On P-partitions Related to Ordinal Sums of Posets

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Abstract

Using the inclusion-exclusion principle, we derive a formula of generating functions for P-partitions related to ordinal sums of posets. This formula simplifies computations for many variations of plane partitions, such as plane partition polygons and plane partitions with diagonals or double diagonals. We illustrate our method by several examples, some of which are new variations of plane partitions.

Keywords: generating function, P-partition, ordinal sum, partition analysis, plane partition

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1 Introduction

A P-partition is an order-reversing map from a poset to non-negative integers [26, Ch. IV]. To be precise, let $(P, \leq_P)$ be a poset and $\mathbb{N}$ the set of non-negative integers. Then $\sigma\colon P \to \mathbb{N}$ is a P-partition related to P if for any two elements $a, b \in P$, $a \leq_P b$ implies that $\sigma(a) \geq \sigma(b)$. The (multivariate) generating function for P-partitions related to a poset $P = \{a_1, \dots, a_n\}$ is given by

$$f_P(\mathbf{x}) = f_P(x_1, x_2, \dots, x_n) := \sum_{\sigma} x_1^{\sigma(a_1)} x_2^{\sigma(a_2)} \cdots x_n^{\sigma(a_n)},$$

where σ runs over all P-partitions related to P.

Stanley [26, Theorem 4.5.4] provided an elegant formula which expresses $f_P(\mathbf{x})$ in terms of descent numbers. However, the formula is a summation over all linear extensions of P. As we know, counting the number of linear extensions is $\#P$-complete [16]. Therefore, we need a more efficient way to compute $f_P(\mathbf{x})$.

On the other hand, P-partitions can be viewed as solutions of a system of linear Diophantine inequalities. In the pioneering book "Combinatory Analysis" [24, Vol. II, pp. 91–170] MacMahon introduced partition analysis as a computational method for solving general systems of linear Diophantine inequalities and equations. The technique was given a new life by Andrews [1] in his study of lecture hall theorem introduced by Bousquet-Mélou and Eriksson [15]. Andrews, Paule and Riese published a series of papers [2–13] to exhibit its various applications to combinatorial problems. Corteel, Savage et. al. [19, 20] presented the "five guidelines" approach to lecture hall type theorems and linear inequalities as a simplification of MacMahon's partition analysis. By this method, Andrews, Corteel and Savage [14] revealed stronger results about lecture hall partitions and anti-lecture hall compositions [17].

The key ingredient of MacMahon's partition analysis is the Omega operator $\Omega_{\geq}$ which is defined by

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}.$$

For the evaluation of the Omega operator and further for the implements of partition analysis, Andrews, Paule and Riese provided the `Mathematica` package `Omega`. Han [22] gave an algorithm by using the coefficients of polynomials. Xin [27] combined the theory of iterated Laurent series and partial fraction decompositions to obtain a fast algorithm. We will use Xin's updated `Maple` package `Ell2` [28] for the examples in this paper.

MacMahon's partition analysis provides us a powerful tool to compute $f_P(\mathbf{x})$ for general posets. It is still interesting to find more efficient algorithms for special types of posets. For example, Ekhad and Zeilberger [21] discussed the posets constructed by "grafting".

Our main goal is to find an efficient method to compute $f_P(\mathbf{x})$ for posets composed of several simple or small blocks by *ordinal sums*. The ordinal sum of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ is the poset $P \oplus Q$ defined on their disjoint union and partially ordered by $x \leq y$ in $P \oplus Q$ if and only if (a) $x, y \in P$ and $x \leq_P y$ or (b) $x, y \in Q$ and $x \leq_Q y$, or (c) $x \in P$ and $y \in Q$. Stanley [25] provided a formula of $f_{P\oplus Q}(\mathbf{x})$ which involves a summation over linear extensions of Q. We use the inclusion-exclusion principle to derive a new formula in Section 2. This formula only involves the minimal elements of Q and is a summation over subsets of these elements. Moreover, it enables us to handle posets composed of several simple or small blocks by ordinal sums, especially $P \oplus P \oplus \cdots \oplus P$. We can deal with small posets by MacMahon's partition analysis and then use the formula iteratively to obtain the final generating functions. This process simplifies the computation for many variations of plane partitions, including plane partition polygons and plane partitions with diagonals or double diagonals. We will illustrate the method by several examples in Sections 3 and 4. Some of the examples are generalizations of known results and some are new variations of plane partitions.

Let us introduce some representations of posets and P-partitions used in this paper. Let $(P, \leq)$ be a poset. For $x, y \in P$, we say y *covers* x, denoted by $x \lessdot y$, if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. Clearly, P is determined by its cover relation set $R(P) := \{(x, y) \colon x \lessdot y\}$ which is taken as one representation of P. Another representation is the *Hasse diagram* of P. Every element of P is represented by a vertex and two vertices x, y are joined by a line with vertex y drawn above vertex x if $x \lessdot y$. To coincide with the descriptions used by Andrews, Paule and Riese [8], we rotate the Hasse diagram by 90 degree clockwise so that smaller elements lie to the left. For example, a diamond poset $P = \{a_1, a_2, a_3, a_4\}$ with $R(P) = \{(a_1, a_2), (a_1, a_3), (a_2, a_4), (a_3, a_4)\}$ can be represented by Figure 1. A P-partition σ related to $P = \{a_1, \ldots, a_n\}$ can be represented by the sequence $(\sigma(a_1), \ldots, \sigma(a_n))$. For convenience, we will omit σ and use also a_i to indicate the integer $\sigma(a_i)$, which will cause no confusion from the context.

It should be noticed that we have also *strict* P-partitions $\sigma \colon P \to \mathbb{N}$ which requires that

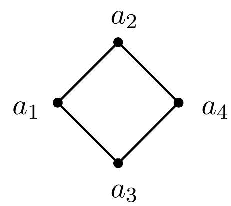

Figure 1: The graph representation of a diamond poset.

$\sigma(a) < \sigma(b)$ for any a covers b in P. The corresponding generating function is given by

$$g_P(\mathbf{x}) = g_P(x_1, x_2, \ldots, x_n) := \sum_{\sigma} x_1^{\sigma(a_1)} x_2^{\sigma(a_2)} \cdots x_n^{\sigma(a_n)},$$

where σ runs over all strict P-partitions related to P. When P is graded with the rank function ρ, it is straightforward to show that

$$g_P(\mathbf{x}) = x_1^{m-\rho(a_1)} \cdots x_n^{m-\rho(a_n)} f_P(\mathbf{x}),$$

where m is the rank of P, i.e., $m = \max_{a \in P} \rho(a)$.

Generally, we have Stanley's reciprocity theorem for P-partitions [26, Theorem 4.5.7] which states that

$$x_1 x_2 \cdots x_n \, g_P(x_1, \ldots, x_n) = (-1)^n f_P\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right).$$

Therefore, we will consider mainly the generating function $f_P(\mathbf{x})$ in this paper.

2 The Main Theorems and Corollaries

Before considering $f_{P\oplus Q}(\mathbf{x})$ for arbitrary posets P and Q, we first look at the case in which Q has a smallest element. To keep expressions as simple as possible, we denote the product $x_1 x_2 \cdots x_k$ by X_k. The following lemma can be viewed as an elementary ingredient in this paper.

Lemma 2.1 *Let $P = \{a_1, a_2, \ldots, a_n\}$ and $Q = \{b_1, b_2, \ldots, b_m\}$ be two posets. If Q contains a smallest element, say b_1, then*

$$f_{P\oplus Q}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = f_P(x_1, x_2, \ldots, x_n) f_Q(y_1 X_n, y_2, \ldots, y_m). \tag{2.1}$$

Proof. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$ be the P-partitions related to P and Q, respectively. Consider the sequence $\mathbf{c} = (a_1 + b_1, a_2 + b_1, \ldots, a_n + b_1, b_1, \ldots, b_m)$. Since b_1 is the smallest element of Q, we have $a_i + b_1 \geq b_1 \geq b_j$ for any $1 \leq i \leq n, 1 \leq j \leq m$. Thus $\mathbf{c}$ is a P-partition related to $P \oplus Q$. Conversely, given a P-partition $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ related to $P \oplus Q$, from the definition of $P \oplus Q$, we immediately derive that $(a_1 - b_1, a_2 - b_1, \ldots, a_n - b_1)$ and $(b_1, \ldots, b_m)$ are P-partitions related to P and Q, respectively. Hence,

$$\begin{aligned} f_{P\oplus Q}(x_1, \ldots, x_n, y_1, \ldots, y_m) &= \sum x_1^{a_1-b_1} \cdots x_n^{a_n-b_1} (X_n y_1)^{b_1} y_2^{b_2} \cdots y_m^{b_m} \\ &= f_P(x_1, \ldots, x_n) f_Q(y_1 X_n, y_2, \ldots, y_m), \end{aligned}$$

which completes the proof. ∎

Lemma 2.1 can also be proved by using partition analysis or Stanley's formula on P-partitions. Its special case in which Q contains only one element will be frequently used:

$$f_{P\oplus\{b\}}(x_1, x_2, \ldots, x_n, y_1) = \frac{f_P(x_1, \ldots, x_n)}{1 - X_n y_1}. \tag{2.2}$$

Denote the ordinal sum of P with itself k times by $k \times P$. By iterative use of Lemma 2.1, we obtain

Theorem 2.2 *Let $P = \{a_1, a_2, \ldots, a_n\}$ be a poset with the smallest element a_1. Then for any positive integer k we have*

$$f_{k\times P}(x_1, x_2, \ldots, x_{kn}) = \prod_{i=0}^{k-1} f_P(X_{in+1}, x_{in+2}, \ldots, x_{(i+1)n}). \tag{2.3}$$

In many cases, we are interested in the specialization $x_i = q$ of $f_{k\times P}(\mathbf{x})$. Noting that the first variable of $f_P(\mathbf{x})$ plays a different role, we need to compute the specialization $f_P(x, q, q, \ldots, q)$ instead of $f_P(q, \ldots, q)$. One will see this trick in the examples.

To set up the formula of $f_{P\oplus Q}(\mathbf{x})$ for general poset $Q = \{b_1, \ldots, b_m\}$, we introduce the notation $Q_{[j_1,\ldots,j_l]}$ which denotes the ordinal sum of the 1-element poset $\{b_0\}$ and the sub-poset (a subset inheriting the order relations) $Q \setminus \{b_{j_1}, \ldots, b_{j_l}\}$ of Q. With this notation, we have

Theorem 2.3 *Let $P = \{a_1, a_2, \ldots, a_n\}$ and $Q = \{b_1, b_2, \ldots, b_m\}$ be two posets. Suppose that the minimal elements of Q are $b_1, \ldots, b_r$. Then we have*

$$f_{P\oplus Q}(x_1, \ldots, x_n, y_1, \ldots, y_m) = f_P(x_1, \ldots, x_n) h_Q(X_n, y_1, \ldots, y_m) \tag{2.4}$$

with

$$\begin{aligned} &h_Q(y_0, y_1, \ldots, y_m) \\ &\qquad = \sum_{l=1}^{r} (-1)^{l-1} \sum_{1\le j_1<\cdots<j_l\le r} f_{Q_{[j_1,\ldots,j_l]}}(y_0 y_{j_1} \cdots y_{j_l}, y_1, \ldots, \hat{y}_{j_1}, \ldots, \hat{y}_{j_l}, \ldots, y_m), \end{aligned}$$

where $\hat{y}_k$ means suppressing the variable y_k.

Proof. We divide the set S of all P-partitions $(a_1,\ldots,a_n,b_1,\ldots,b_m)$ related to $P\oplus Q$ into several groups according to $(b_1,\ldots,b_r)$. In fact, denote by $S_{[j_1,\ldots,j_l]}$ the set

$$\big\{(a_1,\ldots,a_n,b_1,\ldots,b_m)\in S\colon b_{j_1}=b_{j_2}=\cdots=b_{j_l}=\max\{b_1,\ldots,b_m\}\big\}.$$

It follows that $S_{[j_1,\ldots,j_l]}=S_{[j_1]}\cap\cdots\cap S_{[j_l]}$ and $S=S_{[1]}\cup S_{[2]}\cup\cdots\cup S_{[r]}$. Observe that there is a natural bijection between $S_{[j_1,\ldots,j_l]}$ and P-partitions related to $P\oplus Q_{[j_1,\ldots,j_l]}$:

$$(a_1,\ldots,a_n,b_1,\ldots,b_m)\mapsto(a_1,\ldots,a_n,\max\{b_1,\ldots,b_m\},b_1,\ldots,\hat{b}_{j_1},\ldots,\hat{b}_{j_l},\ldots,b_m).$$

Now applying the inclusion-exclusion principle, we derive that

$$\begin{aligned}
&f_{P\oplus Q}(x_1,\ldots,x_n,y_1,\ldots,y_m)\\
&=\sum_{l=1}^{r}(-1)^{l-1}\sum_{1\le j_1<\cdots<j_l\le r}\ \sum_{(a_1,\ldots,a_n,b_1,\ldots,b_m)\in S_{[j_1,\ldots,j_l]}} x_1^{a_1}\cdots x_n^{a_n}y_1^{b_1}\cdots y_m^{b_m}\\
&=\sum_{l=1}^{r}(-1)^{l-1}\sum_{1\le j_1<\cdots<j_l\le r} f_{P\oplus Q_{[j_1,\ldots,j_l]}}(x_1,\ldots,x_n,y_{j_1}\cdots y_{j_l},y_1,\ldots,\hat{y}_{j_1},\ldots,\hat{y}_{j_l},\ldots,y_m).
\end{aligned}$$

Since $Q_{[j_1,\ldots,j_l]}$ are posets with the smallest element b_0, applying Lemma 2.1 to each summand, we arrive at (2.4). ∎

As a direct consequence, we have

Corollary 2.4 *Let $P=\{a_1,a_2,\ldots,a_n\}$ be a poset with minimal elements $a_1,\ldots,a_r$. Then for $k\ge 1$,*

$$f_{k\times P}(x_1,\ldots,x_{kn})=f_P(x_1,\ldots,x_n)\times\prod_{i=1}^{k-1}h_P(X_{in},x_{in+1},x_{in+2},\ldots,x_{(i+1)n}),$$

with

$$h_P(y_0,y_1,\ldots,y_n)=\sum_{l=1}^{r}(-1)^{l-1}\sum_{1\le j_1<\cdots<j_l\le r} f_{P_{[j_1,\ldots,j_l]}}(y_0y_{j_1}y_{j_2}\cdots y_{y_l},y_1,\ldots,\hat{y}_{j_1},\ldots,\hat{y}_{j_l},\ldots,y_n).$$

By Corollary 2.4, in order to compute $f_{k\times P}(\mathbf{x})$, we need only compute $f_P(\mathbf{x})$ and $h_P(y_0,y_1,\ldots,y_n)$, which can be handled by partition analysis or other methods.

3 Applications to Posets with a Smallest Element

In this section, we exhibit some applications of Lemma 2.1 and Theorem 2.2 which focus on posets with a smallest element. We begin with an introductory example, i.e., hexagonal plane partitions with diagonals. Then we deal with plane partition fans which are generalizations of plane partition diamonds. Finally, we introduce solid partition hexahedrons and compute their generating functions.

3.1 An Introductory Example

A hexagonal plane partition with diagonals of length k, first studied by Andrews, Paule and Riese in [11], is a P-partition related to the poset H_k given by Figure 2.

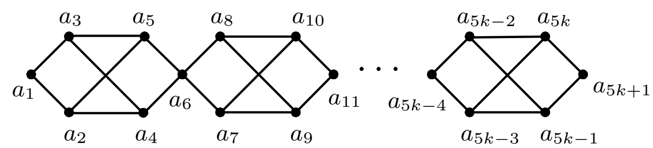

Figure 2: The poset H_k for hexagonal plane partitions with diagonals.

Notice that the poset H_k is isomorphic to $(k \times H) \oplus \{a\}$ with H being the poset given by Figure 3.

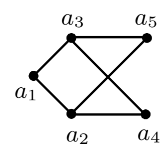

Figure 3: The basic block for the poset H_k.

With the help of the `Maple` package `Ell2`, we find that

$$f_H(x_1, \ldots, x_5) = \frac{(1 - X_1X_3)(1 - X_3X_5)}{(1 - X_3/x_2)(1 - X_5/x_4)} \prod_{i=1}^{5} \frac{1}{(1 - X_i)}.$$

Since H has a smallest element a_1, applying Theorem 2.2, we obtain

$$f_{k\times H}(x_1, \ldots, x_{5k}) = \prod_{i=1}^{5k} \frac{1}{1 - X_i} \prod_{i=0}^{k-1} \frac{(1 - X_{5i+1}X_{5i+3})(1 - X_{5i+3}X_{5i+5})}{(1 - X_{5i+3}/x_{5i+2})(1 - X_{5i+5}/x_{5i+4})}.$$

By Equation (2.2), we finally derive that

$$\begin{aligned} f_{H_k}(x_1, \ldots, x_{5k+1}) &= \frac{f_{k\times P}(x_1, \ldots, x_{5k})}{1 - X_{5k+1}} \\ &= \prod_{i=1}^{5k+1} \frac{1}{1 - X_i} \prod_{i=0}^{k-1} \frac{(1 - X_{5i+1}X_{5i+3})(1 - X_{5i+3}X_{5i+5})}{(1 - X_{5i+3}/x_{5i+2})(1 - X_{5i+5}/x_{5i+4})}, \end{aligned}$$

which coincides with [11, Theorem 4].

Especially, when $x_i = q$ for $1 \le i \le 5k + 1$, we get

$$f_{H_k}(q, \ldots, q) = \frac{(-q^2; q^5)_k(-q^4; q^5)_k}{(q; q)_{5k+1}}.$$

Here and in the follows we use the standard notation $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. Since the poset H_k is graded, we have

$$g_{H_k}(q,\ldots,q) = q^{(15k^2+3k)/2}\frac{(-q^2;q^5)_k(-q^4;q^5)_k}{(q;q)_{5k+1}}.$$

Using the same approach, one can easily recover the generating functions for plane partition diamonds [8], hexagonal plane partitions [11], and plane partition polygons [18].

3.2 Plane Partition Fans

We now generalize plane partition diamonds to *plane partition fans* (of width s and length k), which are P-partitions related to the poset F_k^s given by Figure 4. Notice that the plane

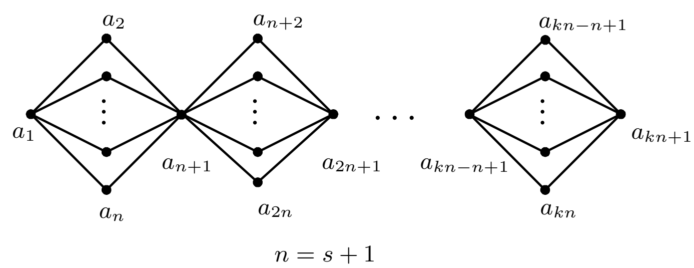

Figure 4: The poset F_k^s for plane partition fans of width s and length k.

partition fans reduce to plane partition diamonds when $s = 2$.

Clearly, the poset F_k^s is isomorphic to $(k\times CL^s)\oplus\{a\}$, where CL^s is the poset $\{a_1,\cdots,a_{s+1}\}$ with the cover relation set

$$R(CL^s) = \{(a_1,a_j)\colon 2\le j\le s+1\}.$$

We are interested in the specialization $x_i = q$ of the generating function $f_{F_k^s}(\mathbf{x})$. From the words after Theorem 2.2, we need a reasonable formula of $f_{CL^s}(x,q,\ldots,q)$. In fact, by the definition of P-partitions, we have

$$\begin{aligned}
f_{CL^s}(x_1,q,\ldots,q) &= \sum_{a_1\ge 0} x_1^{a_1}\sum_{a_1\ge a_2\ge 0} q^{a_2}\sum_{a_1\ge a_3\ge 0} q^{a_3}\cdots\sum_{a_1\ge a_{s+1}\ge 0} q^{a_{s+1}}\\
&= \sum_{a_1\ge 0} x_1^{a_1}\frac{1-q^{a_1+1}}{1-q}\cdots\frac{1-q^{a_1+1}}{1-q}\\
&= \frac{1}{(1-q)^s}\sum_{a_1\ge 0} x_1^{a_1}\sum_{i=0}^{s}(-1)^i\binom{s}{i}q^{(a_1+1)i}\\
&= \frac{1}{(1-q)^s}\sum_{i=0}^{s}(-1)^i\binom{s}{i}\frac{q^i}{1-x_1q^i}.
\end{aligned}$$

Thus, Theorem 2.2 together with Equation (2.2) leads to

Theorem 3.1 *For integers $k \geq 1$ and $s \geq 1$, we have*

$$f_{F_k^s}(q,\ldots,q) = \frac{1}{(1-q)^{ks}(1-q^{(s+1)k+1})} \prod_{j=0}^{k-1} \left(\sum_{i=0}^{s} \frac{(-1)^i \binom{s}{i} q^i}{1-q^{(s+1)j+i+1}} \right). \tag{3.1}$$

The explicit formulaes of $f_{F_k^s}(q,\ldots,q)$ for $s = 2, 3, 4$ are

$$f_{F_k^2}(q,\ldots,q) = \frac{\prod_{i=0}^{k-1}(1+q^{3i+2})}{(q;q)_{3k+1}},$$
$$f_{F_k^3}(q,\ldots,q) = \frac{\prod_{i=0}^{k-1}(1+2q^{4i+2}(1+q)+q^{8i+5})}{(q;q)_{4k+1}},$$
$$f_{F_k^4}(q,\ldots,q) = \frac{\prod_{i=0}^{k-1}(1+(q^{5i+2}+q^{10i+5})(3+5q+3q^2)+q^{15i+9})}{(q;q)_{5k+1}}.$$

3.3 Solid Partition Hexahedrons

MacMahon [23] considered 3-dimensional generalization of plane partition diamonds, whose basic poset is the Boolean poset of order 3 described by Figure 5. Similar to plane partition

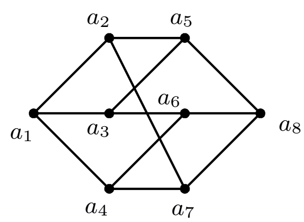

Figure 5: The Boolean poset B_3.

diamonds, we glue B_3 along their extremal elements to obtain a poset S_k represented by Figure 6. We call the P-partitions related to S_k the *solid partition hexahedrons* (of length k).

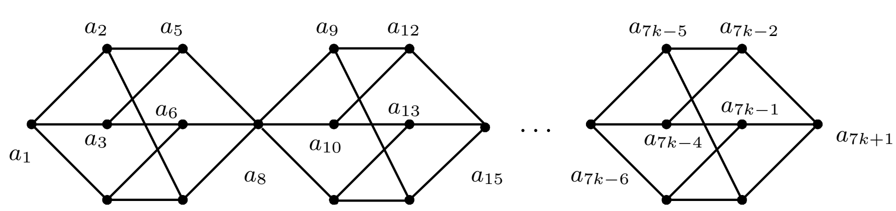

Figure 6: The poset S_k for solid partition hexahedrons.

Let $B_3' = \{a_1,\ldots,a_7\}$ be the poset obtained from B_3 by removing the largest element a_8. Then S_k is isomorphic to $(k \times B_3') \oplus \{a\}$. To compute $f_{S_k}(q,\ldots,q)$, we first use `Ell2` to find out

$$f_{B_3'}(x,q,\ldots,q) = \frac{H(x,q)}{(x;q)_7},$$

where

$$H(x,q) = 1 + (2q + 2q^2 + 3q^3 + 2q^4 + 2q^5)x + (q^3 + 3q^4 + 4q^5 + 8q^6 + 4q^7 + 3q^8 + q^9)x^2 \\ + (2q^7 + 2q^8 + 3q^9 + 2q^{10} + 2q^{11})x^3 + q^{12}x^4.$$

Then by Theorem 2.2 and Equation (2.2) we obtain

$$f_{S_k}(q,\ldots,q) = \frac{\prod_{i=0}^{k-1} H(q^{7i+1},q)}{(q;q)_{7k+1}}.$$

Along this line, we can deal with higher dimensional partitions. D. Zeilberger asked to enumerate the P-partitions related to the four dimension cube (through personal communication), whose basic poset is the Boolean poset of order 4 depicted by Figure 7. Let

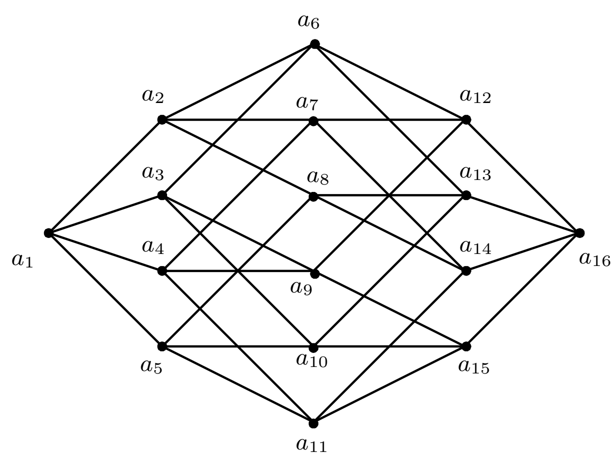

Figure 7: The Boolean poset B_4.

$B_4' = \{a_1,\ldots,a_{15}\}$ be the poset obtained from B_4 by removing the largest element a_{16}. With the help of `Ell2`, we find that

$$f_{B_4'}(x,q,\ldots,q) = \frac{h(x,q)}{(x;q)_{15}},$$

where $h(x,q)$ is a polynomial of degree 11 in x. This leads to

$$f_{(k\times B_4')\oplus\{b\}}(x,q,\ldots,q) = \frac{\prod_{i=0}^{k-1} h(xq^{15k},q)}{(x;q)_{15k+1}}.$$

When $k=1$ and $x=q$, the above formula simplifies to

$$f_{B_4}(q,\ldots,q) = \frac{(1+q^8)N(q)}{(q;q)_{16}},$$

where $N(q)$ is given by the following explicit formula:

$$\begin{aligned}&1+3q^2+5q^3+9q^4+15q^5+28q^6+45q^7+85q^8+124q^9+208q^{10}+287q^{11}+415q^{12}+571q^{13}\\&+789q^{14}+1060q^{15}+1428q^{16}+1872q^{17}+2442q^{18}+3129q^{19}+3978q^{20}+4944q^{21}+6106q^{22}\\&+7361q^{23}+8840q^{24}+10383q^{25}+12176q^{26}+14076q^{27}+16166q^{28}+18321q^{29}+20596q^{30}+22792q^{31}\\&+25027q^{32}+27036q^{33}+28988q^{34}+30554q^{35}+31982q^{36}+33010q^{37}+33804q^{38}+34223q^{39}\\&+34434q^{40}+34223q^{41}+33804q^{42}+33010q^{43}+31982q^{44}+30554q^{45}+28988q^{46}+27036q^{47}\\&+25027q^{48}+22792q^{49}+20596q^{50}+18321q^{51}+16166q^{52}+14076q^{53}+12176q^{54}+10383q^{55}\\&+8840q^{56}+7361q^{57}+6106q^{58}+4944q^{59}+3978q^{60}+3129q^{61}+2442q^{62}+1872q^{63}+1428q^{64}\\&+1060q^{65}+789q^{66}+571q^{67}+415q^{68}+287q^{69}+208q^{70}+124q^{71}+85q^{72}+45q^{73}+28q^{74}+15q^{75}\\&+9q^{76}+5q^{77}+3q^{78}+q^{80}.\end{aligned}$$

4 Applications to General Posets

In this section, we show the applications of Theorem 2.3 and Corollary 2.4 by two examples. In the first example, we provide a simple solution for plane partitions with double diagonals. Furthermore, we generalize them to complete plane partitions. In the second one, we recover the generating functions for plane partitions with diagonals, studied by Andrews, Paule and Riese [10].

4.1 Complete Plane Partitions

In [20], Davis, Souza, Lee and Savage introduced plane partitions with double diagonals whose corresponding poset is given by Figure 8. They used the "digraph method" to derive a recurrence relation on the generating functions and then proved the formulaes. We will see that Theorem 2.3 enables us to find out the generating functions directly.

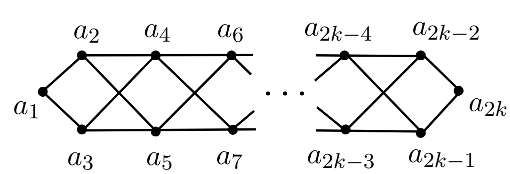

Figure 8: The poset for plane partitions with double diagonals.

Let A^i denote the anti-chain poset with i elements, i.e., an i-element poset with the empty cover relation set. Then the plane partitions with double diagonals are exactly the P-partitions related to the poset $K^2 = A^1 \oplus ((k-1) \times A^2) \oplus A^1$, whose generating function is given by the following theorem.

Theorem 4.1 *For any integer $k > 1$, we have*

$$f_{K^2}(x_1, \ldots, x_{2k}) = \prod_{i=1}^{2k} \frac{1}{1 - X_i} \prod_{j=0}^{k-2} \frac{1 - X_{2j+1} X_{2j+3}}{1 - X_{2j+3}/x_{2j+2}}.$$

Especially,

$$f_{K^2}(q, \ldots, q) = \frac{(-q^2; q^2)_{k-1}}{(q; q)_{2k}}.$$

Proof. Let $Q = A^2$. Then $Q_{[1]}$ and $Q_{[2]}$ are both isomorphic to the poset $\{a_1, a_2\}$ with partial order $a_1 \leq a_2$. While $Q_{[1,2]}$ is isomorphic to A^1. Thus, $f_{Q_{[1,2]}}(x_0) = 1/(1 - x_0)$, $f_{Q_{[1]}}(x_0, x_2) = 1/(1 - x_0)(1 - x_0 x_2)$ and $f_{Q_{[2]}}(x_0, x_1) = 1/(1 - x_0)(1 - x_0 x_1)$, so that

$$\begin{aligned} h_Q(y_0, y_1, y_2) &= f_{Q_{[1]}}(y_0 y_1, y_2) + f_{Q_{[2]}}(y_0 y_2, y_1) - f_{Q_{[1,2]}}(y_0 y_1 y_2) \\ &= \frac{1 - y_0^2 y_1 y_2}{(1 - y_0 y_1)(1 - y_0 y_2)(1 - y_0 y_1 y_2)}. \end{aligned}$$

By iterative use of Theorem 2.3, we derive that

$$\begin{aligned} f_{K^2}(\mathbf{x}) &= \frac{1}{1 - x_1} \cdot \left(\prod_{i=0}^{k-2} h_Q(X_{2i+1}, x_{2i+2}, x_{2i+3}) \right) \cdot \frac{1}{1 - X_{2k}} \\ &= \prod_{i=1}^{2k} \frac{1}{1 - X_i} \prod_{k=0}^{k-2} \frac{1 - X_{2k+1} X_{2k+3}}{1 - X_{2k+3}/x_{2k+2}}, \end{aligned}$$

as desired. ∎

Notice that Theorem 4.1 is the case $n = 1$ of Theorem 6 in [12], which provides the generating functions for k-elongated partition diamonds of length n.

For $K^3 = A^1 \oplus ((k-1) \times A^3) \oplus A^1$, one can prove in a similar way the following result.

Theorem 4.2 *For any integer $k > 1$, we have*

$$\begin{aligned} & f_{K^3}(x_1, \ldots, x_{3k-1}) \\ &= \prod_{i=1}^{3k-1} \frac{1}{1 - X_i} \prod_{j=0}^{k-2} \frac{h_j(\mathbf{x})}{(1 - x_{3j+3} X_{3j+1})(1 - x_{3j+4} X_{3j+1})(1 - X_{3j+4}/x_{3j+2})(1 - X_{3j+4}/x_{3j+3})}, \end{aligned}$$

where

$$\begin{aligned} h_j(\mathbf{x}) &= 1 + 2(1 + x_{3j+2})(1 + x_{3j+3})(1 + x_{3j+4}) X_{3j+1}^2 X_{3j+4} + X_{3j+1}^3 X_{3j+4}^3 \\ &\quad - \left(x_{3j+2} + x_{3j+3} + x_{3j+4} + \frac{1}{x_{3j+2}} + \frac{1}{x_{3j+3}} + \frac{1}{x_{3j+4}} + 3 \right) (1 + X_{3j+1} X_{3j+4}) X_{3j+1} X_{3j+4}. \end{aligned}$$

More generally, for a sequence $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ of positive integers, we define the *complete plane partitions* of type $\mathbf{s}$ to be the P-partitions related to the poset

$$CP_{\mathbf{s}} = A^1 \oplus A^{s_1} \oplus A^{s_2} \oplus \cdots \oplus A^{s_k} \oplus A^1.$$

We are interested in the specialization $x_i = q$ of $f_{CP_{\mathbf{s}}}(\mathbf{x})$. Noting that $A^r_{[j_1,\ldots,j_l]}$ is isomorphic to the poset CL^{r-l} discussed in subsection 3.2, we have

$$\begin{aligned}
h_{A^r}(y_0, q, \ldots, q) &= \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} \frac{1}{(1-q)^{r-l}} \sum_{i=0}^{r-l} \frac{(-1)^i \binom{r-l}{i} q^i}{1 - y_0 q^{i+l}} \\
&= \sum_{l=1}^{r} (-1)^{l-1} \frac{\binom{r}{l}}{(1-q)^{r-l}} \sum_{j=l}^{r} \frac{(-1)^{j-l} \binom{r-l}{j-l} q^{j-l}}{1 - y_0 q^j} \\
&= \frac{1}{(1-q)^r} \sum_{j=1}^{r} \frac{(-1)^{j-1} \binom{r}{j} q^j}{1 - y_0 q^j} \sum_{l=1}^{j} \binom{j}{l} \left(\frac{1-q}{q} \right)^l \\
&= \frac{1}{(1-q)^r} \sum_{j=1}^{r} \frac{(-1)^{j-1} \binom{r}{j} (1 - q^j)}{1 - y_0 q^j}.
\end{aligned}$$

Thus we derive

Theorem 4.3 *The specialization $x_i = q$ of the generating function for complete plane partitions of type $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ is given by*

$$f_{CP_{\mathbf{s}}}(q, \ldots, q) = \frac{1}{(1-q)(1-q^{l_k+2})} \prod_{i=1}^{k} h_{s_i}(q^{l_{i-1}+1}),$$

where $l_0 = 0$, $l_i = s_1 + \cdots + s_i$ $(1 \le i \le k)$ and

$$h_r(x) = \frac{1}{(1-q)^r} \sum_{j=1}^{r} \frac{(-1)^{j-1} \binom{r}{j} (1 - q^j)}{1 - x q^j}.$$

As an example, for $K^r = CP_{(r,\ldots,r)} = A^1 \oplus ((k-1) \times A^r) \oplus A^1$, we have

$$\begin{aligned}
f_{K^3}(q, \ldots, q) &= \frac{\prod_{i=0}^{k-2} (1 + 2q^{3i+2} + 2q^{3i+3} + q^{6i+5})}{(q;q)_{3k-1}}, \\
f_{K^4}(q, \ldots, q) &= \frac{\prod_{i=0}^{k-2} (1 + q^{4i+3})(1 + 3q^{4i+2} + 4q^{4i+3} + 3q^{4i+4} + q^{8i+6})}{(q;q)_{4k-2}}.
\end{aligned}$$

Unfortunately there are no elegant formulaes for $f_{K^r}(q, \ldots, q)$ when $r \ge 5$.

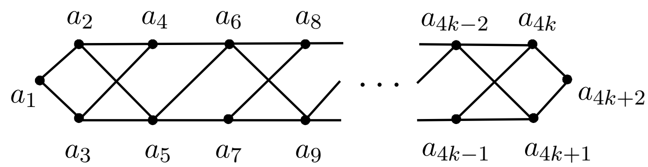

Figure 9: The poset for plane partitions with diagonals.

4.2 Plane Partitions with Diagonals

In [10], Andrews, Paule and Riese studied plane partitions with diagonals whose corresponding poset PD_k can be depicted as Figure 9. We will recover the generating function $f_{PD_k}(\mathbf{x})$ using Theorem 2.3 and Corollary 2.4.

It is clear that PD_k is isomorphic to $A \oplus ((k-1) \times B) \oplus C$ with the posets A, B and C given by Figure 10.

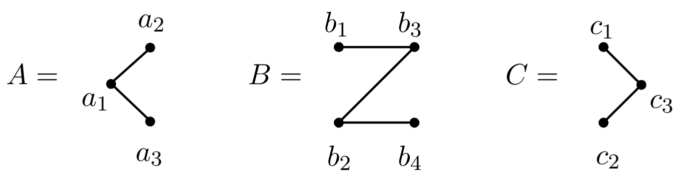

Figure 10: Three basic blocks of the poset PD_k.

To compute $h_B(y_0, y_1, y_2, y_3, y_4)$, we observe that $B_{[1]}, B_{[2]}, B_{[1,2]}$ are given by Figure 11. With the aid of the `Maple` package `Ell2`, we can find the generating functions $f_{B_{[1]}}(\mathbf{x}), f_{B_{[2]}}(\mathbf{x})$

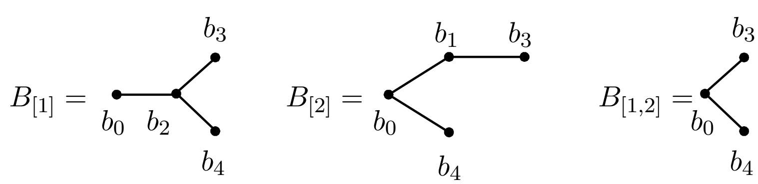

Figure 11: The posets related to B.

and $f_{B_{[1,2]}}(\mathbf{x})$. After simplifying, we obtain that $h_B(\mathbf{y}) = p_1(\mathbf{y})/p_2(\mathbf{y})$ with

$$\begin{aligned} p_1(y_0, y_1, \ldots, y_4) = 1 &- y_0^2 y_1 y_2 (1 + y_4 + y_2 y_4 + y_2 y_3 y_4 + y_1 y_2 y_3 y_4) \\ &+ y_0^3 y_1 y_2^2 y_4 (1 + y_1 + y_1 y_3 + y_1 y_2 y_3 + y_1 y_2 y_3 y_4) - y_0^5 y_1^3 y_2^4 y_3 y_4^2. \end{aligned}$$

and

$$p_2(y_0, y_1, \ldots, y_4) = (1 - y_0 y_2)(1 - y_0 y_2 y_4)(1 - y_0 y_1 y_2 y_4) \prod_{i=1}^{4} (1 - y_0 y_1 \cdots y_i).$$

Similarly, we have

$$h_C(y_0, y_1, y_2, y_3) = \frac{1 - y_0^2 y_1 y_2}{(1 - y_0 y_2) \prod_{i=1}^{3} (1 - y_0 y_1 \cdots y_i)}.$$

Now by Theorem 2.3 and Corollary 2.4, we recover Andrews-Paule-Riese's formula:

Theorem 4.4 *For any integer $k > 1$, we have*

$$\begin{aligned} f_{PD_k}(x_1, \ldots, x_{4k+2}) = {} & \frac{(1 - X_1 X_3)(1 - X_{4k-1} X_{4k+1})}{(1 - X_3/x_2)(1 - X_{4k+1}/x_{4k})} \prod_{i=1}^{4k+2} \frac{1}{1 - X_i} \\ & \times \prod_{j=0}^{k-2} \frac{h_j(\mathbf{x})}{(1 - X_{4j+5}/x_{4j+4})(1 - X_{4j+7}/x_{4j+6})(1 - X_{4j+7}/x_{4j+4}x_{4j+6})}, \end{aligned}$$

where

$$\begin{aligned} h_j(\mathbf{x}) = {} & 1 - x_{4j+5} x_{4j+7} X_{4j+3}^2 X_{4j+5}^2 X_{4j+7} + (1 + 1/x_{4j+4} x_{4j+6}) X_{4j+5} X_{4j+7} (X_{4j+3} - 1) \\ & + (x_{4j+5} + 1/x_{4j+6}) X_{4j+3} X_{4j+7} (X_{4j+5} - 1) + (x_{4j+5} x_{4j+7} X_{4j+7} - 1) X_{4j+3} X_{4j+5}. \end{aligned}$$

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References

[1] G. E. Andrews, MacMahon's partition analysis I: the lecture hall theorem, Progr. Math., **161** (1998) 1–22.

[2] G. E. Andrews, MacMahon's partition analysis II: fundamental theorems, Ann. Comb., **4** (2000) 327–338.

[3] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis III: the Omega package, Europ. J. Combin., **22** (2001) 887–904.

[4] G. E. Andrews and P. Paule, MacMahon's partition analysis IV: hypergeometric multisums, Sém. Lothar. Combin., **42** (1999) Art. B42, 24 pp.

[5] G. E. Andrews, P. Paule, A. Riese, and V. Strehl, MacMahon's partition analysis V: bijections, recursions, and magic squares, In: Algebraic Combinatorics and Applications, Lecture Notes in Mathmatics, A. Betten et. al. Eds., Springer-Verlag, 2001, pp. 1–39.

[6] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis VI: a new reduction algorithm, Ann. Comb., **5** (2001) 251–270.

[7] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis VII: constrained compositions, In: q-Series with Applications to Combinatorics, Numbers Theory, and Physics, Contemp. Math., vol. 291, B. C. Berndt and K. Ono, Eds., Amer. Math. Soc., 2001, pp. 11–27.

[8] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis VIII: plane partition diamonds, Adv. in Appl. Math., **27** (2001) 231–242.

[9] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis IX: k-gon partitions, Bull. Austral. Math. Soc., **64** (2001) 321–329.

[10] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis X: plane partitions with diagonals, South East Asian J. Math. Sci., **3** (2004) 3–14.

[11] G. E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis XI: hexagonal plane partitions, 2004. SFB Report n. 2004–4, J. Kepler University, Linz.

[12] G. E. Andrews, P. Paule, MacMahon's partition analysis XI: broken diamonds and modular forms. Acta Arith., **126** (2007) 281–294.

[13] G. E. Andrews and P. Paule, MacMahon's partition analysis XII: plane partitions, J. London Math. Soc., **76** (2007) 647–666

[14] G. E. Andrews, S. Corteel and C. D. Savage, On q-series identities arising from lecture hall partitions, International Journal of Number Theory, to appear.

[15] M. Bousquet-Mélou and K. Eriksson, Lecture hall partitions, Ramanujan J., **1** (1997) 101–111.

[16] G. Brightwell and P. Winkler, Counting linear extensions is $\#P$-complete, Order, **8** (1992) 225–242.

[17] S. Corteel and C. D. Savage, Anti-lecture hall compositions, Discrete Math., **263** (2003) 275–280.

[18] S. Corteel and C. D. Savage, Plane partition diamonds and generalizations, Integers, **3** (2003) A9.

[19] S. Corteel, S. Lee and C. D. Savage, Five guidelines for partition analysis with applications to lecture hall-type theorems, Integers, **7** (2007) A9.

[20] J. W. Davis, E. D. Souza, S. Lee and C. D. Savage, Enumeration of integer solutions to linear inequalities defined by digraphs, Contemporary Mathematics, to appear.

[21] S. B. Ekhad and D. Zeilberger, Using Rota's umbral calculus to enumerate to Stanley's P-partitions, Adv. in Appl. Mathematics, **41**(2008), 206–213.

[22] G. N. Han, A general algorithm for the MacMahon Omega operator, Ann. Comb., **7** (2003) 467–480.

[23] P. A. MacMahon, Memoir on the theory of the partition of numbers – Part VI, Phil. Trans., **211** (1912) 345–373.

[24] P. A. MacMahon, Combinatory Analysis, 2 Vols, Cambridge University Press, Cambridge, 1915–1916. (Reprinted: Chelsea, New York, 1960).

[25] R. P. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. **119**, Provedence R. I., 1972.

[26] R. P. Stanley, Enumerative Combinatorics Vol. I, Cambridge University Press, Cambridge, 1997.

[27] G. Xin, A fast algorithm for MacMahon's partition analysis, Electron. J. Combin., **11** (2004) R58.

[28] G. Xin, `http://www.combinatorics.net.cn/homepage/xin/MPA/Ell2.txt`.