Note on Bipartite Unicyclic Graphs of Given Bipartition with Minimal Energy*

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Abstract

The energy of a graph G is defined as the sum of the absolute values of all the eigenvalues of the graph. Let $\mathcal{UB}(p,q)$ denote the set of all bipartite unicyclic graphs of a given (p,q)-bipartition, where $q \geq p \geq 2$. $B(p,q)$ denotes the graph formed by attaching $p-2$ and $q-2$ vertices to two adjacent vertices of a quadrangle C_4, respectively, and $H(3,q)$ denotes the graph formed by attaching $q-2$ vertices to the pendent vertex of $B(2,3)$. In the paper "F. Li and B. Zhou, Minimal energy of bipartite unicyclic graphs of a given bipartition, MATCH Commun. Math. Comput. Chem. 54(2005), 379–388", the authors proved that either $B(3,q)$ or $H(3,q)$ is the graph with minimal energy in $\mathcal{UB}(3,q)(q \geq 3)$. At the end of the paper they conjectured that $H(3,q)$ achieves the minimal energy in $\mathcal{UB}(3,q)$ and checked that this is true for $q = 3, 4$. However, they could not find a proper way to prove it generally. This short note is to give a confirmative proof to the conjecture.

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Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G of order n. The energy of G is defined as $E(G) = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|$. For more information on the energy of graphs, we refer to [1]. Let $\mathcal{UB}(p, q)$ denote the set of all bipartite unicyclic graphs of a given (p, q)-bipartition, where $q \geq p \geq 2$. $B(p, q)$ denotes the graph formed by attaching $p - 2$ and $q - 2$ vertices to two adjacent vertices of a quadrangle C_4, respectively, and $H(3, q)$ denotes the graph formed by attaching $q - 2$ vertices to the pendent vertex of $B(2, 3)$. See Figure 1 for the graphs $B(p, q)(q \geq p \geq 2)$ and $H(3, q)(q \geq 3)$. For terminology and notations not defined here, we refer to [1, 2] and the references therein.

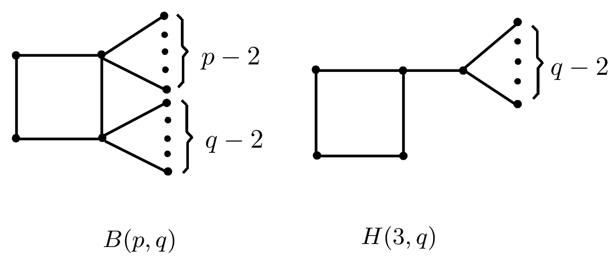

Figure 1: Graphs $B(p, q)$ $(q \geq p \geq 2)$ and $H(3, q)$ $(q \geq 3)$.

In [2] the authors proved that either $B(3, q)$ or $H(3, q)$ achieves the minimal energy in the class $\mathcal{UB}(3, q)$ of bipartite unicyclic graphs of a $(3, q)$-bipartition $(q \geq 3)$. But, they could not determine which one is smaller. At the end of paper [2] they conjectured that $H(3, q)$ *achieves the minimal energy in* $\mathcal{UB}(3, q)$ and checked that this is true for $q = 3, 4$. However, they could not find a proper way to prove it generally. In this short note we will give a confirmative proof to the conjecture.

Theorem 1 $H(3, q)$ $(q \geq 3)$ *achieves the minimal energy in* $\mathcal{UB}(3, q)$.

Proof. Since from [2] either $B(3, q)$ or $H(3, q)$ achieves the minimal energy in $\mathcal{UB}(3, q)$, we only need to prove that $E(B(3, q)) > E(H(3, q))$.

In fact, for $B(3, q)$ and $H(3, q)$ we have from [2] the characteristic polynomials:

$$\begin{aligned} \phi(B(3, q)) &= x^{q-3}(x^6 - (q + 3)x^4 + (3q - 4)x^2 - (q - 2)), \\ \phi(H(3, q)) &= x^{q-1}(x^4 - (q + 3)x^2 + (4q - 6)). \end{aligned}$$

Suppose that

$$\begin{aligned} f(x) &= x^6 - (q+3)x^4 + (3q-4)x^2 - (q-2) \\ &= (x-\sqrt{x}_1)(x-\sqrt{x}_2)(x-\sqrt{x}_3)(x+\sqrt{x}_1)(x+\sqrt{x}_2)(x+\sqrt{x}_3). \\ g(y) &= y^4 - (q+3)y^2 + (4q-6) \\ &= (x-\sqrt{y}_1)(x-\sqrt{y}_2)(x+\sqrt{y}_1)(x+\sqrt{y}_2). \end{aligned}$$

Then, from the relations between the roots and the coefficients of a polynomial equation, we have that $x_1+x_2+x_3 = q+3$, $x_1x_2+x_2x_3+x_1x_3 = 3q-4$ and $x_1x_2x_3 = q-2$, and $y_1+y_2 = q+3$ and $y_1y_2 = 4q-6$.

Let $f_0(x) = x^3 - (q+3)x^2 + (3q-4)x - (q-2)$. It is easy to check that $f_0(0) < 0$, $f_0(0.6) > 0$, $f_0(q) < 0$, $f_0(q^{10}) > 0$, since $q \geq 3$. Suppose that $x_1 \leq x_2 \leq x_3$. Then, clearly $x_3 > q$ and $\sqrt{x}_1 + \sqrt{x}_2 + \sqrt{x}_3 > \sqrt{x}_3 > \sqrt{q}$. So,

$$\begin{aligned} &(\sqrt{x_1x_2} + \sqrt{x_2x_3} + \sqrt{x_1x_3})^2 \\ = \; & x_1x_2 + x_2x_3 + x_1x_3 + 2\sqrt{x_1x_2x_3}(\sqrt{x}_1 + \sqrt{x}_2 + \sqrt{x}_3) \\ > \; & 3q - 4 + 2\sqrt{q-2}\sqrt{q} > 4q - 6. \end{aligned}$$

Thus,

$$\begin{aligned} &(\sqrt{x}_1 + \sqrt{x}_2 + \sqrt{x}_3)^2 \\ = \; & x_1 + x_2 + x_3 + 2(\sqrt{x_1x_2} + \sqrt{x_2x_3} + \sqrt{x_1x_3}) \\ > \; & q + 3 + 2\sqrt{4q-6} \\ = \; & y_1 + y_2 + 2\sqrt{y_1y_2} \\ = \; & (\sqrt{y}_1 + \sqrt{y}_2)^2. \end{aligned}$$

Finally, we get that for $q \geq 3$,

$$E(B(3,q)) \;=\; 2(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) > 2(\sqrt{y_1} + \sqrt{y_2}) = E(H(3,q)).$$

The theorem is thus proved. ∎

References

[1] I. Gutman, X. Li, J. B. Zhang, Graph Energy, in: M. Dehmer, F. Emmert–Streib (Eds.), *Analysis of Complex Networks: From Biology to Linguistics*, Wiley–VCH Verlag, Weinheim, 2009, pp. 145–174.

[2] F. Li, B. Zhou, Minimal energy of bipartite unicyclic graphs of a given bipartition, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 379–388.