg(n)$ for sufficiently large n.

Remark 1.2. **(i)** Similarly to Theorem 1.1, one can prove that for any $p \in (0,1)$ and natural number k, the following *a.a.s* holds:

$$a \le \gamma_k(G(n,p)) - \{\log_b n - \log_b \log n + (k-2)\log_b \log_b n\} \le c,$$

where $a = a(k,p)$ and $c = c(k,p)$ are two constants. This is left to the interested readers. The magnitude $\log_b n - \log_b \log n + (k-2)\log_b \log_b n$ comes from computing $\mathbf{E}\left(X_r^k\right)$ for a suitable r, where X_r^k is the number of k-tuple dominating sets of size r. For instance, for $p = 1/2$,

$$\mathbf{E}\left(X_r^k\right) = \binom{n}{r}\left\{1 - \binom{r}{0}\left(\frac{1}{2}\right)^r - \binom{r}{1}\left(\frac{1}{2}\right)^r - \cdots - \binom{r}{k-1}\left(\frac{1}{2}\right)^r\right\}^{n-r};$$

and $\mathbf{E}\left(X_r^k\right) \to 0$ or ∞ depends on $(n-r)\binom{r}{k-1}\left(\frac{1}{2}\right)^r \ge r\log n$ or not, $\mathbf{V}\left(X_r^k\right) = o\left(\mathbf{E}^2\left(X_r^k\right)\right)$ if $\mathbf{E}\left(X_r^k\right) \to \infty$; and roughly, r is chosen satisfying $(n-r)\binom{r}{k-1}\left(\frac{1}{2}\right)^r \approx r\log n$.

(ii) Theorem 1.1 also holds when $p = p(n)$ approaches zero sufficiently slowly, which is left to the interested readers.

2. Proof of Theorem 1.1

Due to similarity, we only prove Theorem 1.1 for $p = 1/2$ for simplicity. The lower bound is proven in subsection 2.1 by the Markov inequality, and the upper bound is verified in subsection 2.2 by the Chebyschev inequality.

2.1. The lower bound

For $r > 1$, let X_r denote the number of 2-tuple dominating sets of size r. To get the lower bound, we only need to show that $\mathbf{E}(X_r) \to 0$ when $r = \lfloor \log_2 n - \log_2 \log n \rfloor$ by the Markov inequality.

Let $S_1, \cdots, S_{\binom{n}{r}}$ be all subsets of vertices with size r, and each A_k the event that S_k is a 2-tuple dominating set, and each I_k the indicator random variable of A_k. Clearly,

$$X_r = \sum_{k=1}^{\binom{n}{r}} I_k.$$

It is easy to see that

$$\mathbf{E}(X_r) = \binom{n}{r} \left\{1 - \left(1 - \frac{1}{2}\right)^r - r \cdot \left(1 - \frac{1}{2}\right)^{r-1} \cdot \left(\frac{1}{2}\right)\right\}^{n-r},$$

where $\left(1 - \frac{1}{2}\right)^r$ is the probability of a vertex outside D is not connected to any vertex in D, and $r\left(1 - \frac{1}{2}\right)^{r-1}\left(\frac{1}{2}\right)$ is the probability of a vertex outside D is connected to only one vertex in D. Thus

$$\begin{aligned}
\mathbf{E}(X_r) &= \binom{n}{r} \left\{1 - (r+1) \cdot \left(\frac{1}{2}\right)^r\right\}^{n-r} \\
&\leq \binom{n}{r} \exp\left\{-(n-r)(r+1) \cdot \left(\frac{1}{2}\right)^r\right\} \\
&= \exp\left\{-(n-r)(r+1) \cdot \left(\frac{1}{2}\right)^r + r + r\log n - r\log r\right\} \\
&\leq \exp\left\{-n \cdot r \cdot \left(\frac{1}{2}\right)^r + r \cdot (r+1) \cdot \left(\frac{1}{2}\right)^r + r + r\log n - r\log r\right\} \\
&= \exp\left\{\frac{\log_2 n(\log_2 n + 1) \cdot \log n}{n} + (1 - o(1))\log_2 n - \log_2 n \cdot \log\log_2 n\right\} \to 0.
\end{aligned}$$

By the Markov inequality,

$$\mathbf{P}(X_r > 0) \leq \mathbf{E}(X_r) \to 0,$$

which implies *a.a.s.*,

$$\gamma_2(G(n, 1/2)) \geq \lfloor \log_2 n - \log_2 \log n \rfloor + 1.$$

2.2. The upper bound

To obtain the upper bound, we need to show that if $r = \lfloor \log_2 n - \log_2 \log n \rfloor + 2$, then

$$\mathbf{E}(X_r) \to \infty \text{ and } \mathbf{V}(X_r) = o(\mathbf{E}^2(X_r)).$$

Notice for $0 < x < 1$,

$$1 - x \geq \exp\{-x/(1-x)\}.$$

It is easy to see that

$$\begin{aligned}
\mathbf{E}(X_r) &= \binom{n}{r}\left\{1-(r+1)\cdot\left(\frac{1}{2}\right)^r\right\}^{n-r} \\
&\geq \binom{n}{r}\exp\left\{-\frac{(n-r)(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r}\right\} \\
&\geq (1-o(1))\left(\frac{en}{r}\right)^r\frac{1}{\sqrt{2\pi r}}\exp\left\{-\frac{(n-r)(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r}\right\} \\
&\geq \exp\left\{-\frac{(n-r)(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r}+r+r\log n-(r+1/2)\log r-\log\sqrt{2\pi}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
-\frac{(n-r)(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r} &= -\frac{n(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r}+\frac{r(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r} \\
&= -(1-o(1))\frac{n\log_2 n\cdot\frac{\log n}{4n}}{1-\frac{\log n}{4n}\cdot\log n} = -(1/4-o(1))\log_2 n\log n,
\end{aligned}$$

we have

$$\mathbf{E}(X_r) \geq \exp\left\{-(1/4-o(1))\log_2 n\log n+r+r\log n-(r+1/2)\log r-\log\sqrt{2\pi}\right\}.$$

Clearly, $r = (1-o(1))\log_2 n$; and the significant factor in

$$r + r\log n - (r+1/2)\log r - \log\sqrt{2\pi}$$

is $r\log n$, which is $(1-o(1))\log_2 n\cdot\log n$. Therefore,

$$\mathbf{E}(X_r) \geq \exp\left\{(3/4+o(1))\log_2 n\log n+r-(r+1/2)\log r-\log\sqrt{2\pi}\right\} \to \infty.$$

Now we estimate $\mathbf{V}(X_r)$. Note

$$\begin{aligned}
\mathbf{V}(X_r) &= \sum_{j=1}^{\binom{n}{r}}\mathbf{V}(I_i)+\sum_{i\neq j}\mathbf{COV}[I_i,I_j] \\
&= \sum_{i=1}^{\binom{n}{r}}\mathbf{E}(I_i)(1-\mathbf{E}(I_i))+2\sum_{i=1}^{\binom{n}{r}}\sum_{j<i}[\mathbf{E}(I_i)(I_j)-\mathbf{E}(I_i)\mathbf{E}(I_j)] \\
&= \mathbf{E}(X_r)+\binom{n}{r}\sum_{s=0}^{r-1}\binom{r}{s}\binom{n-r}{r-s}\mathbf{E}(I_iI_j)-\mathbf{E}^2(X_r),
\end{aligned}$$

where $s = |S_i \cap S_j|$ (see S_i in subsection 2.1); and

$$\begin{aligned}
\mathbf{E}(I_iI_j) &= \mathbf{P}\{S_i \text{ and } S_j \text{ are 2-tuple domination sets}\} \\
&\leq \mathbf{P}\left\{S_i \text{ and } S_j \text{ 2-tuple dominate } \overline{S_i \cup S_j}\ \right\} \\
&= \mathbf{P}\left\{\text{each } x \in \overline{S_i \cup S_j} \text{ has at least two neighbors both in } S_i \text{ and in } S_j\right\},
\end{aligned}$$

where $\overline{S_i \cup S_j}$ is the set of all vertices outside $S_i \cup S_j$. For $x \in \overline{S_i \cup S_j}$, let $B_{ij}(x)$ be the event that x has exactly 1 neighbor both in $S_i \backslash S_j$ and in $S_j \backslash S_i$; $C_{ij}(x)$ the event that x has at most 1 neighbor in $S_i \cup S_j$, and $D_{ij}(x)$ the event that x has at most 1 neighbor in S_i but has at least 2 neighbors in $S_j \backslash S_i$. Then

$$\begin{aligned}
\mathbf{P}(B_{ij}(x)) &= (r-s)\left(\frac{1}{2}\right)^{r-s}(r-s)\left(\frac{1}{2}\right)^{r-s}\left(1-\frac{1}{2}\right)^{s} = (r-s)^2\left(\frac{1}{2}\right)^{2r-s}; \\
\mathbf{P}(C_{ij}(x)) &= \left(1-\frac{1}{2}\right)^{2r-s} + (2r-s)\left(1-\frac{1}{2}\right)^{2r-s-1}\left(\frac{1}{2}\right) = (2r-s+1)\left(\frac{1}{2}\right)^{2r-s}; \\
\mathbf{P}(D_{ij}(x)) &= \left\{1-\left(1-\frac{1}{2}\right)^{r-s} - (r-s)\left(1-\frac{1}{2}\right)^{r-s-1}\left(\frac{1}{2}\right)\right\} \\
&\qquad \times\left\{\left(1-\frac{1}{2}\right)^{r} + r\left(1-\frac{1}{2}\right)^{r-1}\left(\frac{1}{2}\right)\right\} \\
&= (r+1)\left(\frac{1}{2}\right)^{r} - (r+1)(r-s+1)\left(\frac{1}{2}\right)^{2r-s}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{E}(I_iI_j) &\leq \prod_{x \notin S_i \cup S_j}\left(1-\mathbf{P}(B_{ij}(x)) - \mathbf{P}(C_{ij}(x)) - \mathbf{P}(D_{ij}(x)) - \mathbf{P}(D_{ji}(x))\right) \\
&= \left(1-2(r+1)\left(\frac{1}{2}\right)^{r} + (r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}\right)^{n-2r+s}.
\end{aligned}$$

Let

$$J_1 = \binom{n}{r}\sum_{s=1}^{r-1}\binom{r}{s}\binom{n-r}{r-s}\left(1-2(r+1)\left(\frac{1}{2}\right)^{r} + (r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}\right)^{n-2r+s},$$

$$J_2 = \binom{n}{r}\binom{r}{0}\binom{n-r}{r}\left(1-2(r+1)\left(\frac{1}{2}\right)^{r} + (r^2+2r+1)\left(\frac{1}{2}\right)^{2r}\right)^{n-2r} - \mathbf{E}^2(X_r) + \mathbf{E}(X_r).$$

Then

$$\mathbf{V}(X_r) \leq J_1 + J_2.$$

Notice

$$\begin{aligned}
f(s) &:= \binom{r}{s}\binom{n-r}{r-s}\left(1-2(r+1)\left(\frac{1}{2}\right)^r+(r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}\right)^{n-2r+s} \\
&\le \binom{r}{s}\frac{n^{r-s}}{(r-s)!}\left(1-2(r+1)\left(\frac{1}{2}\right)^r+(r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}\right)^{n-2r+s} \\
&\le 2\binom{r}{s}\frac{n^{r-s}}{(r-s)!}\left(1-2(r+1)\left(\frac{1}{2}\right)^r+(r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}\right)^{n} \\
&\le 2\binom{r}{s}\frac{n^{r-s}}{(r-s)!}\exp\left\{n\left((r^2+2r-s^2-s+1)\left(\frac{1}{2}\right)^{2r-s}-2(r+1)\left(\frac{1}{2}\right)^r\right)\right\} \\
&:= g(s);
\end{aligned}$$

and for $s=\Omega(r)$,

$$g(s+1)/g(s)=\frac{(r-s)^2}{n(s+1)}\exp\left\{\left(\frac{1}{2}\right)^{2r-s}n(r^2+2r-s^2-5s-3)\right\}>1;$$

and for $s=o(r)$, $g(s+1)/g(s)<1$. So $g(s)$ is first decreasing then increasing.

Since for large enough n,

$$\begin{aligned}
g(1)/g(r-1) &= \frac{n^{r-2}}{(r-1)!}\exp\left\{-n\left(\frac{3r+1}{2}\right)\left(\frac{1}{2}\right)^r+n(2r^2+4r-2)\left(\frac{1}{2}\right)^{2r}\right\} \\
&= \frac{n^{r-2}}{(r-1)!}\exp\left\{-3/8r\log n(1+o(1))\right\}=\frac{n^{5/8r(1+o(1))}}{(r-1)!}>1,
\end{aligned}$$

we have that

$$f(s)\le g(1),\ \sum_{s=1}^{r-1}f(s)\le rg(1).$$

Now we can estimate J_1.

$$\begin{aligned}
&\frac{J_1}{\mathbf{E}^2(X_r)}=\frac{\binom{n}{r}\sum_{s=1}^{r-1}f(s)}{\mathbf{E}^2(X_r)}\le\frac{\binom{n}{r}rg(1)}{\mathbf{E}^2(X_r)} \\
&\le(1+o(1))2\cdot\frac{r^3}{n}\cdot\exp\left\{n(2r^2+4r-2)\left(\frac{1}{2}\right)^{2r}-2n(r+1)\left(\frac{1}{2}\right)^r-\frac{2(n-r)(r+1)\cdot\left(\frac{1}{2}\right)^r}{1-(r+1)\cdot\left(\frac{1}{2}\right)^r}\right\} \\
&\le(1+o(1))2\cdot\frac{r^3}{n}\cdot\exp\left\{(1+o(1))r^2\log^2 n/(8n)-r\log n/2-r\log n/2(1-o(1))\right\}\to 0.
\end{aligned}$$

Combining with

$$\begin{aligned}
J_2 &= \binom{n}{r}\binom{n-r}{r}\left(1-2(r+1)(1/2)^r+(r^2+2r+1)\,(1/2)^{2r}\right)^{n-2r} \\
&\quad -\left(\binom{n}{r}(1-(r+1)\cdot(1/2)^r)^{n-r}\right)^2+\mathbf{E}(X_r) \\
&= \mathbf{E}^2(X_r)\left\{\frac{\binom{n-r}{r}}{\binom{n}{r}}\frac{[1-2(r+1)\left(\frac{1}{2}\right)^r+(r^2+2r+1)\left(\frac{1}{2}\right)^{2r}]^{n-2r}}{[1-(r+1)\cdot\left(\frac{1}{2}\right)^r]^{2n-2r}}-1+o(1)\right\} \\
&\le \mathbf{E}^2(X_r)\left\{\frac{\binom{n-r}{r}}{\binom{n}{r}}\frac{e^{-2n(r+1)(1/2)^r+4r(r+1)(1/2)^r+nr^2(1/2)^{2r}(1+o(1))}}{e^{-2(n-r)(r+1)(1/2)^r/(1-(r+1)(1/2)^r)}}-1+o(1)\right\} \\
&\le \mathbf{E}^2(X_r)\left\{(1-o(1))\cdot\frac{e^{-2n(r+1)(1/2)^r+4r(r+1)(1/2)^r+nr^2(1/2)^{2r}(1+o(1))}}{e^{-2(n-r)(r+1)(1/2)^r(1+(r+1)(1/2)^r)}}-1+o(1)\right\} \\
&= \mathbf{E}^2(X_r)[(1-o(1))(1+o(1))-1+o(1)]=o\left(\mathbf{E}^2(X_r)\right),
\end{aligned}$$

we see that

$$\mathbf{V}(X_r)=J_1+J_2=o\left(\mathbf{E}^2(X_r)\right).$$

By the Chebychev's inequality,

$$\mathbf{P}[\gamma_2(G(n,1/2))>r]\le\mathbf{P}[X_r=0]\le\mathbf{P}[|X_r-\mathbf{E}X_r|>\mathbf{E}X_r]\le\mathbf{V}(X_r)/\mathbf{E}^2(X_r)\to 0.$$

So *a.a.s.*,

$$\gamma_2(G(n,1/2))\le\lfloor\log_2 n-\log_2\log n\rfloor+2.$$

□

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