# Component Factors with Large Components in Graphs

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**Abstract**

In this paper we obtain sufficient conditions using isolated vertices for component factors with each component of order at least three. In particular, we show that if a graph $G$ satisfies $iso(G-S) \leq |S|/2$ for all $S \subset V(G)$, then $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, where $iso(G-S)$ denotes the number of isolated vertices in $G-S$.

## 1 Introduction

In this paper we consider component factors of graphs, which are defined as follows. For a set $\mathcal{S}$ of connected graphs, a spanning subgraph $F$ of a graph $G$ is called an *$\mathcal{S}$-factor* of $G$ if every component of $F$ is an element of $\mathcal{S}$. An $\mathcal{S}$-factor is also referred as a *component factor*. There have been many papers on component factors of graphs, but in most cases, $\mathcal{S}$ contains $K_2$ (i.e., a single edge), but it is relatively rare that $\mathcal{S}$ contains no small component. As well, it is known that if $\mathcal{S}$ does not contain $K_2$, then in most cases finding a criterion for a graph to have an $\mathcal{S}$-factor is very difficult since finding a maximum $\mathcal{S}$-subgraph of a given graph is an $NP$-complete problem. In this paper we obtain several sufficient conditions in terms of the number of isolated vertices for a graph to have a component factor such that each component has order at least three.

We begin with some notation and definitions. We consider a finite simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. We denote by $|G|$ the order of $G$. For a subset $S \subseteq V(G)$, $G-S$ denotes the subgraph of $G$ induced by $V(G)-S$. For a vertex $v$ of $G$, the degree of $v$ and the neighborhood of $v$ in $G$ are denoted by $d_G(v)$ and $N_G(v)$, respectively. In particular, $d_G(v) = |N_G(v)|$. The minimum degree and the maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Denote by $\alpha(G)$ the independence number of $G$, which is the maximum cardinality among the

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independent sets of vertices of $G$. Let $iso(G)$ and $Iso(G)$ denote the number of isolated vertices and the set of isolated vertices of $G$, respectively. In particular, $iso(G) = |Iso(G)|$. For sets $X$ and $Y$, $X \subset Y$ means that $X$ is a proper subset of $Y$.

We denote the complete graph, the path and the cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. We denote the complete bipartite graph by $K_{n,m}$. A criterion for a graph to have a star-factor is given below.

**Theorem 1.** (Amahashi and Kano [1]) *A graph $G$ has a star-factor, i.e., $\{K_{1,1}, \ldots, K_{1,n}\}$-factor, if and only if $iso(G-S) \le n|S|$ for all $S \subset V(G)$.*

A graph $R$ is called *factor-critical* if for every vertex $x$ of $R$, $R - x$ has a 1-factor ($K_2$-factor). A graph $H$ is called a *sun* if $H = K_1$, $H = K_2$ or $H$ is the corona of a factor critical graph $R$ with order at least three, i.e., $H$ is obtained from $R$ by adding a new vertex $w = w(v)$ together with a new edge $vw$ for every vertex $v$ of $R$ (Figure 1). A sun with order at least 6 is called a *big sun*. The number of sum components of $G$ is denoted by $sun(G)$. The next theorem gives a criterion for a graph to have a path-factor each of whose components is of order at least three. Note that a shorter proof of the following theorem and a formula for a maximum $\{P_3, P_4, P_5\}$-subgraph of a graph was given in [3].

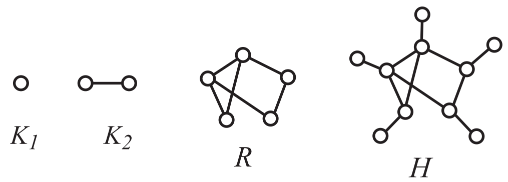


Figure 1: A factor-critical graph $R$ and the sun $H$ obtained from $R$.

**Theorem 2.** (Kaneko [2]) *A graph $G$ has a $\{P_3, P_4, P_5\}$-factor (i.e., $P_{\ge 3}$-factor) if and only if $sun(G-S) \le 2|S|$ for all $S \subset V(G)$.*

In this paper we consider the following problem, and give partial answers to the problem.

**Problem 1.** *Let $G$ be a graph and $\lambda$ be a positive rational number. If $iso(G-S) \le \lambda|S|$ for all $\emptyset \ne S \subset V(G)$, what factor does $G$ have?*

## 2 Component Factors with Large Components

In this section, we first prove the next theorem.

**Theorem 3.** *If a graph $G$ satisfies*

$$iso(G-S) \le \frac{2}{3}|S| \qquad \textit{for all} \quad S \subset V(G),$$

*then $G$ has a $\{P_3, P_4, P_5\}$-factor.*

**Proof.** Suppose that $G$ satisfies the condition but has no $\{P_3, P_4, P_5\}$-factor. By Theorem 2, there exists a subset $S \subset V(G)$ such that $sun(G-S) > 2|S|$. Assume that there exist $a$ isolated vertices, $b$ $K_2$'s and $c$ big sun components $H_1, H_2, \ldots, H_c$, where $|H_i| \geq 6$, in $G-S$. We choose one vertex from each $K_2$ component of $G-S$, and denote the set of such vertices by $X$. Then $|X| = b$. For each $H_i$, let $R_i$ denote the factor-critical subgraph of $H_i$ and let $Y_i = V(R_i)$. Then $iso(H_i - Y_i) = |Y_i| = |H_i|/2$. Let $Y = \cup_{i=1}^{r} Y_i$. So we have

$$iso(G - (S \cup X \cup Y)) = a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}.$$

Moreover, it follows that

$$\begin{aligned} |S \cup X \cup Y| &< \frac{sun(G-S)}{2} + |X| + |Y| \qquad (\text{from } sun(G-S) > 2|S|) \\ &= \frac{a+b+c}{2} + b + \sum_{i=1}^{c} \frac{|H_i|}{2} \\ &\leq \frac{3}{2}\left(a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}\right) = \frac{3}{2} iso(G - (S \cup X \cup Y)). \end{aligned}$$

This contradicts the condition that $iso(G - S') \leq (2/3)|S'|$ for all $S' \subset V(G)$. ■

Let $m \geq 1$ be an integer Let $G = K_m + (2m+1)K_2$, which is a graph obtained from $K_m$ and $(2m+1)K_2$ by joining every vertex of $K_m$ to every vertex of $(2m+1)K_2$. Then $G$ has no $\{P_3, P_4, P_5\}$-factor. Let $T \subseteq V(G)$ be an independent set with $|T| \geq 2$. Then $T \subseteq V((2m+1)K_2)$ and so $|N_G(T)| = |T| + m$. If $|T| \leq 2m$, then $i(G - N_G(T)) \leq 2|N_G(T)|/3$, otherwise $i(G - N_G(T)) = 2|N_G(T)|/3 + 1 = 2m+1$. Since $\delta(G) \geq m+1 \geq 2$, so $i(G-S) \leq 2|S|/3 + 1$ for all $S \subseteq V(G)$. Therefore the condition of Theorem 3 is sharp.

The next lemma is knows as Harlem Theorem, which is a generalization of Hall's Theorem.

**Lemma 1.** *Let $G$ be a bipartite graph with bipartition $(U, W)$, and $f : U \to \{1, 2, 3, \ldots\}$. If $|W| = \sum_{x \in U} f(x)$ and*

$$|N_G(S)| \geq \sum_{x \in S} f(x) \qquad \text{for all} \quad \emptyset \neq S \subseteq U,$$

*then $G$ has a star-factor $F$ such that each vertex $u$ of $U$ satisfies $d_F(u) = f(u)$, that is, every $u$ is the center of a star $K_{1,f(u)}$ in $F$.*

We next consider graphs satisfying $iso(G-S) \leq |S|/2$ for all $S \subset V(G)$.

**Lemma 2.** *If $|G| \leq 6$ and $iso(G-S) \leq |S|/2$ for all $S \subset V(G)$, then $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

**Proof.** It is clear that if $G$ satisfies the condition, then $\delta(G) \geq 2$ and $|G| \geq 3$. If $|G| = 3$, then $G$ is connected and has a $K_{1,2}$-factor. If $|G| = 4$, then $\Delta(G) = 3$, which implies that

$G$ has a $K_{1,3}$-factor. Assume $|G| = 5$. If $G$ has two non-adjacent vertices $x$ and $y$, then $2 = |\{x, y\}| = iso(G - (V(G) - \{x, y\})) \leq |V(G) - \{x, y\}|/2 = 3/2$, a contradiction. Hence $G$ is a complete graph $K_5$, and so it has a $K_5$-factor. Now we consider the case of $|G| = 6$. By Theorem 2, $G$ has a $\{P_3, P_4, P_5\}$-factor, say $F$. Then $F$ must be a $P_3$-factor, which is a $K_{1,2}$-factor. Therefore the lemma holds. ∎

**Theorem 4.** *If a graph $G$ satisfies*

$$iso(G - S) \leq \frac{|S|}{2} \qquad \textit{for all} \quad S \subseteq V(G),$$

*then $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

**Proof.** It is clear that $|G| \geq 3$ and $\delta(G) \geq 2$. Use induction on the lexicographic order of $(|G|, |E(G)|)$. So we assume that the theorem holds for a graph $H$ with either $|H| < |G|$ or $|H| = |G|$ and $|E(H)| < |E(G)|$. Moreover, we may assume that $G$ is connected and $|G| \geq 7$ by Lemma 2. Let

$$\beta = \min\left\{ \frac{|S|}{2} - iso(G - S) \;\middle|\; S \subset V(G) \text{ and } iso(G - S) \geq 1 \right\}.$$

Then $\beta \geq 0$ as $iso(G - S) \leq |S|/2$. For a vertex $x$ with $d_G(x) = \delta(G)$, we have $\beta \leq |N_G(x)|/2 - iso(G - N_G(x))$ and so

$$\delta(G) = d_G(x) = |N_G(x)| \geq 2(\beta + iso(G - N_G(x))) \geq 2(\beta + 1). \tag{1}$$

Take a maximal vertex subset $S$ such that $|S|/2 - iso(G - S) = \beta$. Then

$$\frac{|S'|}{2} - iso(G - S') > \beta \qquad \text{for all} \qquad S \subset S' \subset V(G). \tag{2}$$

*Claim 1. $G - S$ has no component of order two or three.*

Assume that $G - S$ has a component $D$ isomorphic to $K_2$. Let $V(D) = \{x, y\}$. Then

$$\begin{aligned} & \frac{|S \cup \{x\}|}{2} - iso(G - (S \cup \{x\})) \\ = \ & \frac{|S| + 1}{2} - (iso(G - S) + 1) < \beta, \end{aligned}$$

a contradiction.

Assume that $G - S$ has a component $D$ of order three. Let $V(D) = \{x, y, z\}$. Then

$$\begin{aligned} & \frac{|S \cup \{x, y\}|}{2} - iso(G - (S \cup \{x, y\})) \\ = \ & \frac{|S| + 2}{2} - (iso(G - S) + 1) = \beta, \end{aligned}$$

a contradiction to the maximality of $S$.

*Claim 2. Every component $D$ of $G - S$ with $|D| \geq 4$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

Let $X$ be a non-empty subset of $V(D)$. Then by (2), we have

$$\frac{|S \cup X|}{2} - iso(G - (S \cup X)) > \beta = \frac{|S|}{2} - iso(G - S).$$

Thus $|X|/2 > iso(D - X)$, which implies that $D$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor by the induction hypothesis.

By Claim 1, let $G - S = aK_1 \cup \big(D_1 \cup \cdots \cup D_c\big)$, where $V(aK_1) = Iso(G - S) = \{u_1, \ldots, u_a\}$ and each $D_i$ is a component of $G - S$ with $|D_i| \geq 4$. It is immediate that

$$a = iso(G - S) = |S|/2 - \beta \geq 1. \tag{3}$$

We construct a bipartite graph $B$ with vertex set $V(B) = S \cup U$, where $U = \{u_1, u_2, \ldots, u_a\}$, such that two vertices $u_i \in U$ and $x \in S$ are adjacent in $B$ if and only if $u_i$ and $x$ are joined by an edge of $G$.

*Claim 3. For every $\emptyset \neq Y \subseteq U$, we have $|N_B(Y)| \geq 2|Y| + 2\beta$, and $|N_B(U)| = 2|U| + 2\beta = |S|$.*

It follows from (3) and the choice of $S$ that $|N_B(U)| = |S| = 2a + 2\beta = 2|U| + 2\beta$. Assume that there exists a subset $\emptyset \neq Y' \subset U$ such that $N_B(Y') < 2|Y'| + 2\beta$. Then, by the definition of $\beta$, $N_B(Y') = N_G(Y') \subset S$ satisfies

$$|Y'| \leq iso(G - N_G(Y')) \leq \frac{|N_G(Y')|}{2} - \beta < |Y'|,$$

a contradiction. Hence the claim holds.

*Claim 4. If $\beta \geq 2$, then the theorem holds.*

Assume $\beta \geq 2$. Then $\delta(G) \geq 6$ by (1). It is obvious that $G$ has an edge $e$ such that $G - e$ is connected. Let $X \subset V(G - e) = V(G)$. If $iso(G - X) \geq 1$, then

$$iso(G - e - X) \leq iso(G - X) + 2 \leq \frac{|X|}{2} - \beta + 2 \leq \frac{|X|}{2}.$$

If $iso(G - X) = 0$, then $iso(G - e - X) \leq 2$. Further $iso(G - e - X) \geq 1$ implies $|X| \geq 5$ as $\delta(G - e) \geq 5$. Hence if $iso(G - X) = 0$, then $iso(G - e - X) \leq 2 \leq |X|/2$. Therefore by the induction hypothesis, $G - e$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which is of course the desired factor of $G$.

From Claim 4 and the definition of $\beta$, it remains to consider the cases of $\beta \in \{0, 1/2, 1, 3/2\}$. Note that $|S| = 2|U| + 2\beta$.

*Case 1.* $\beta = 0$.

Define $f : U \to \{1, 2, 3 \ldots\}$ by $f(u) = 2$ for all $u \in U$. Then by Lemma 1 and Claim 3, $B$ has a $K_{1,2}$-factor with centers in $U$. Hence by Claim 2, $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

*Case 2.* $\beta = 1/2$.

In this case, $|S| = 2|U| + 1$. Choose a vertex $u_1 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1\}$. Then $|N_B(Y)| \geq \sum_{x \in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence by Lemma 1, $B$ has a $\{K_{1,2}, K_{1,3}\}$-factor. Therefore we can obtain a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of $G$.

*Case 3.* $\beta = 1$.

Clearly, $\delta(G) \geq 4$ by (1). We consider two subcases.

*Subcase 3.1.* $|U| \geq 2$.

In this case, $|S| = 2|U| + 2$. Choose two vertex $u_1, u_2 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = f(u_2) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1, u_2\}$. Then $|N_B(Y)| \geq \sum_{x \in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence, by Lemma 1, $B$ has a $\{K_{1,2}, K_{1,3}\}$-factor and so $G$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

*Subcase 3.2.* $|U| = 1$.

It is clear that $|S| = 2|U| + 2 = 4$ and $V(G) \neq S \cup U$ as $|G| \geq 7$. Let $H = G - (S \cup U)$, $U = \{u\}$ and $S = \{s_1, s_2, s_3, s_4\}$. Consider $G - \{s_1, u, s_2\}$. If $iso(G - \{s_1, u, s_2\} - X) \leq |X|/2$ for all $X \subseteq V(G) - \{s_1, u, s_2\}$, then the theorem follows by the induction hypothesis. So we may assume there exists a subset $R \subseteq V(G) - \{s_1, u, s_2\}$ such that $iso(G - \{s_1, s, s_2\} - R) > |R|/2$. However it follows that

$$\frac{3}{2} = \frac{|R| + 3}{2} - \frac{|R|}{2} < \frac{|R \cup \{s_1, u, s_2\}|}{2} - iso(G - \{s_1, u, s_2\} - R) \leq \beta = 1,$$

a contradiction.

*Case 4.* $\beta = 3/2$.

By (1), we have $\delta(G) \geq 5$. Let $uv, vw \in E(G)$. Then for every $X \subseteq V(G) - \{u, v, w\}$ with $iso(G - \{u, v, w\} - X) \geq 1$, it follows that

$$iso(G - \{u, v, w\} - X) \leq \frac{|X \cup \{u, v, w\}|}{2} - \beta \leq \frac{|X|}{2}.$$

If $iso(G - \{u, v, w\} - X) = 0$, then obviously $iso(G - \{u, v, w\} - X) \leq |X|/2$. Hence by the induction hypothesis, $G - \{u, v, w\}$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of $G$.

Consequently the theorem is proved. ■

We now show that the condition in Theorem 4 is sharp. Consider a graph $G$ given in Figure 2. Then $G$ satisfies $iso(G - S) \leq (|S| + 1)/2$ for all $S \subset V(G)$, but has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Hence the condition of the theorem is sharp in this sense. The condition of Theorem 4 is sufficient but not necessary. For example, let $G = K_{1,3}$ (or $C_{3m}$, where $m \geq 2$). Then $G$ contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor but dissatisfies the condition of Theorem 4.

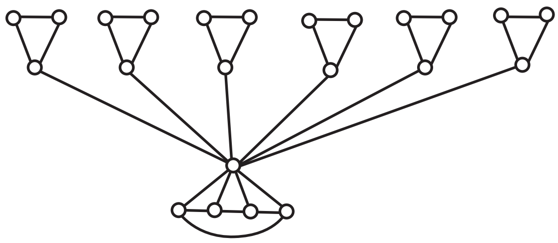

Figure 2: A graph has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

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