Non Symmetric Cauchy Kernels for the Classical Groups

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Abstract

We give non-symmetric versions of the Cauchy kernel and Littlewood's kernels, corresponding to the types A, B, C and D, of the classical groups. Defining two families of key polynomials (one of them being the Demazure characters), we show that these new kernels are diagonal in the basis of key polynomials. We define scalar products such that the two families of key polynomials are adjoint to each other.

Keywords: Cauchy kernel; Littlewood's kernels; Classical groups; Key polynomials; Isobaric divided differences.

1 Introduction

Key polynomials occur naturally in geometry and representation theory. They were defined by Demazure [3] as characters of the action of a complex torus on spaces of sections of ample line bundles over Schubert subvarieties of a flag variety (the case where the Schubert variety is the full flag variety give the irreducible characters of the linear, symplectic and orthogonal groups over $\mathbb{C}$). In this text, we shall adopt a purely combinatorial point of view, keeping only from the work of Demazure the definition of isobaric divided differences.

Given two sets of indeterminates $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, the classical *Cauchy kernel* $\widetilde{\Omega}^A$ diagonalizes in the basis of Schur functions :

$$\widetilde{\Omega}^A = \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(\mathbf{x})\, s_\lambda(\mathbf{y}). \tag{1}$$

The Cauchy kernel may be considered as the generating function of all characters of the symmetric groups. Multiplying the kernel $\widetilde{\Omega}^A$ by the factor

$$\prod_{1\le i<j\le n}(1-x_ix_j)\prod_{i,j=1}^{n}(1-x_i/y_j)^{-1} \quad \text{or} \quad \prod_{1\le i\le j\le n}(1-x_ix_j)\prod_{i,j=1}^{n}(1-x_i/y_j)^{-1},$$

Littlewood [11] obtained expansions for the following kernels $\widetilde{\Omega}^C$ and $\widetilde{\Omega}^D$, in terms of symplectic Schur functions and orthogonal Schur functions (see below for the precise definitions):

$$\widetilde{\Omega}^C = \frac{\prod_{1\le i<j\le n}(1-x_ix_j)}{\prod_{i,j=1}^{n}(1-x_iy_j)(1-x_i/y_j)} = \sum_{\lambda} s_\lambda(\mathbf{x})\, Sp_\lambda(\mathbf{y}'), \tag{2}$$

$$\widetilde{\Omega}^D = \frac{\prod_{1\le i\le j\le n}(1-x_ix_j)}{\prod_{i,j=1}^{n}(1-x_iy_j)(1-x_i/y_j)} = \sum_{\lambda} s_\lambda(\mathbf{x})\, \mathcal{O}_\lambda(\mathbf{y}')\,, \tag{3}$$

where $\mathbf{y}' = \{y_1,\ldots,y_n,y_1^{-1},\ldots,y_n^{-1}\}$.

In this paper, we shall study the following non-symmetric versions of the kernels $\widetilde{\Omega}^A$, $\widetilde{\Omega}^C$ and $\widetilde{\Omega}^D$:

$$\begin{aligned}
\Omega^A &:= \frac{1}{\prod_{i+j\le n+1}(1-x_iy_j)},\\
\Omega^B &:= \frac{\prod_{1\le i<j\le n}(1-x_ix_j)\prod_{i=1}^{n}(1+x_i)}{\prod_{i,j=1}^{n}(1-x_iy_j)\ \prod_{i=1}^{n}\prod_{j=i}^{n}(1-x_i/y_j)},\\
\Omega^C &:= \frac{\prod_{1\le i<j\le n}(1-x_ix_j)}{\prod_{i,j=1}^{n}(1-x_iy_j)\ \prod_{i=1}^{n}\prod_{j=i}^{n}(1-x_i/y_j)},\\
\Omega^D &:= \frac{\prod_{1\le i\le j\le n-1}(1-x_ix_j)}{\prod_{i=1}^{n-1}\prod_{j=1}^{n}(1-x_iy_j)\ \prod_{i=1}^{n-1}\prod_{j=i}^{n}(1-x_i/y_j)}.
\end{aligned}$$

It will be convenient to interpolate between Ω^B and Ω^C, choosing an arbitrary parameter β, and defining :

$$\Omega^{BC} = \frac{\prod_{1\le i<j\le n}(1-x_ix_j)\prod_{i=1}^{n}(1+\beta x_i)}{\prod_{i,j=1}^{n}(1-x_iy_j)\ \prod_{i=1}^{n}\prod_{j=i}^{n}(1-x_i/y_j)}.$$

For each type A,B,C,D,BC, there exist two families of isobaric divided differences, which allow, starting from all dominant monomials, to define two families of key polynomials, one of them being the Demazure characters. Our main result (Th. 6) is that all kernels $\Omega^A,\ldots,\Omega^{BC}$ diagonalize in the corresponding basis of key polynomials.

Notice that in type A, one also has a polynomial kernel, which is the resultant $\prod_i \prod_j (x_i - y_j)$ of two z-polynomials $\prod_i (z - x_i)$ and $\prod_j (z - y_j)$. It still decomposes without multiplicity in the basis of products of Schur functions in $\mathbf{x}$ and $\mathbf{y}$. The non-symmetric version of the resultant, $\prod_{i+j \le n+1} (x_i - y_j)$, decomposes in the basis of products of *Schubert polynomials* in $\mathbf{x}$ and $\mathbf{y}$, and the main properties of Schubert polynomials are easy consequences of the fact that $\prod_{i+j \le n+1} (x_i - y_j)$ is a reproducing kernel [7].

In the present article, we have rather taken in the case of type A the inverse function $\prod_{i+j \le n+1} (1 - x_i y_j)^{-1}$. The corresponding polynomials are no more the Schubert polynomials, though there are interesting relationships between them and the Demazure characters.

The Cauchy kernel may be used to define a scalar product on the ring of symmetric polynomials with coefficients in $\mathbb{Z}$, with respect to which Schur functions constitute an orthonormal basis [12]. Starting from Weyl's denominators, we also define scalar products with respect to which, for all classical types, the basis of key polynomials are adjoint of each other (Th. 15). However, Bogdan Ion [5, 6] has shown that key polynomials can be obtained as a limit case of Macdonald polynomials. Thus the definition of the scalar product and the orthogonality property of key polynomials result from the theory of Macdonald polynomials. Nevertheless, we are giving an independent derivation in sections 6, 7, because this approach relies only on simple properties of divided differences and does not require double affine Hecke algebras.

2 Weyl Groups

We shall realize the classical groups as groups operating on vectors, or, equivalently, on Laurent polynomials, when considering the vectors to be exponents of monomials. For more informations about Coxeter groups, see [1].

Fixing a positive integer n, we define the operators s_i $(1 \le i \le n)$, and τ_n acting on vectors $v \in \mathbb{Z}^n$ as follows (*operators are noted on the right*):

$$\begin{aligned} v s_i &= [\ldots, v_{i+1}, v_i, \ldots]\ ,\ 1 \le i < n, \\ v s_n &= [\ldots, v_{n-1}, -v_n], \\ v \tau_n &= [\ldots, -v_n, -v_{n-1}]\,. \end{aligned}$$

Denoting a Laurent monomial $x_1^{v_1} \cdots x_n^{v_n}$ by x^v, we extend by linearity the preceding operators to operators on Laurent polynomials in indeterminates $x_1, \ldots, x_n$. The simple transpositions s_i $(i = 1, \ldots, n-1)$ interchanges x_i and x_{i+1}, s_n transforms x_n into x_n^{-1}, and τ_n sends x_{n-1} onto x_n^{-1}, x_n onto x_{n-1}^{-1}.

The group generated by $s_1, \ldots, s_{n-1}$ is isomorphic to the symmetric group $\mathfrak{S}_n$ (type A_{n-1}). Adding the generator s_n gives the Weyl group of type B_n or C_n (which will be distinguished later), while $s_1, \ldots, s_{n-1}, \tau_n$ induce a faithful representation of the type D_n.

An element w of any of these groups can be identified with the image under w of the vector $v = [1, 2, \ldots, n]$. For type A_{n-1}, one gets *permutations*; for type B_n, C_n, one gets the *bar-permutations*, writing $\bar{r}$ rather than $-r$; and for type D_n, one gets the *bar-permutations with an even number of bars*. The *length* $\ell(w)$ of w is the length of a reduced decomposition of w.

There is a unique element of maximal length for each type, usually denoted w_0. For A_{n-1}, it is $\omega^A := [n, \ldots, 1]$. For B_n, C_n, it is $\omega^B = \omega^C := [-1, \ldots, -n]$. For D_n, it is $\omega^D := [-1, \ldots, -n]$ if n is even, and otherwise, it is $\omega^D := [-1, \ldots, -n+1, n]$. Reduced decompositions for these elements are

$$\begin{aligned}
\omega^A &= (s_1)\,(s_2 s_1)\,\cdots(s_{n-1}\cdots s_1)\,, \\
\omega^B &= \omega^C = (s_n)\,(s_{n-1}s_n s_{n-1})\,\cdots(s_1\cdots s_{n-1}s_n s_{n-1}\cdots s_1)\,, \\
\omega^D &= (s_{n-1}\tau_n)\,(s_{n-2}s_{n-1}\tau_n s_{n-2})\,\cdots(s_1\cdots s_{n-2}s_{n-1}\tau_n s_{n-2}\cdots s_1)\,.
\end{aligned}$$

A *partition* λ is a decreasing element of $\mathbb{N}^n$: $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. In the case of type A_{n-1}, B_n, C_n, a vector v is *dominant* (resp. a monomial x^v is *dominant*) if v is a partition. In the case of type D_n a vector v is *dominant* if it is a partition, or if $[v_1, \ldots, v_{n-1}, -v_n]$ is a partition. For a given type, we define the *length* $\ell(v)$ of $v \in \mathbb{Z}^n$ to be the minimum number of generators of the group that must be applied to pass from v to a dominant vector. Thus dominant vectors have length 0.

3 The Weyl character formula

In this section, we give a brief review of the Weyl character formula, from an algebraic point of view only.

Let $\rho^A = \rho^D := [n-1, \ldots, 1, 0]$, $\rho^B := [n - \frac{1}{2}, \ldots, 2 - \frac{1}{2}, 1 - \frac{1}{2}]$, $\rho^C := [n, \ldots, 2, 1]$. The sums

$$\sum_w (-1)^{\ell(w)} \left(x^{\rho^\heartsuit}\right)^w ,$$

$\heartsuit = A, B, C, D$, under the appropriate group, can be written as determinants :

$$\begin{aligned}
\Delta^A &= \det\Big(x_i^{n-j}\Big)_{1\le i,j\le n}\,, \\
\Delta^B &= \det\Big(x_i^{n+1/2-j} - x_i^{j-n-1/2}\Big)_{1\le i,j\le n}\,, \\
\Delta^C &= = \det\Big(x_i^{n+1-j} - x_i^{j-n-1}\Big)_{1\le i,j\le n}\,, \\
2\Delta^D &= \det\Big(x_i^{n-j} + x_i^{j-n}\Big)_{1\le i,j\le n}\,.
\end{aligned}$$

These determinants are easily factorized :

$$\Delta^A = \prod_{1\le i<j\le n} (x_i - x_j)\,, \tag{4}$$

$$\Delta^B = \prod_{i=1}^{n}\Big(x_i^{1/2} - x_i^{-1/2}\Big) \prod_{1\le i<j\le n} (x_i - x_j)(1 - \frac{1}{x_i x_j})\,, \tag{5}$$

$$\Delta^C = \prod_{i=1}^{n}\big(x_i - x_i^{-1}\big) \prod_{1\le i<j\le n} (x_i - x_j)(1 - \frac{1}{x_i x_j})\,, \tag{6}$$

$$\Delta^D = \prod_{1\le i<j\le n} (x_i - x_j)(1 - \frac{1}{x_i x_j})\,. \tag{7}$$

Taking now the images of general dominant monomials, one obtains Weyl's expressions of the characters of the linear, symplectic or orthogonal groups [15] . For a partition $\lambda \in \mathbb{N}^n$, (with $\lambda_n = 0$ in type D) the quotient

$$\left(\sum_w (-1)^{\ell(w)} x^{(\lambda+\rho)w}\right)\left(\sum_w (-1)^{\ell(w)} x^{\rho w}\right)^{-1}$$

is equal to

$$s_\lambda(\mathbf{x}) \quad , \quad \text{type } A\,, \tag{8}$$

$$Sp_\lambda(\mathbf{x}') \quad , \quad \text{type } C\,, \tag{9}$$

$$\mathcal{O}_\lambda(\mathbf{x}'') \quad , \quad \text{type } B\,, \tag{10}$$

$$\mathcal{O}_\lambda(\mathbf{x}') \quad , \quad \text{type } D\,, \tag{11}$$

where $\mathbf{x}' = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$, $\mathbf{x}'' = \{x_1, \ldots, x_n, 1, x_1^{-1}, \ldots, x_n^{-1}\}$.

For a combinatorial interpretation in terms of lattice paths, we refer to Chen, Li and Louck [4].

In the remainder of this text, we shall be concerned with the generalization of these characters by Demazure.

4 Divided differences and key polynomials

Restricting to $n = 1, 2$, one can interpret Weyl's formulas as operators on the ring of polynomials in one or two variables. These operators are similar to Newton's divided differences. They are called *Demazure operators* [3], or *isobaric divided differences* [7].

More specifically, for each type A, B, C, D, one defines two families of divided differences acting on functions of $x_1, \ldots, x_n$, and written on the right.

The first family is

$$\begin{aligned}
\pi_i \quad &: \quad f \longmapsto f\,\pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}}\ , \, 1 \le i < n\,, \\
\pi_n^C \quad &: \quad f \longmapsto f\,\pi_n^C := \frac{x_n f - x_n^{-1} f^{s_n}}{x_n - x_n^{-1}}\ , \\
\pi_n^B \quad &: \quad f \longmapsto f\,\pi_n^B := \frac{x_n f - f^{s_n}}{x_n - 1}\ , \\
\pi_n^D \quad &: \quad f \longmapsto f\,\pi_n^D := \frac{f - x_{n-1}^{-1} x_n^{-1} f^{\tau_n}}{1 - x_{n-1}^{-1} x_n^{-1}}\ .
\end{aligned}$$

It is convenient to interpolate between the operators π_n^B and π_n^C and define

$$\pi_n^{BC} : \ f(x_1, \ldots, x_n) \longmapsto \frac{(x_n + \beta) f - (x_n^{-1} + \beta) f^{s_n}}{x_n - x_n^{-1}}\ .$$

One sees that π_n^C is recovered by putting $\beta = 0$, while π_n^B corresponds to $\beta = 1$. This operator results from the representation of the Hecke algebra of type $\widetilde{C}_n$, defined by Noumi (cf. Sahi [14, 2.4]).

The second family is

$$\widehat{\pi}_i^{\heartsuit} = \widehat{\pi}_i := \pi_i - 1\,,\ 1 \le i < n\,,$$

and

$$\widehat{\pi}_n^{\heartsuit} = \pi_n^{\heartsuit} - 1\ , \heartsuit = B, C, D, BC\,.$$

Each family satisfies the braid relations for type A, B, C, D respectively [3]. Notice that the operators π_i (resp. $\widehat{\pi}_i$) , $1 \le i \le n$, commute with the multiplication by functions invariant under s_i, and that π_n^D (resp. $\widehat{\pi}_n^D$) commutes with the multiplication by functions invariant under τ_n. Thus, computations with a single π_i, $i \le n$ are reduced to an action on the linear span of $1, x_i$. In particular, it is immediate to obtain that each operator satisfies the following quadratic relations (which are degenerate cases of the Hecke relations).

Lemma 1 *The squares of the isobaric divided differences satisfy*

$$\pi_i \pi_i = \pi_i\,, \qquad \widehat{\pi}_i \widehat{\pi}_i = -\widehat{\pi}_i\,,\ 1 \le i < n\,,$$
$$\pi_n^\heartsuit \pi_n^\heartsuit = \pi_n^\heartsuit\,, \qquad \widehat{\pi}_n^\heartsuit \widehat{\pi}_n^\heartsuit = -\widehat{\pi}_n^\heartsuit\,, \heartsuit = B, C, D, BC\ .$$

We define the *key polynomials of type* $\heartsuit$, for $\heartsuit = A, B, C, D, BC$, to be the images of dominant monomials under products of isobaric divided differences. For type A, B, C, D, these are the *Demazure characters*. Using the divided differences $\widehat{\pi}_i$ instead of π_i, one obtains a second family of key polynomials.

In more details, we start with all dominant monomials x^v and put

$$x^v = K_v^\heartsuit = \widehat{K}_v^\heartsuit\,.$$

The other polynomials are defined recursively by

$$K_v^\heartsuit\,\pi_i = K_{v\,s_i}^\heartsuit \ \ \&\ \widehat{K}_v^\heartsuit\,\widehat{\pi}_i = \widehat{K}_{v\,s_i}^\heartsuit\,, \text{ when } v_i > v_{i+1}\,,\ i < n\,. \tag{12}$$

$$K_v^\heartsuit\,\pi_n^\heartsuit = K_{v\,s_n}^\heartsuit \ \ \&\ \widehat{K}_v^\heartsuit\,\widehat{\pi}_n^\heartsuit = \widehat{K}_{v\,s_n}^\heartsuit\,, \text{ when } v_n > 0\,, \text{ for } \heartsuit = B, C, BC\,. \tag{13}$$

$$K_v^D\,\pi_n^D = K_{v\,\tau_n}^D \ \&\ \widehat{K}_v^D\,\widehat{\pi}_n^D = \widehat{K}_{v\,\tau_n}^D\,,\ v_{n-1} + v_n > 0\,. \tag{14}$$

The definition is consistent since the operators satisfy the braid relations. Notice that, when $v \in \mathbb{N}^n$, then all $K_v^\heartsuit$ (resp. $\widehat{K}_v^\heartsuit$), $\heartsuit = A, B, C, D, BC$ coincide with each other, since the exceptional generators s_n or τ_n are not used in the computation. In that case, we shall write K_v, $\widehat{K}_v$, ignoring the types. We shall also need to use at the same time operators acting on $x_1, \ldots, x_n$, and operators acting on $y_1, \ldots, y_n$. In that case, we use superscripts.

The images of a dominant monomial x^v under the maximal divided difference $\pi_\omega^\heartsuit$, for $\heartsuit = A, C, B, D$, are respectively the RHS of Eq. (8), (9), (10), (11).

For $\heartsuit = BC$, and $\beta = -1$, one recovers the *odd symplectic characters* of Proctor [13, Prop. 7.3].

Divided differences can be extended to operators on paths. We refer specially to the work of Littelmann [9, 10].

5 Cauchy-type Kernels

In this section, we shall show that all the kernels $\Omega^\heartsuit$, $\heartsuit = A, B, C, D, BC$, are diagonal in the basis of key polynomials. In fact, our computations will essentially be reduced to the following cases, the verifications of which are immediate.

$$\begin{aligned}
(1-ax_i)^{-1}\,\pi_i &= (1-ax_i)^{-1}(1-ax_{i+1})^{-1} && (15)\\
(1-ax_i)^{-1}\,\widehat{\pi}_i &= ax_{i+1}(1-ax_i)^{-1}(1-ax_{i+1})^{-1} && (16)\\
(1-ax_{i+1})\,\pi_i &= (1-a/x_i)\,\pi_i \;=\; 1\,,\; 1\le i<n\,, && (17)\\
(1-ax_{i+1})(1-b/x_i)\,\pi_i &= 1-ab\,,\; 1\le i<n\,, && (18)\\
(1-b/x_n)\,\pi_n^{BC} &= 1+\beta b\,, && (19)\\
(1-b/x_{n-1})(1-b/x_n)\,\pi_n^{D} &= 1-b^2. && (20)
\end{aligned}$$

We introduce the operator

$$\Xi_n := \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}^x_\sigma\,\pi^y_{\sigma\omega}\,,$$

where ω is the maximal element in $\mathfrak{S}_n$. Filtering the set of permutations according to the position of n, one gets the following factorization.

Lemma 2 *We have*

$$\Xi_n = \Xi_{n-1}\left(\sum_{i=0}^{n-1} \widehat{\pi}^x_{[n-1:i]}\,\pi^y_{[n-1:n-1-i]}\right)\,, \tag{21}$$

where

$$\pi_{[n-1:i]} := \pi_{n-1}\,\pi_{n-2}\cdots\pi_{n-i}\,.$$

For example, the element Ξ_4 factorizes as

$$\Xi_4 = \Xi_3\,\left(\pi^y_3\pi^y_2\pi^y_1 + \widehat{\pi}^x_3\pi^y_3\pi^y_2 + \widehat{\pi}^x_3\widehat{\pi}^x_2\pi^y_3 + \widehat{\pi}^x_3\widehat{\pi}^x_2\widehat{\pi}^x_1\right)\,.$$

The next proposition shows that the operator Ξ_n allows to obtain the kernel Ω^A from the generating function of dominant monomials.

Proposition 3 *We have*

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\cdots(1-x_1\cdots x_ny_1\cdots y_n)}\,\Xi_n = \frac{1}{\prod_{i+j\le n+1} 1-x_iy_j} \;=\; \Omega^A\,. \tag{22}$$

Proof. The factor $(1-x_1\cdots x_n y_1\cdots y_n)^{-1}$ commutes with all the divided differences $\widehat{\pi}_i^x$, π_i^y, $1\le i\le n-1$. Using the above factorization of Ξ_n, and supposing the proposition true for $n-1$, one has to compute the image of $\prod_{i+j\le n}(1-x_iy_j)^{-1}$ under the sum

$$\sum_{i=0}^{n-1} \widehat{\pi}^x_{[n-1:i]}\,\pi^y_{[n-1:n-1-i]}.$$

By (16), one obtains

$$\prod_{i+j\le n}(1-x_iy_j)^{-1}\,\widehat{\pi}^x_{n-1}\cdots\widehat{\pi}^x_k = \prod_{i+j\le n}(1-x_iy_j)^{-1}\frac{x_ny_1}{1-x_ny_1}\cdots\frac{x_{k+1}y_{n-k}}{1-x_{k+1}y_{n-k}}\,.$$

Thanks to (15), the action of $\pi^y_{n-1}\cdots\pi^y_{n-k+1}$ on this last function reduces to multiplication by

$$\frac{1}{1-x_1y_n}\,\frac{1}{1-x_2y_{n-1}}\cdots\frac{1}{1-x_{k-1}y_{n-k+2}}\,.$$

Reducing now the sum to a common denominator, it can be rewritten as the product of Ω^A times the factor

$$\sum_{k=1}^{n-1} x_n\ldots x_{k+1}y_1\ldots y_{n-k}(1-x_ky_{n-k+1}) + (1-x_ny_1).$$

This last factor is equal to $(1-x_1\cdots x_ny_1\cdots y_n)$ which commutes with all the divided differences. This completes the proof. ∎

Lemma 4 *Let Φ^{BC} be the following operator acting on the variables $y_1,\ldots,y_n$:*

$$\Phi^{BC} := \left(\pi_n^{BC}\pi_{n-1}\cdots\pi_1\right)\left(\pi_n^{BC}\pi_{n-1}\cdots\pi_2\right)\cdots\left(\pi_n^{BC}\pi_{n-1}\right)\left(\pi_n^{BC}\right).$$

Then

$$\Omega^A\,\Phi^{BC} = \Omega^{BC}.$$

Proof. Thanks to (18) and (19), one has

$$\begin{aligned}\frac{f}{(1-x'y_i)(1-x/y_{i+1})}\pi_i &= \frac{(1-x'y_{i+1})(1-x/y_i)\pi_i\,f}{(1-x'y_i)(1-x'y_{i+1})(1-x/y_{i+1})(1-x/y_i)}\\ &= \frac{(1-xx')\,f}{(1-x'y_i)(1-x'y_{i+1})(1-x/y_{i+1})(1-x/y_i)}\,,\end{aligned}$$

where $i < n$, and $f = f^{s_i}$, and

$$\frac{f}{1 - xy_n}\,\pi_n^{BC} = (1 - x/y_n)\,\pi_n^{BC}\,\frac{f}{(1 - xy_n)(1 - x/y_n)}$$
$$= (1 + \beta x)\,\frac{f}{(1 - xy_n)(1 - x/y_n)}$$

where $f = f^{s_n}$.

By above two types of computations, we have

$$\begin{aligned}
&\Omega^A\left(\pi_n^{BC}\pi_{n-1}\cdots\pi_1\right) = \left(\frac{1}{1 - x_1y_n}\pi_n^{BC}\right)\frac{1 - x_1y_n}{\prod_{i+j\le n+1}(1 - x_iy_j)}\left(\pi_{n-1}\cdots\pi_1\right)\\
= \;& \Omega^A\,\frac{(1 + \beta x_1)}{(1 - x_1/y_n)}\left(\pi_{n-1}\cdots\pi_1\right)\\
= \;& \left(\frac{1}{(1 - x_1/y_n)(1 - x_2y_{n-1})}\pi_{n-1}\right)\frac{(1 - x_2y_{n-1})(1 + \beta x_1)}{\prod_{i+j\le n+1}(1 - x_iy_j)}\left(\pi_{n-2}\cdots\pi_1\right)\\
= \;& \Omega^A\,\frac{(1 + \beta x_1)(1 - x_1x_2)}{(1 - x_1/y_n)(1 - x_1/y_{n-1})(1 - x_2y_n)}\left(\pi_{n-2}\cdots\pi_1\right)\\
= \;& \left(\frac{1}{(1 - x_1/y_{n-1})(1 - x_3y_{n-2})}\pi_{n-2}\right)\\
&\cdot\frac{(1 - x_3y_{n-2})(1 + \beta x_1)(1 - x_1x_2)}{\prod_{i+j\le n+1}(1 - x_iy_j)(1 - x_1/y_n)(1 - x_2y_n)}\left(\pi_{n-3}\cdots\pi_1\right)\\
= \;& \cdots\\
= \;& \Omega^A\,\frac{(1 + \beta x_1)\prod_{i=2}^{n-1}(1 - x_1x_i)}{\prod_{i=2}^{n}(1 - x_1/y_i)\prod_{i=2}^{n-1}(1 - x_iy_{n-i+2})}\left(\pi_1\right)\\
= \;& \left(\frac{1}{(1 - x_ny_1)(1 - x_1/y_2)}\pi_1\right)\frac{1 - x_ny_1}{\prod_{i+j\le n+1}(1 - x_iy_j)}\\
&\cdot\frac{(1 + \beta x_1)\prod_{i=2}^{n-1}(1 - x_1x_i)}{\prod_{i=3}^{n}(1 - x_1/y_i)\prod_{i=2}^{n-1}(1 - x_iy_{n-i+2})}\\
= \;& \Omega^A\,\frac{(1 + \beta x_1)\prod_{i=2}^{n}(1 - x_1x_i)}{\prod_{i=1}^{n}(1 - x_1/y_i)\prod_{i=2}^{n}(1 - x_iy_{n-i+2})},
\end{aligned}$$

which implies that

$$\Omega^A\left(\pi_n^{BC}\pi_{n-1}\cdots\pi_1\right) = \left(\frac{1}{\prod_{i=1}^{n}(1 - x_iy_{n-i+1})}\pi_n^{BC}\pi_{n-1}\cdots\pi_1\right)\frac{1}{\prod_{i+j\le n}(1 - x_iy_j)}.$$

Therefore, we have

$$
\begin{aligned}
\Omega^A\,\Phi^{BC} &= \frac{1}{\prod_{i=2}^n(1-x_iy_{n-i+2})}\big(\pi_n^{BC}\pi_{n-1}\cdots\pi_2\big)\cdots\big(\pi_n^{BC}\big)\\
&\quad\cdot\Omega^A\,\frac{(1+\beta x_1)\prod_{i=2}^n(1-x_1x_i)}{\prod_{i=1}^n(1-x_1/y_i)}\\
&= \frac{(1+\beta x_2)\prod_{i=3}^n(1-x_2x_i)}{\prod_{i=2}^n(1-x_2/y_i)\prod_{i=3}^n(1-x_iy_{n-i+3})}\big(\pi_n^{BC}\pi_{n-1}\cdots\pi_3\big)\cdots\big(\pi_n^{BC}\big)\\
&\quad\cdot\Omega^A\,\frac{(1+\beta x_1)\prod_{i=2}^n(1-x_1x_i)}{\prod_{i=1}^n(1-x_1/y_i)\prod_{i=2}^n(1-x_iy_{n-i+2})}\\
&= \frac{1}{\prod_{i=3}^n(1-x_iy_{n-i+3})}\big(\pi_n^{BC}\pi_{n-1}\cdots\pi_3\big)\cdots\big(\pi_n^{BC}\big)\\
&\quad\cdot\Omega^A\,\frac{\prod_{i=1}^2(1+\beta x_i)\prod_{i=1}^2\prod_{j=i+1}^n(1-x_ix_j)}{\prod_{i=2}^n(1-x_iy_{n-i+2})\prod_{i=1}^2\prod_{j=i}^n(1-x_i/y_j)}\\
&= \cdots\\
&= \left(\frac{1}{1-x_ny_n}\pi_n^{BC}\right)\Omega^A\,\frac{\prod_{i=1}^{n-1}(1+\beta x_i)\prod_{1\le i<j\le n}(1-x_ix_j)}{\prod_{i=2}^{n-1}\prod_{j=i}^n(1-x_jy_{n-j+i})\prod_{\substack{1\le i\le j\le n\\ i\ne n}}(1-x_i/y_j)}\\
&= \Omega^A\,\frac{\prod_{i=1}^n(1+\beta x_i)\prod_{1\le i<j\le n}(1-x_ix_j)}{\prod_{i=2}^n\prod_{j=i}^n(1-x_jy_{n-j+i})\prod_{1\le i\le j\le n}(1-x_i/y_j)} = \Omega^{BC}.
\end{aligned}
$$

∎

To treat the type D, we define recursively the following operators (still acting on $y_1,\ldots,y_n$ only) :

$$
\Phi_2^D = \pi_1\pi_2^D,\ \Phi_3^D = \big(\pi_2\pi_3^D\big)\ \pi_1\pi_2\pi_3^D\,,\ldots,
$$

$$
\Phi_n^D = \big(\Phi_{n-1}^D\big)^+\ \pi_1\pi_2\cdots\pi_{n-1}\pi_n^D\,, \quad (23)
$$

where the symbol $(\,)^+$ denotes the shift $i\to i+1$ of all indices inside the parentheses. For example, taking $n=4$, we have

$$
\Phi_4^D = \big(\pi_3\pi_4^D\big)\ \big(\pi_2\pi_3\pi_4^D\big)\ \big(\pi_1\pi_2\pi_3\pi_4^D\big)\ .
$$

Lemma 5 *Let*

$$
\Omega_{n-1}^A = \prod_{i+j\le n}(1-x_iy_j)^{-1}.
$$

Then

$$
\Omega_{n-1}^A\,\Phi_n^D = \Omega^D\,.
$$

Proof. The computation is similar to Lemma 4 except that the successive steps in the computation of the image of Ω^A_{n-1} are of three possible types.

Step π_{n-1}. The current function is $(1-xy_{n-1})^{-1}f$, where f is symmetrical in y_n and y_{n-1}. Thanks to (15), we have

$$\frac{f}{1-xy_{n-1}}\pi_{n-1} = \frac{f}{(1-xy_n)(1-xy_{n-1})}\,.$$

We have just created a factor $(1-xy_n)^{-1}$.

Step π^D_n. The current function is $(1-xy_{n-1})^{-1}(1-xy_n)^{-1}f$, where f is invariant under τ_n. Thanks to (20), we have

$$\begin{aligned}\frac{f}{(1-xy_{n-1})(1-xy_n)}\,\pi^D_n &= \frac{(1-x/y_{n-1})(1-x/y_n)\pi^D_n\, f}{(1-xy_{n-1})(1-xy_n)(1-x/y_{n-1})(1-x/y_n)}\\ &= \frac{(1-x^2)\, f}{(1-xy_{n-1})(1-xy_n)(1-x/y_{n-1})(1-x/y_n)}\end{aligned}$$

and the transformation is just multiplication by

$$(1-x^2)\,(1-x/y_{n-1})^{-1}(1-x/y_n)^{-1}\,.$$

Step π_i, $i<n-1$. The current function is $(1-x'y_i)^{-1}(1-x/y_{i+1})^{-1}\,f$, where $f=f^{s_i}$. Thanks to (18), one has

$$\begin{aligned}\frac{f}{(1-x'y_i)(1-x/y_{i+1})}\,\pi_i &= \frac{(1-x'y_{i+1})(1-x/y_i)\pi_i\, f}{(1-x'y_i)(1-x'y_{i+1})(1-x/y_i)(1-x/y_{i+1})}\\ &= \frac{(1-xx')\, f}{(1-x'y_i)(1-x'y_{i+1})(1-x/y_i)(1-x/y_{i+1})}\,.\end{aligned}$$

The function has been multiplied by

$$(1-xx')\,(1-x'y_{i+1})^{-1}(1-x/y_i)^{-1}\,.$$

Finally, the product of all the above factors is

$$\prod_{1\le i\le j\le n-1}(1-x_ix_j)\prod_{i=1}^{n-1}\prod_{j=i}^{n-1}(1-x_jy_{n-j+i})^{-1}\prod_{i=1}^{n-1}\prod_{j=i}^{n}(1-x_i/y_j)^{-1}\,,$$

which is indeed equal to the quotient to Ω^D/Ω^A_{n-1}. ∎

The preceding relations between the different kernels, and the function $(1-x_1y_1)^{-1}(1-x_1x_2y_1y_2)^{-1}\cdots$ allow to expand these kernels.

Theorem 6 *We have*

$$\Omega^A = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K_{v\omega}(\mathbf{y}), \tag{24}$$

$$\Omega^{BC} = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K^{BC}_{-v}(\mathbf{y}), \tag{25}$$

$$\Omega^D = \sum_{v\in\mathbb{N}^n:\, v_n=0} \widehat{K}_v(\mathbf{x})\, K^{D}_{-v}(\mathbf{y}), \tag{26}$$

where x_n is specialized to 0 *in the last equation.*

Proof. Note that

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\cdots(1-x_1\cdots x_ny_1\cdots y_n)}$$

is the generating function of dominant monomials $x^\lambda y^\lambda$ in n indeterminates $x_1y_1,\ldots,x_ny_n$.

From the definitions of Ξ_n and of key polynomials, one has

$$\sum_\lambda x^\lambda y^\lambda\, \Xi_n = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K_{v\omega}(\mathbf{y})\,,$$

where the sum ranges over all partitions λ of length at most n. Thus, Proposition 3 entails (24).

The image of a key polynomial $K_{v_n,\ldots,v_1}(\mathbf{y})$, $v\in\mathbb{N}^n$ under Φ^{BC} is $K_{-v}(\mathbf{y})$. Therefore, the image of the RHS of (24) under Φ^{BC} is the RHS of (25), and Lemma 4 gives (25).

Similarly, the image of $K_{v_{n-1},\ldots,v_1,0}(\mathbf{y})$ under Φ^D_n is $K_{-v_1,-v_2,\ldots,-v_{n-1},0}(\mathbf{y})$. Therefore, the image of the expansion of Ω^A_{n-1} under Φ^D_n is the RHS of (26), and Lemma (5) completes the proof of (26) and of the theorem. ∎

Note that (24) has been established combinatorially in [8], using the Schensted bijection and double crystal graphs.

Let us conclude this section by showing that the identities (24), (25) and (26) imply the Cauchy formula and Littlewood's formulas respectively. Indeed, $\widehat{\pi}_i\pi_i = 0$, $1\le i<n$, a fortiori, $\widehat{\pi}_i\pi_\omega = 0$, where ω is the maximal element of $\mathfrak{S}_n$. Therefore all the summands in the right hand sides of (24), (25) and (26) are sent to 0 under π^x_ω, except the terms

$$\widehat{K}_\lambda(\mathbf{x})\,\pi^x_\omega = K_{\lambda\omega}(\mathbf{x}) = s_\lambda(\mathbf{x}).$$

On the other hand,

$$\prod_{i+j\le n+1}(1-x_iy_j)^{-1}\,\pi^x_\omega \;=\; \prod_{i,j=1}^{n}(1-x_iy_j)^{-1}$$
$$\prod_{1\le i\le j\le n}(1-x_i/y_j)^{-1}\,\pi^x_\omega \;=\; \prod_{i,j=1}^{n}(1-x_i/y_j)^{-1}$$

Specializing β, we get Cauchy's formula (1) and Littlewood's identities (2), (3), as images of (24), (25) and (26) respectively.

6 Scalar products

Bogdan Ion [5, 6] has shown how to obtain the two families of Demazure characters $K^\heartsuit_v$, $\widehat{K}^\heartsuit_v$, $\heartsuit = A, B, C, D$, by degeneration of Macdonald polynomials. Degenerating Cherednik's scalar product [2], one gets a scalar product for each of the types $\heartsuit$, with respect to which the bases $\{K^\heartsuit_v\}$, $\{\widehat{K}^\heartsuit_v\}$ are adjoint of each other. But instead of having recourse to the elaborate theory of non symmetric Macdonald polynomials, we shall directly define scalar products on polynomials, and check orthogonality properties by simple recursions.

Recall that in the theory of Schubert polynomials [7], one defines a scalar product by using the maximal divided difference; as a consequence divided differences are self-adjoint. This scalar product can also be written as

$$(f,g) = CT\left(fg\prod_{1\le i,j\le n}(x_i^{-1}-x_j^{-1})\right)\,,$$

where CT means "constant term".

It is easy to adapt this definition to our present needs, keeping the compatibility of the scalar product with the isobaric divided differences.

One first replaces the Vandermonde determinant by Weyl's denominators Δ^B, Δ^C and Δ^D multiplied by x^ρ. We add to their list

$$\Delta^{BC} := \Delta^C \prod\nolimits_{i=1}^{n}(1+\beta x_i)^{-1}\,, \tag{27}$$

keeping $\rho^{BC} = \rho^C = [n,\ldots,1]$.

Definition 7 *For $\heartsuit = B, C, BC$, and for Laurent polynomials f, g in $x_1, \ldots, x_n$, let*

$$(f,g)^{\heartsuit} = CT\left(fg\,(-1)^n x^{\rho^{\heartsuit}}\,\Delta^{\heartsuit}\right), \tag{28}$$

$$(f,g)^{D} = CT\left(fg\,x^{\rho^{D}}\,\Delta^{D}\right), \tag{29}$$

$$(f,g)^{A} = CT\Big(f(x_1,\ldots,x_n)g(x_n^{-1},\ldots,x_1^{-1}) \prod_{1\le i<j\le n}(1-x_ix_j^{-1})\Big), \tag{30}$$

where one expands $(1+\beta x_i)^{-1}$ as a formal series in the variable $x_i\beta$.

For example, taking $n = 2$, one has

$$\begin{aligned}
(f,g)^{A} &= CT\left(f(x_1,x_2)\,g(x_2^{-1},x_1^{-1})\,(1-x_1x_2^{-1})\right),\\
(f,g)^{BC} &= CT\left(\frac{fg\,x_1^2x_2\,(x_1-x_1^{-1})(x_2-x_2^{-1})(x_1-x_2)(1-x_1^{-1}x_2^{-1})}{(1+x_1\beta)(1+x_2\beta)}\right),\\
(f,g)^{D} &= CT\Big(fg\,x_1\,(x_1-x_2)(1-\frac{1}{x_1x_2})\Big).
\end{aligned}$$

Notice that the scalar product $(f,g)^{BC}$ does specialize to $(f,g)^B$ for $\beta = 1$, and to $(f,g)^C$ for $\beta = 0$:

$$\begin{aligned}
(f,g)^{B} &= CT\Big(fgx_1^{3/2}x_2^{1/2}(x_1^{1/2}-x_1^{-1/2})(x_2^{1/2}-x_2^{-1/2})(x_1-x_2)(1-\frac{1}{x_1x_2})\Big)\\
&= CT\Big(fg\,x_1(x_1-1)(x_2-1)(x_1-x_2)(1-\frac{1}{x_1x_2})\Big),\\
(f,g)^{C} &= CT\left(fg\,x_1^2x_2\,(x_1-\frac{1}{x_1})(x_2-\frac{1}{x_2})(x_1-x_2)(1-\frac{1}{x_1x_2})\right).
\end{aligned}$$

The crucial property of the scalar products (28) and (29) is the following compatibility with isobaric divided differences.

Theorem 8 *Write $\pi_n = \pi_n^{\heartsuit}$, $\widehat{\pi}_n = \widehat{\pi}_n^{\heartsuit}$, for $\heartsuit = B, C, BC, D$. Then the operators π_i and $\widehat{\pi}_i$ $(1 \le i \le n)$ are self-adjoint with respect to $(\,,\,)^{\heartsuit}$, i.e. for every pair of Laurent polynomials f, g, one has*

$$\big(f\pi_i\,,\,g\big)^{\heartsuit} = \big(f\,,\,g\pi_i\big)^{\heartsuit}, \quad \big(f\widehat{\pi}_i\,,\,g\big)^{\heartsuit} = \big(f\,,\,g\widehat{\pi}_i\big)^{\heartsuit}.$$

In the case of type A, for $1 \le i \le n-1$, π_i (resp. $\widehat{\pi}_i$) is adjoint to π_{n-i} (resp. $\widehat{\pi}_{n-i}$), i.e. for every pair of Laurent polynomials f, h, one has

$$\big(f\pi_i\,,\,h\big)^A = \big(f\,,\,h\pi_{n-i}\big)^A, \quad \big(f\widehat{\pi}_i\,,\,h\big)^A = \big(f\,,\,h\widehat{\pi}_{n-i}\big)^A.$$

Proof. To treat all types in a uniform way, we write $h(x_n^{-1},\ldots,x_1^{-1}) = g(x_1,\ldots,x_n)$. Then $(h\pi_{n-i})(x_n^{-1},\ldots,x_1^{-1}) = (g\pi_i)(x_1,\ldots,x_n)$, $1 \le i < n$, and

$$(f, h\pi_{n-i})^A = CT\Big(f\,(g\pi_i) \prod_{1\le i<j\le n} (1 - x_i x_j^{-1})\Big)\,.$$

For all types, and $i < n$, the scalar product can be written as

$$CT\Big(CT_{x_i,x_{i+1}}\,(fg(1 - x_i/x_{i+1})\clubsuit)\Big),$$

where $\clubsuit$ is a function symmetrical in x_i, x_{i+1} and $CT_{x_i,x_{i+1}}$ is the constant term in the variables x_i, x_{i+1} only.

Let us write f, g as $f = f_1 + x_{i+1}f_2$, $g = g_1 + x_{i+1}g_2$, with f_1, f_2, g_1, g_2 invariant under s_i. The difference $f\pi_i g - g\pi_i f = f\widehat{\pi}_i g - g\widehat{\pi}_i f$ is equal to $(f_1g_2 - g_1f_2)x_{i+1}$. Therefore the constant term

$$\begin{aligned} &CT_{x_i,x_{i+1}}\Big((f\pi_i g - g\pi_i f)\,(1 - x_i/x_{i+1})\clubsuit\Big) \\ &\qquad = CT_{x_i,x_{i+1}}\Big((f\widehat{\pi}_i g - g\widehat{\pi}_i f)\,(1 - x_i/x_{i+1})\clubsuit\Big) \\ &\qquad = CT_{x_i,x_{i+1}}\Big((x_{i+1} - x_i)\,(f_1g_2 - g_1f_2)\clubsuit\Big) \end{aligned}$$

is null, because the function inside parentheses is antisymmetrical in x_i, x_{i+1}.

In the case $i = n$, $\heartsuit = BC$, one writes

$$(f\,,\,g)^{BC} = CT\Big(CT_{x_n}\Big(fg\,\frac{x_n}{1+\beta x_n}(x_n - {x_n}^{-1})\clubsuit\Big)\Big)\,,$$

where $\clubsuit$ is a function invariant under s_n. Therefore, to evaluate $(f\widehat{\pi}_n\,,\,g)^{BC} - (f\,,\,g\widehat{\pi}_n)^{BC} = (f\pi_n\,,\,g)^{BC} - (f\,,\,g\pi_n)^{BC}$ one can first compute

$$CT_{x_n}\Big(\,(f\widehat{\pi}_n g - g\widehat{\pi}_n f)\,\frac{x_n}{1+\beta x_n}(x_n - x_n^{-1})\clubsuit\Big) = CT_{x_n}\Big((g^{s_n}f - f^{s_n}g)\clubsuit\Big)$$

which is null, because the function under parentheses is alternating under s_n.

Similarly, for $\heartsuit = D$, neglecting a function invariant under τ_n, to determine $(f\widehat{\pi}_n\,,\,g)^D - (f\,,\,g\widehat{\pi}_n)^D = (f\pi_n\,,\,g)^D - (f\,,\,g\pi_n)^D$, one can first compute

$$CT_{x_{n-1},x_n}\Big((f\widehat{\pi}_n g - g\widehat{\pi}_n f)(1 - x_{n-1}x_n)\Big) = CT_{x_{n-1},x_n}\Big(f^{\tau_n}g - g^{\tau_n}f\Big)$$

which is also null, because the function $f^{\tau_n}g - g^{\tau_n}f$ is alternating under τ_n. This completes the proof. ∎

7 Orthogonality

Let us extend the usual dominance order on partitions [12] to an order on vectors in $\mathbb{Z}^n$. Given two vectors $u = [u_1, u_2, \ldots, u_n]$ and $v = [v_1, v_2, \ldots, v_n]$ in $\mathbb{Z}^n$, $u \le v$ means the following inequalities

$$u_1 \le v_1,\ u_1 + u_2 \le v_1 + v_2,\ u_1 + u_2 + u_3 \le v_1 + v_2 + v_3,\ \ldots.$$

One also extends the notation $|\lambda|$ to vectors: $|v| := v_1 + \cdots + v_n$.

We give in the following lemmas some easy properties of the scalar products.

Lemma 9 *For every dominant vector λ, for every group element w, then every monomial x^u appearing in the expansion of $x^\lambda \pi_w$ is such that $u \ge \lambda\omega^\heartsuit$.*

Proof. By recursion on length, one sees that $K_v^\heartsuit$ is equal to $x^v + \sum c_u^v x^u$, with $v < u$. ∎

Notice that for type B, C, D with n even, then $\lambda\omega^\heartsuit = -\lambda$. When n is odd, then $\lambda\omega^D = [-\lambda_1, \ldots, -\lambda_{n-1}, \lambda_n]$.

Lemma 10 *For $u, v \in \mathbb{Z}^n$, $\heartsuit \ne A$,*

$$(x^v, x^u)^\heartsuit \ne 0 \quad \textit{implies that} \quad v \le -u\,.$$

For $v, u \in \mathbb{N}^n$,

$$(x^v, x^u)^A \ne 0 \quad \textit{implies that} \quad v \le u\omega, \quad \textit{and} \quad |v| = |u|\,.$$

Proof. Rewrite $x^\rho \Delta^C$ as the determinant

$$\det\left(x_i^{j-i}(x_i^{2n-2j+2} - 1)\right)_{i,j=1}^n.$$

If one expands the determinant by rows, then the powers of x_1 are nonnegative, the term x_2^{-1} is multiplied by strictly positive powers of x_1, the term x_3^{-2} is mutliplied by monomials in x_1, x_2 of degree at least 2, $\cdots$. Therefore, the scalar product $(x^v, x^u)^C$ can have a constant term only if

$$v_1 + u_1 \le 0, v_1 + v_2 + u_1 + u_2 \le 0, v_1 + v_2 + v_3 + u_1 + u_2 + u_3 \le 0, \ldots,$$

i.e. $v \le -u$.

For $(x^v, x^u)^\heartsuit$, $\heartsuit = B, D$, the proof is similar. For $(x^v, x^u)^{BC}$, we have multiplied $(x^v, x^u)^C$ by formal series in $x_1, x_2, \ldots$ with positive exponents. Therefore $(x^v, x^u)^{BC} \ne 0$ still implies $v \le -u$.

For $(x^v, x^u)^A$, rewrite the product $\prod_{1\le i<j\le n}(1-x_i x_j^{-1})$ as the determinant $\det(x_i^{j-i})_{i,j=1}^n$. One obtains that the scalar product $(x^v, x^u)^A$ has a constant term only if

$$v_1 - u_n \le 0, v_1 + v_2 - u_n - u_{n-1} \le 0, \ldots,$$

i.e. $v \le u\omega$. Moreover, to have a non-zero constant term, the total degree must be 0, i.e. $|v| = |u|$. ∎

Lemma 11 *Let λ and μ be two dominant vectors, such that there exists w such that $(x^\lambda \pi_w , x^\mu)^\heartsuit \neq 0$.*

Then if $\heartsuit = D$ and n is odd, one has $\lambda_1 = \mu_1, \ldots, \lambda_{n-1} = \mu_{n-1}$ and $\lambda_n = -\mu_n$. In all the other cases ($\heartsuit = D$ and n even, or $\heartsuit = A, B, C, BC$), one has $\lambda = \mu$.

Proof. In case that $(x^\lambda \pi_w , x^\mu)^\heartsuit \neq 0$, there exists at least one monomial x^v in $x^\lambda \pi_w$ such that $(x^v, x^\mu)^\heartsuit \neq 0$. When $\heartsuit \neq A$, Lemma 10 implies that $v \le -\mu$, and Lemma 9 implies that $v \ge \lambda\omega^\heartsuit$. In final $\lambda\omega^\heartsuit \le -\mu$. Reversing the role λ and μ thanks to Th. 8, one also has $\mu\omega^\heartsuit \le -\lambda$, hence $\mu = \lambda$ when $\heartsuit = B, C, BC, D$ with even n. For type D with n odd, the inequalities give $\lambda_1 = \mu_1 = -v_1, \ldots, \lambda_{n-1} = \mu_{n-1} = -v_{n-1}$, $\lambda_n \le v_n \le -\mu_n$. Such v can occur as an exponent in $x^\lambda \pi_w$ only if $v_n = \lambda_n$. However the scalar product $(x^v, x^\mu)^D = (x_n^{\lambda_n}, x_n^{\mu_n})^D$ is non-zero only for $\lambda_n + \mu_n = 0$, and this gives the required relation between λ and μ for type D, n odd. The proof for type A is similar to the proof for types B, C. ∎

Notce that if all the components of λ are different, then w must be the maximal element of the group.

Corollary 12 *Let μ be a dominant vector, then*

$$\heartsuit \neq A\,,\ (K_v\,,\, x^\mu)^\heartsuit \neq 0 \quad \textit{implies that } v = -\mu\,. \tag{31}$$

In that case $(K_{-\mu}\,,\, x^\mu)^\heartsuit = 1$.

$$(K_v\,,\, x^\mu)^A \neq 0 \quad \textit{implies that } v = \mu\omega\,. \tag{32}$$

In that case $(K_{\mu\omega}\,,\, x^\mu)^A = 1$.

For example, $[2,1,1]\omega^D = [-2,-1,1] = -[2,1,-1]$, $\left(K^D_{-2,-1,1}, x^{2,1,-1}\right) = 1$, and $[2,1]\omega^D = [-2,-1]$, $\left(K^D_{-2,-1}, x^{2,1}\right) = 1$.

Lemma 13 *Let $\heartsuit \neq A$ and $i : 1 \leq i \leq n$. Given four polynomials f_1, $f_2 = f_1\pi_i$, g_1, $g_2 = g_1\widehat{\pi}_i$, then $(f_1, g_1)^\heartsuit = 0$ & $(f_2, g_1)^\heartsuit = 1$, implies that*

$$(f_1, g_2)^\heartsuit = 1 \qquad \& \qquad (f_2, g_2)^\heartsuit = 0\,.$$

Moreover, any space V stable under π_i which is orthogonal to g_1 is orthogonal to g_2.

Proof. We have

$$(f_2, g_2)^\heartsuit = (f_1\pi_i\,,\, g_1\widehat{\pi}_i)^\heartsuit = (f_1\pi_i\widehat{\pi}_i, g_1)^\heartsuit = 0\,,$$

and

$$(f_1, g_2)^\heartsuit = (f_1,\, g_1\widehat{\pi}_i)^\heartsuit = (f_1, g_1\pi_i - g_1)^\heartsuit = (f_1\pi_i, g_1)^\heartsuit = (f_2, g_1)^\heartsuit = 1\,.$$

The last statement is immediate. ∎

The next lemma has a similar proof.

Lemma 14 *Given an integer i $(1 \leq i \leq n-1)$ and four polynomials f_1, $f_2 = f_1\pi_{n-i}$, g_1, $g_2 = g_1\widehat{\pi}_i$, then $(f_1, g_1)^A = 0$ & $(f_2, g_1)^A = 1$ implies that*

$$(f_1, g_2)^A = 1 \qquad \& \qquad (f_2, g_2)^A = 0\,.$$

Moreover, any space V stable under π_{n-i} which is orthogonal to g_1 is orthogonal to g_2.

We are now ready to conclude.

Theorem 15 *Let $u, v \in \mathbb{Z}^n$, and $\heartsuit \neq A$. Then*

$$(K_v\,,\, \widehat{K}_u)^\heartsuit = \delta_{-v,u}\,, \tag{33}$$

where, as usual, $\delta_{-v,u}$ is the Kronecker delta.

In the case of type A, for $u, v \in \mathbb{N}^n$, we have

$$(K_v\,,\, \widehat{K}_u)^A = \delta_{v\omega,u}\,. \tag{34}$$

Proof. When u is dominant, (31) implies that $(K_v\,,\, x^u)^\heartsuit = \delta_{-v,u}$. By induction on length, suppose that u is such that $\widehat{K}_u$ is orthogonal to every K_v, except $(K_{-u}, \widehat{K}_u)^\heartsuit = 1$. Take i such that the linear span of $\widehat{K}_u, \widehat{K}_u\widehat{\pi}_i$ is two-dimensional. Then one uses Lemma 13, with $f_1 = K_{-us_i}$, $f_2 = K_{-u} = f_1\pi_i$, $g_1 = \widehat{K}_u$, $g_2 = \widehat{K}_{us_i} = g_1\widehat{\pi}_i$, and V generated by all K_v, $v \neq -u, -us_i$, to conclude that $\widehat{K}_{us_i}$ is orthogonal to all K_v, except for $(K_{-us_i}, \widehat{K}_{us_i})^\heartsuit = 1$.

For type A, one replaces Lemma 13 by Lemma 14 to arrive to a similar conclusion.

∎

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