Minimal Universal Denominators for Linear Difference Equations

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Abstract

We provide minimal universal denominators for linear difference equations with fixed leading and trailing coefficients. In the case of first order equations, they are factors of Abramov's universal denominators. While in the case of higher order equations, we show that Abramov's universal denominators are minimal.

Keywords: linear difference equation, universal denominator, Abramov's universal denominator, minimal universal denominator.

1. Introduction

Finding rational solutions of linear difference equations with polynomial coefficients plays an important role in computer algebra. Many problems reduce to it, such as the generalization of Gosper's algorithm [12] and the problem of finding hypergeometric solutions of a non-homogenous difference equation [11].

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Let K be a field of characteristic zero. We denote by $K[n]$ and $K(n)$ the sets of polynomials in n and the rational functions of n, respectively. Consider the linear difference equation

$$\sum_{i=0}^{d} p_i(n+i)y(n+i) = p(n), \tag{1.1}$$

where $p_0(n), \ldots, p_d(n), p(n) \in K[n]$ are given polynomials such that $p_0(n)$ and $p_d(n)$ are non-zero. A polynomial $g(n) \in K[n]$ is called a *universal denominator* for (1.1) if for every rational solution $y(n) \in K(n)$ to (1.1), $g(n)y(n)$ is a polynomial. Once a universal denominator is found, then it is easy to find the rational solutions to (1.1) by finding polynomial solutions using the techniques in [1, 7, 11].

Abramov firstly developed an algorithm to find a universal denominator in [2, 3]. The algorithm given there is quite complicated and relies on all the coefficients $p_0(n), \ldots, p_d(n)$ and $p(n)$. In [4] (see also [5, 6]) Abramov presented an improved version of his algorithm which requires only the leading and tailing coefficients ($p_0(n)$ and $p_d(n)$). The resulting universal denominators are usually called Abramov's universal denominators. Barkatou [8] (see also [15]) provided an explicit formula for Abramov's universal denominators in a generalized form. Chen, Paule and Saad [9] obtained the same formula by convergence properties. Khmel'nov [10] presented a variant of van Hoeij's algorithm [14] for the scalar case which improves the Abramov's bounds but uses all coefficients.

In this article, we consider the possibility of the improvement of Abramov's algorithm. Suppose that we have an algorithm to compute a universal denominator $u(n) = u(n \,|\, p_0, p_d)$ for (1.1) using $p_0(n)$ and $p_d(n)$. If for any two polynomials $p_0(n), p_d(n)$, there exist $p_1(n), \ldots, p_{d-1}(n)$ and $p(n)$ such that (1.1) has a rational solution whose denominator is $u(n)$ (written in reduced form), then the algorithm is optimal. We call the corresponding $u(n)$ the *minimal universal denominator*. To get a universal denominator smaller than $u(n)$, we have to use more information.

For first order linear difference equations, we provide an improvement of Abramov's algorithm and prove that the resulting polynomials are the minimal universal denominators. While in the case of higher order, we show that Abramov's universal denominators are already minimal.

2. First Order Linear Difference Equations

Throughout the paper, we use $\gcd(a(n), b(n))$ to denote the monic greatest common divisor of polynomials $a(n), b(n) \in K[n]$. If $\gcd(a(n), b(n)) = 1$, then we say that the rational function $a(n)/b(n) \in K(n)$ is *reduced*.

Let $\mathbb{N}$ be the set of nonnegative integers. For two polynomials $a(n), b(n)$, we call

$$Dis(a, b) = \{h \in \mathbb{N} \mid \gcd(a(n), b(n+h)) \neq 1\}$$

the *dispersion set*. Observe that $Dis(a, b)$ can be computed as the nonnegative integer roots of the resultant polynomial $R(h) = Resultant_n(a(n), b(n+h))$.

In the case of first order linear difference equations, (1.1) becomes

$$p_0(n)y(n) + p_1(n+1)y(n+1) = p(n). \tag{2.1}$$

2.1 Universal Denominator

The following lemma is the key observation to get the universal denominator.

Lemma 2.1 *Suppose* $u_0, v_0, u_1, v_1 \in K[n]$ *be polynomials such that* $\gcd(u_0, v_0) = \gcd(u_1, v_1) = 1$. *Then*

$$\frac{u_0}{v_0} + \frac{u_1}{v_1} \in K[n] \Longrightarrow v_0/v_1 \in K.$$

Proof. Denote $g = \gcd(v_0, v_1)$. Then

$$\frac{u_0}{v_0} + \frac{u_1}{v_1} = \frac{u_0v_1/g + u_1v_0/g}{v_0v_1/g}.$$

Since

$$\gcd(u_0v_1/g + u_1v_0/g, v_0/g) = \gcd(u_0v_1/g, v_0/g) = 1,$$

$\frac{u_0}{v_0} + \frac{u_1}{v_1} \in K[n]$ implies that v_0/g is a constant. Similarly, v_1/g is also a constant. The assertion follows immediately. ∎

Now suppose $f(n)/g(n)$ be a reduced rational solution to (2.1), where $f, g \in K[n]$. Lemma 2.1 implies that when written in reduced forms, the denominators of $p_0(n)f(n)/g(n)$ and $p_1(n+1)f(n+1)/g(n+1)$ are the same up to a scalar multiple. Therefore, we may assume that

$$p_0(n)\frac{f(n)}{g(n)} = \frac{r(n)}{s(n)}, \quad p_1(n)\frac{f(n)}{g(n)} = \frac{r'(n)}{s(n-1)}, \tag{2.2}$$

where $\gcd(r(n), s(n)) = \gcd(r'(n), s(n-1)) = 1$. Note that $s(n)$ is a factor of $g(n)$, we may further assume that $g(n) = u(n)s(n)$ for a certain polynomial $u(n)$. We will see that upper bounds of $u(n)$ and $s(n)$ can be computed by Gosper's algorithm. For this, recall that in Gosper's algorithm, we do the following computation:

Let $Dis(p_1, p_0) = \{h_1, \ldots, h_N\}$ $(h_1 < h_2 < \ldots < h_N)$ be the dispersion set of $p_1(n)$ and $p_0(n)$. Set $p_0^{(0)} = p_0$, $p_1^{(0)} = p_1$ and for $1 \le i \le N$,

$$\begin{aligned} s_i(n) &= \gcd(p_1^{(i-1)}(n), p_0^{(i-1)}(n+h_i)), \\ p_1^{(i)}(n) &= p_1^{(i-1)}(n)/s_i(n), \qquad p_0^{(i)}(n) = p_0^{(i-1)}(n)/s_i(n-h_i). \end{aligned} \tag{2.3}$$

Finally, let

$$c(n) = \prod_{i=1}^{N}\prod_{j=1}^{h_i} s_i(n-j). \tag{2.4}$$

Using these notations, we have

Lemma 2.2 *The polynomials $s(n)$ and $u(n)$ are bounded by the following divisibilities:*

$$s(n-1) \mid c(n) \quad \textit{and} \quad u(n) \mid s_1(n-h_1)\cdots s_N(h-h_N).$$

Proof. By (2.2), we have

$$\frac{p_1(n)}{p_0(n)} = \frac{r'(n)}{r(n)}\frac{s(n)}{s(n-1)}.$$

Since the Gosper-Petkovšek representation [11] of $p_1(n)/p_0(n)$ is

$$\frac{p_1^{(N)}(n)}{p_0^{(N)}(n)}\frac{c(n+1)}{c(n)},$$

we derive from the maximality of $c(n)$ that $s(n-1) \mid c(n)$ ([13, Lemma 5.3.1]).

Since $g(n) = u(n)s(n)$ and $\gcd(f, g) = 1$, (2.2) implies that

$$u(n) \mid p_0(n) \quad \text{and} \quad u(n) \mid p_1(n)s(n-1).$$

Therefore, $u(n) \mid \gcd(p_0(n), p_1(n)c(n))$. On the other hand, Proposition 5.3.1 of [13] states that for any $1 \le i \le N$,

$$\gcd\left(p_0^{(N)}(n), p_1^{(i-1)}(n-h)\right) = 1, \quad \forall\, h < h_i.$$

Since $s_i(n)$ is a factor of $p_1^{(i-1)}(n)$, we derive that

$$\gcd\left(p_0^{(N)}(n), \prod_{j=0}^{h_i-1} s_i(n-j)\right) = 1,$$

which implies that

$$\gcd(p_0(n), p_1(n)c(n)) = s_1(n-h_1) \cdots s_N(n-h_N).$$

This completes the proof. ∎

Combining Lemma 2.1 and Lemma 2.2, we obtain

Theorem 2.3 *Let $y(n) = f(n)/g(n)$ be a solution to* (2.1)*, where $f, g \in K[n]$ and* $\gcd(f, g) = 1$*. Then*

$$g(n) \;\Bigg|\; \prod_{i=1}^{N} \prod_{j=0}^{h_i} s_i(n-j)\,, \tag{2.5}$$

where $s_i(n)$'s are defined by (2.3)*.*

We see that Theorem 2.3 is noting but saying that $\prod_{i=1}^{N} \prod_{j=0}^{h_i} s_i(n-j)$ is a universal denominator for (2.1).

Remark. Let us recall Abramov's algorithm. The universal denominator $u(n)$ is given by the following process.

Let $Dis(p_d, p_0) = \{h_1, \ldots, h_N\}$ $(h_1 > h_2 > \ldots > h_N)$ be the dispersion set of $p_d(n)$ and $p_0(n)$. Let $\tilde{p}_0^{(0)} = p_0$, $\tilde{p}_d^{(0)} = p_d$ and for $1 \leq i \leq N$,

$$\begin{aligned} &\tilde{s}_i(n) = \gcd(\tilde{p}_d^{(i-1)}(n), \tilde{p}_0^{(i-1)}(n+h_i)), \\ &\tilde{p}_d^{(i)}(n) = \tilde{p}_d^{(i-1)}(n)/\tilde{s}_i(n), \qquad \tilde{p}_0^{(i)}(n) = \tilde{p}_0^{(i-1)}(n)/\tilde{s}_i(n-h_i). \end{aligned} \tag{2.6}$$

Finally,

$$\tilde{u}(n) = \prod_{i=1}^{N} \prod_{j=0}^{h_i} \tilde{s}_i(n-j).$$

Note that the only difference between the two algorithms lies in the order of h_i's. In (2.6), the loop starts from the largest h in $Dis(p_d, p_0)$, while in (2.3), the loop starts from the smallest one. In the next subsection, we will show the minimality of the universal denominators given by Theorem 2.3. Therefore, they are factors of Abramov's universal denominators. The following example shows that the algorithm improve the Abramov's algorithm, even the gcd-improvement of Abramov's algorithm given by Abramov and Barkatou [6].

Example 1 *Suppose* $p_1(n) = n+2$, $p_0(n) = n(n+1)$. *By the gcd-improvement of Abramov's algorithm, we obtain the universal denominator* $(n+2)(n+1)n$. *While by* (2.5), *we get a smaller universal denominator* $(n+2)(n+1)$.

2.2 Minimality

To prove that the universal denominator given by (2.5) is minimal, we need the following lemmas to construct suitable solutions to (2.1).

Lemma 2.4 *Let* $f, g \in K[n]$ *be two polynomials which are relatively prime. Then for any polynomials* $h_1, \ldots, h_r \in K[n]$, *there exists an integer* M *such that for all integers* $m > M$,

$$\gcd(h_i, f + mg) = 1, \quad i = 1, 2, \ldots, r.$$

Proof. Suppose that there exist $m_1 < m_2 < \cdots$ and $i_1, i_2, \ldots$ such that $\gcd(h_{i_k}, f + m_k g) \neq 1$ for all $k \geq 1$. Since $\{h_1, \ldots, h_r\}$ is a finite set, there is $1 \leq j \leq r$ such that $i_k = j$ holds for infinite many k. Note that the irreducible factor of h_j is also finite, there exists an irreducible factor p of h_j such that $p \mid (f + mg)$ holds for infinitely many m, say, for m_s and m_t with $m_s \neq m_t$. Then $p \mid (m_s - m_t)g$, and hence $p \mid \gcd(f, g)$, which contradicts to the hypothesis $\gcd(f, g) = 1$. ∎

Corollary 2.5 *Let $f, g \in K[n]$ be two polynomials. Then for any polynomial $c \in K[n]$, there exists an integer M such that for all integers $m > M$,*

$$\gcd(c, f + mg) = \gcd(c, f, g).$$

Proof. Let $d = \gcd(f, g)$ and $f' = f/d, g' = g/d$. Then $\gcd(f', g') = 1$ and hence by Lemma 2.4, there exists M such that for all integers $m > M$, $\gcd(c, f' + mg') = 1$. Therefore,

$$\gcd(c, f + mg) = \gcd(c, d(f' + mg')) = \gcd(c, d) = \gcd(c, f, g), \quad \forall\, m > M.$$

This completes the proof. ∎

This corollary enable us to prove inductively the following fact.

Lemma 2.6 *Let $a, b, c \in K[n]$ be polynomials such that*

$$\gcd(a, c) = \gcd(b, c) = 1.$$

Then there exists a polynomial $f \in K[n]$ such that

$$\gcd(f(n), c(n)) = \gcd(f(n), c(n-1)) = 1,$$

and

$$\frac{a(n)f(n) + b(n)f(n+1)}{c(n)} \in K[n].$$

Proof. We firstly prove that the assertion holds for $c(n)$ of the form

$$c(n) = p(n)^{\alpha_0} p(n+1)^{\alpha_1} \cdots p(n+r)^{\alpha_r}, \tag{2.7}$$

where $p(n)$ is an irreducible polynomial and $\alpha_0, \alpha_1, \ldots, \alpha_r$ are positive integers.

Let

$$\gamma = \max\{\alpha_0, \ldots, \alpha_r\} \quad \text{and} \quad q(n) = \big(p(n)p(n+1)\cdots p(n+r)\big)^{\gamma}.$$

Furthermore, for $k = 0, \ldots, r+1$, let

$$f_k(n) = (-1)^k q(n+k-r-2) \cdot q(n+k) \cdot \prod_{i=1}^{r+1-k} a(n-i) \cdot \prod_{i=0}^{k-1} b(n+i).$$

We claim that $f = \sum_{k=0}^{r+1} f_k(n)$ is a polynomial required.

In fact, we have

$$\begin{aligned} & a(n)f(n) + b(n)f(n+1) \\ = \ & a(n)f_0(n) + \sum_{k=1}^{r+1} \big(a(n)f_k(n) + b(n)f_{k-1}(n+1)\big) + b(n)f_{r+1}(n+1) \\ = \ & a(n)f_0(n) + b(n)f_{r+1}(n+1). \end{aligned}$$

Noting that $q(n) \mid f_0(n)$ and $q(n) \mid f_{r+1}(n+1)$, we immediately derive that

$$\frac{a(n)f(n) + b(n)f(n+1)}{c(n)} \in K[n].$$

Since $p(n)$ is irreducible, for any integer i,

$$\gcd(p(n+i), q(n+i-r-1)) = \gcd(p(n+i), q(n+i+1)) = 1.$$

If $p(n+i) \mid a(n-j)$ for some $1 \le j \le r-i$, then $p(n+i+j) \mid a(n)$, which contradicts to the hypothesis $\gcd(c(n), a(n)) = 1$. If $p(n+i) \mid b(n+j)$ for some $0 \le j \le i$, then $p(n+i-j) \mid b(n)$, which contradicts to $\gcd(c(n), b(n)) = 1$. Therefore,

$$\gcd\big(p(n+i), f_{i+1}(n)\big) = 1, \quad \forall -1 \le i \le r.$$

Notice that for any $-1 \le i \le r$, we have $p(n+i) \mid f_j(n)$ whenever $j \ne i+1$ and $0 \le j \le r+1$. Therefore, we finally obtain that

$$\gcd\big(p(n+i), f(n)\big) = 1, \quad \forall -1 \le i \le r,$$

which implies that

$$\gcd(f(n), c(n)) = \gcd(f(n), c(n-1)) = 1.$$

Thus $f(n)$ satisfies the conditions we required.

Now we prove that the assertion holds for general $c(n)$ by induction on the degree of $c(n)$. The assertion becomes trivial for $c(n)$ of degree 0 by taking $f(n) = 1$.

Suppose that $c(n)$ is of degree greater than 0. If $c(n)$ is of form (2.7), we are done. Otherwise, there exist non-constant polynomials $c_1(n)$ and $c_2(n)$ such that $c(n) = c_1(n)c_2(n)$ and

$$\gcd(c_1(n+1), c_2(n)) = \gcd(c_1(n), c_2(n)) = \gcd(c_1(n-1), c_2(n)) = 1.$$

Clearly, the degrees of $c_1(n)$ and $c_2(n)$ are less than that of $c(n)$. By induction, there exists $f_1(n)$ and $f_2(n)$ such that

$$\frac{a(n)c_2(n-1)\cdot f_1(n) + b(n)c_2(n+1)\cdot f_1(n+1)}{c_1(n)} \in K[n],$$
$$\frac{a(n)c_1(n-1)\cdot f_2(n) + b(n)c_1(n+1)\cdot f_2(n+1)}{c_2(n)} \in K[n],$$

and that

$$\gcd(f_1(n), c_1(n)c_1(n-1)) = \gcd(f_2(n), c_2(n)c_2(n-1)) = 1.$$

For any integer m, set

$$f^{(m)}(n) = c_2(n)c_2(n-1)f_1(n) + mc_1(n)c_1(n-1)f_2(n).$$

Then

$$\begin{aligned}
&\frac{a(n)f^{(m)}(n) + b(n)f^{(m)}(n+1)}{c(n)} \\
&= \frac{a(n)c_2(n-1)f_1(n) + b(n)c_2(n+1)f_1(n+1)}{c_1(n)} \\
&\quad + m\frac{a(n)c_1(n-1)f_2(n) + b(n)c_1(n+1)f_2(n+1)}{c_2(n)} \in K[n].
\end{aligned}$$

By Corollary 2.5, for sufficient large m, we have

$$\begin{aligned}
&\gcd(c(n)c(n-1), f^{(m)}(n)) \\
&= \gcd\big(c(n)c(n-1), c_2(n)c_2(n-1)f_1(n), c_1(n)c_1(n-1)f_2(n)\big) \\
&= \gcd\big(c_2(n)c_2(n-1)\gcd(c_1(n)c_1(n-1), f_1(n)), \\
&\qquad c_1(n)c_1(n-1)\gcd(c_2(n)c_2(n-1), f_2(n))\big) \\
&= \gcd(c_2(n)c_2(n-1), c_1(n)c_1(n-1)) = 1.
\end{aligned}$$

Therefore, $f^{(m)}(n)$ is the desired polynomial for sufficient large m, which completes the proof. ∎

Now we are ready to prove the minimality of the universal denominators.

Theorem 2.7 *Let $p_0(n), p_1(n)$ be two polynomials in n. Let $s_i(n)$'s and $c(n)$ be given by* (2.3) *and* (2.4)*, respectively. Denote*

$$g(n) = c(n) \prod_{i=1}^{N} s_i(n) = \prod_{i=1}^{N} \prod_{j=0}^{h_i} s_i(n-j).$$

Then there exists a polynomial $f(n)$ such that $\gcd(f(n), g(n)) = 1$ *and*

$$p_0(n)\frac{f(n)}{g(n)} + p_1(n+1)\frac{f(n+1)}{g(n+1)} \in K[n].$$

Proof. By definition,

$$\frac{p_0(n)}{g(n)} = \frac{p_0^{(N)}(n)}{c(n+1)} \quad \text{and} \quad \frac{p_1(n+1)}{g(n+1)} = \frac{p_1^{(N)}(n+1)}{c(n+1)}.$$

Since

$$\frac{p_1^{(N)}(n)}{p_0^{(N)}(n)}\frac{c(n+1)}{c(n)}$$

is the Gosper-Petkovšek representation of $p_1(n)/p_0(n)$,

$$\gcd(p_0^{(N)}(n), c(n+1)) = \gcd(p_1^{(N)}(n+1), c(n+1)) = 1.$$

By Lemma 2.6, there exist a polynomial $f(n)$ such that

$$\frac{p_0^{(N)}(n)f(n) + p_1^{(N)}(n+1)f(n+1)}{c(n+1)} \in K[n],$$

and

$$\gcd\big(f(n), c(n+1)\big) = \gcd\big(f(n), c(n)\big) = 1.$$

Since $g(n) \mid c(n)c(n+1)$, we have $\gcd(f(n), g(n)) = 1$, as desired. ∎

Example 2 *Suppose* $p_1(n) = n+2$, $p_0(n) = n(n+1)$ *as in Example 1. The universal denominator is* $u(n) = (n+2)(n+1)$. *We notice that when* $f(n) = n+3$,

$$p_0(n)\frac{f(n)}{u(n)} + p_1(n+1)\frac{f(n+1)}{u(n+1)} = n+2 \in K[n].$$

Therefore, $(n+3)/(n+1)(n+2)$ *is a reduced solution to the difference equation*

$$p_0(n)y(n) + p_1(n+1)y(n+1) = n+2.$$

Noting further that $(n+1)(n+2)$ *is a universal denominator, we derive that it is the minimal universal denominator.*

2.3 Connection with Gosper's Algorithm

Viewing Gosper's algorithm as a special case of linear difference equations of order one, we find that the universal denominator $u(n)$ given by (2.5) coincides with the denominator given by Gosper's algorithm.

Recall that [13, Section 5.2] the kernel equation of Gosper's algorithm is

$$r(n)y(n+1) - y(n) = 1. \tag{2.8}$$

Write $r(n)$ in reduced form $r(n) = f(n)/g(n)$. The denominator $c(n)$ of $y(n)$ is given by the following process.

Let $Dis(f,g) = \{h_1, \ldots, h_N\}$ $(h_1 < h_2 < \ldots < h_N)$ be the dispersion set of $f(n)$ and $g(n)$. Set $p_0^{(0)} = f$, $p_1^{(0)} = g$ and for $1 \le i \le N$,

$$\begin{aligned} &c_i(n) = \gcd(p_1^{(i-1)}(n), p_0^{(i-1)}(n+h_i)), \\ &p_1^{(i)}(n) = p_1^{(i-1)}(n)/c_i(n), \qquad p_0^{(i)}(n) = p_0^{(i-1)}(n)/c_i(n-h_i). \end{aligned} \tag{2.9}$$

Finally, set

$$c(n) = \prod_{i=1}^{N}\prod_{j=1}^{h_i} c_i(n-j).$$

On the other hand, written (2.8) in the form of (2.1), we have

$$p_0(n) = g(n), \quad p_1(n) = f(n-1), \quad \text{and} \quad p(n) = g(n).$$

Since $\gcd(f, g) = 1$, the smallest element h_1 in $Dis(f, g)$ is greater than 1, and hence

$$Dis(p_1, p_0) = \{h - 1 \mid h \in Dis(f, g)\} = \{h_1 - 1, h_2 - 1, \ldots, h_N - 1\}.$$

Comparing (2.9) and (2.3), we derive that $s_i(n) = c_i(n-1)$ and hence $u(n) = c(n)$.

3. Higher Order Linear Difference Equations

The following Theorem shows that for linear difference of order greater than one, $u(n)$ given by Abramov's algorithm is the minimal universal denominator.

Theorem 3.1 *Let $p_0(n), p_d(n)$ $(d \geq 2)$ be two polynomials and $u(n)$ be given by* (2.6)*. Then there exist polynomials $p_1(n), \ldots, p_{d-1}(n)$ such that $y(n) = 1/u(n)$ satisfies*

$$p_0(n)y(n) + p_1(n+1)y(n+1) + \cdots + p_d(n+d)y(n+d) = 0.$$

Proof. Let $p'(n) = -p_0^{(N)}(n-1)\prod_{i=1}^{N} s_i(n)$. Then

$$\begin{aligned} & p_0(n)y(n) + p'(n+1)y(n+1) \\ = & \frac{p_0^{(N)}(n)}{\prod_{i=1}^{N}\prod_{j=0}^{h_i-1} s_i(n-j)} - \frac{p_0^{(N)}(n)\prod_{i=1}^{N} s_i(n+1)}{\prod_{i=1}^{N}\prod_{j=0}^{h_i} s_i(n+1-j)} \\ = & 0. \end{aligned} \tag{3.1}$$

Similarly, denoting $p''(n) = -p_d^{(N)}(n+1)\prod_{i=1}^{N} s_i(n-h_i)$, we have

$$p''(n+d-1)y(n+d-1) + p_d(n+d)y(n+d) = 0. \tag{3.2}$$

For $d = 2$, we may take $p_1(n) = p'(n) + p''(n)$. For $d > 2$, we may take $p_1(n) = p'(n)$, $p_{d-1}(n) = p''(n)$ and $p_i(n) = 0$ for $1 < i < d-1$. ∎

Example 3 *Suppose $d = 2$ and $p_0(n) = n(n+1)$, $p_2(n) = n+2$. We have $s_1(n) = n+2$ and hence $u(n) = (n+2)(n+1)n$. Taking $p_1(n) = -n(n+3)$, we have*

$$p_0(n)/u(n) + p_1(n+1)/u(n+1) + p_2(n+2)/u(n+2) = 0.$$

Thus, $u(n)$ is the minimal universal denominator.

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References

[1] S.A. Abramov, Problems in computer algebra that are connected with a search for polynomial solutions of linear differential and difference equations, *Moscow Univ. Comput. Math. Cybernet.* **3** (1989) 63–68.

[2] S.A. Abramov, Rational solutions of linear differential and difference equations with polynomial coefficients, *Comput. Maths. Math. Phys.* **29** (1991) 7–12; Transl. from *Zh. Vychisl. Mat. Mat. Fiz.* **29** (1989) 1611–1620.

[3] S.A. Abramov, Denominators of rational solutions of linear difference equations, *Programmirovanie* **1** (1994) 49–52.

[4] S.A. Abramov, Rational solutions of linear difference and q-difference equations with polynomial coefficients, in: *Proc. ISSAC'95*, ACM Press, New York, 1995, 285–289.

[5] S.A. Abramov, Rational solutions of linear difference and q-difference equations with polynomial coefficients, *Program. Comput. Software* **21(6)** 273–278, Transl. from *Programmirovanie* **(6)** (1995) 3–11.

[6] S.A. Abramov and M.A. Barkatou, Rational solutions of first order linear difference systems, in: *Proc. ISSAC'98*, ACM Press, New York, 1998, 124–131.

[7] S.A. Abramov, M. Bronstein, and M. Petkovšek, On polynomial solutions of linear operator equations, in: *Proc. ISSAC'95*, ACM Press, 1995, 290–296.

[8] M.A. Barkatou, Rational solutions of matrix difference equations: the problem of equivalence and factorization, in: *Proc. ISSAC'99*, ACM Press, 1999, 277–282.

[9] W.Y.C. Chen, P. Paule, and H.L. Saad, Convergence to universal denominators, *Adv. Appl. Math.* **41** (2008) 351–364.

[10] D.E. Khmel'nov, Improved algorithms for solving difference and q-difference equations, *Program. Comput. Software* **26(2)** (1999) 107–115.

[11] M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symbolic Comput.* **14** (1992) 243–264.

[12] M. Petkovšek, A generalization of Gosper's algorithm, *Discrete Math.* **134** (1994) 125–131.

[13] M. Petkovšek, H.S. Wilf, and D. Zeilberger, A=B, A K Peters, Wellesley, Massachusetts, 1996.

[14] M. van Hoeij, Rational solutions of linear difference equations, in: *Proc. ISSAC'98*, ACM press, 1998, 120–123.

[15] C. Weixlbaumer, Solutions of difference equations with polynomial coefficients, Master Thesis, RISC, J. Kepler University, Linz, 2001.