Constructive proof of deficiency theorem of (g, f)-factor *

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Abstract

Berge [2] gave a formula for computing the deficiency of maximum matchings of a graph. More generally, Lovász obtained a deficiency formula of (g, f)-optimal graphs and consequently a criterion for the existence of (g, f)-factors. Moreover, Lovász proved that there is one of these decompositions which is "canonical" in a sense. In this paper, we present a short constructive proof for the deficiency formula of (g, f)-optimal graphs, and the proof implies an efficient algorithm of time complexity $O(g(V)|E|)$ for computing the deficiency. Furthermore, this proof implies this canonical decomposition (that is, in polynomial time) via efficient algorithms.

1 Introduction

In this paper, we consider finite undirected simple graphs without loop and multiple edge. For a graph $G = (V, E)$, the degree of x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. Let T and H be two graphs and $R = T \cup H$ denote a graph with $E(R) = E(T) \cup E(H)$ and $V(R) = V(H) \cup V(T)$. A set M of edges in a graph G is a *matching* if no two members of M share a vertex. A matching M is a *maximum matching* of G if there doesn't exist a matching M' of G such that $|M'| > |M|$. A matching M is *perfect* if every vertex of G is covered by an edge of M.

Let f and g be two nonnegative integer-valued functions on $V(G)$ with $g(x) \le f(x)$ for every $x \in V(G)$. A spanning subgraph F of G is a (g, f)*-factor* if $g(v) \le d_F(v) \le f(v)$ for all $v \in V(G)$. When $g \equiv f \equiv 1$, a (g, f)-factor is called a 1*-factor* (or perfect matching).

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For 1-factors in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem. In 1947, Tutte gave a characterization (i.e., so-called Tutte's 1-Factor Theorem) for the existence of 1-factors in arbitrary graphs. Berge [2] discovered the deficiency formula of maximum matchings, which is often referred as Berge's formula.

The more general version of deficiency formula for (g, f)-optimal subgraphs was investigated by Lovász [6]. In this paper, we present a short proof to Lovász's deficiency formula by using alternating trail.

Theorem 1.1 (Lovász, [6]) *Let G be a graph and $g, f : V(G) \to \mathbb{Z}$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then*

$$def(G) = \max\{\sum_{t\in T} d_{G-S}(t) - g(T) + f(S) - q_G(S,T) \geq 0 \mid \ S, T \subseteq V(G) \text{ and } S \cap T = \emptyset\},$$

where $q_G(S, T)$ denotes the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(V(C), T) + \sum_{x\in V(C)} f(x) \equiv 1$ (mod 2).

2 The short proof of deficiency formula

Given two integer-valued functions f and g with $g \leq f$ and a subgraph H of G, we define the *deficiency* of H with respect to $g(v)$ as

$$def_H(G) = \sum_{v\in V} \max\{0, g(v) - d_H(v)\}.$$

Suppose that G contains no (g, f)-factor. Choose a spanning subgraph F of G satisfying $d_F(v) \leq f(v)$ for every vertex $v \in V$ such that the deficiency is minimized over all such choices. Then F is called as a (g, f)-*optimal subgraph* of G. Necessarily, there is a vertex $v \in V$ such that $d_F(v) < g(v)$ and so the deficiency of F is positive.

In the rest of the paper, F always denotes a (g, f)-optimal subgraph. The deficiency of G, $def(G)$, is defined as $def_F(G)$ and the deficiency of an induced subgraph $G[S]$ of G for a vertex subset $S \subseteq V$ by $def_F(S)$.

Let $B_0 = \{v \mid d_F(v) < g(v)\}$. An F-*alternating trail* is a trail $P = v_0v_1 \ldots v_k$ with $v_iv_{i+1} \notin F$ for i even and $v_iv_{i+1} \in F$ for i odd.

We define

$$D^* = \{v \mid \exists \text{ both an even and an odd } F\text{-alternating trails from vertices of } B_0 \text{ to } v\},$$
$$B^* = \{v \mid \exists \text{ an even } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D^*,$$
$$A^* = \{v \mid \exists \text{ an odd } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D^*,$$

and $C^* = V(G) - A^* - B^* - D^*$. Clearly, D^*, B^*, A^*, and C^* are disjoint. We call an F-alternating trail M an *augmenting trail* if $def_F(G) > def_{F\triangle M}(G)$.

For any $v \in B^*$, then $d_F(v) \leq g(v)$, or else by exchanging edges of F along an even alternating trail ending in v, we decrease the deficiency. Similarly, $d_F(v) = f(v)$ for any $v \in A^*$, or else we can decrease the deficiency by exchanging edges of F along an odd alternating trail ending in v. By the above discussion, we have $d_F(v) = g(v) = f(v)$ for every $v \in V(D^*) - B_0$.

From the definitions stated above, we can easily see the following lemma.

Lemma 2.1 *If F is an optimal subgraph, then F cannot contain an augmenting trail.*

Let τ denote the number of components of $G[D^*]$ and $D_1, \ldots, D_\tau$ be the components of $G[D^*]$.

Lemma 2.2 *$def_F(D_i) \leq 1$ for $i = 1, \ldots, \tau$ and $g(v) = f(v)$ for any $v \in D^*$.*

Proof. Suppose the result does not hold. Let $v \in D_i$ and $def_F(v) \geq 1$. By the definition of D^*, there exists an odd alternating trail C from a vertex x of B_0 to v. Then $x = v$, otherwise, $def_F(G) > def_{F\Delta C}(G)$, a contradiction. Furthermore, if $def_F(v) \geq 2$, then $def_F(G) > def_{F\Delta C}(G)$, a contradiction. So we have $def_F(v) = 1$ and $def_F(u) \leq 1$ for any $u \in D_i - v$. Moreover, $d_F(v) + 1 = f(v) = g(v)$. Set

$$D_i^1 = \{w \in D_i \mid \exists \text{ both an odd alternating trail } T_1 \text{ and an even alternating trail } T_2 \text{ from } v \text{ to } w \text{ such that } V(T_1 \cup T_2) \subseteq D_i\}.$$

Now we choose a maximal subset D_i^2 of D_i^1 such that $C \subseteq D_i^2$ and there exist both an odd alternating trail T_1 and an even alternating trail T_2 from v to w, where $V(T_1 \cup T_2) \subseteq D_i^2$, for any $w \in D_i^2$.

Claim. $D_i^2 = D_i$.

Otherwise, since D_i is connected, then there exists an edge $xy \in E(G)$ such that $x \in D_i - V(D_i^2)$ and $y \in V(D_i^2)$. We consider two cases.

Case 1. $xy \in E(F)$.

Then there exists an even alternating trail P_1 from v to x, where $xy \in P_1$ and $V(P_1) - x \subseteq V(D_i^2)$. Since $x \in D_i$, there exists an odd alternating trail P_2 from a vertex t of B_0 to x. Then $t \neq v$, otherwise, we have $V(P_1 \cup P_2) \subseteq D_i^2$, a contradiction. If $E(P_1) \cap E(P_2) = \emptyset$, then $def_F(G) > def_{F\triangle(P_1 \cup P_2)}(G)$, a contradiction. Let $z \in P_2$ be the first vertex which also belongs to D_i^2 and denote the subtrail of P_2 from t to z by P_3. By the definition, there exist both an odd alternating trail P_4 from v to z and an even alternating trail P_5 from v to z such that $V(P_4 \cup P_5) \subseteq V(D_i^2)$. Thus either $P_4 \cup P_3$ or $P_5 \cup P_3$ is an augmenting trail, a contradiction.

Case 2. $xy \notin E(F)$.

The proof is similar to that of Case 1.

Let $u \in V(D_i) - v$ and $def_F(u) = 1$, then there exists an odd alternating trail P_6 from v to u. We have $def_F(G) > def_{F\triangle P_6}(G)$, a contradiction.

We complete the proof. $\square$

By the proof of above lemma, we have the following result.

Lemma 2.3 *If* $def_F(D_i) = 1$, *then* $E[D_i, B^*] \subseteq F$ *and* $E[D_i, A^*] \cap E(F) = \emptyset$ *for* $i = 1, \ldots, \tau$.

Proof. Let $def_F(r) = 1, r \in V(D_i)$. Suppose the result does not hold. Let $uv \notin E(F)$, where $u \in V(D_i), v \in V(B^*)$. By the proof of Lemma 2.2, there exists an even alternating trail $P \subseteq G[D_i]$ from r to u. Then $P \cup uv$ be an odd alternating trail from r to v, a contradiction.

Let $xy \in E(F)$, where $x \in D_i$ and $y \in A^*$. Then there exists an odd alternating trail $P_1 \subseteq G[D_i]$ from r to x. Hence $P_1 \cup xy$ is an even alternating trail from r to y, a contradiction. $\square$

From the definitions of B^*, C^* and D^*, the following lemma follows immediately.

Lemma 2.4 $E[B^*, C^* \cup B^*] \subseteq F$, $E[D^*, C^*] = \emptyset$.

Lemma 2.5 *F misses at most an edge from D_i to B^* and contains at most an edge from D_i to A^*; if F misses an edge from D_i to B^*, then $E[D_i, A^*] \cap F = \emptyset$; if F contains an edge from D_i to A^*, then $E[D_i, B^*] \subseteq F$.*

Proof. By Lemma 2.3, we may assume $def_F(D_i) = 0$. Let $u \in V(D_i)$, by the definition of D^*, there exists an alternating trail P from a vertex x of B_0 to u. Without loss of generality, suppose that the first vertex in P belonging to D_i is y, and the sub-trail of P from x to y is denoted by P_1, which is an odd alternating trail. Then $y_1 \in B^*$, $y_1y \in E(P_1)$ and $y_1y \notin F$. Since $y \in D^*$, there exists an even alternating trail P_2 from a vertex x_1 of B_0 to y. Hence we have $y_1y \in E(P_2)$, otherwise, $P_2 \cup y_1y$ is an odd alternating trail from x_1 to y_1, a contradiction. Let P_3 be a sub-trail of P_2 from y to y. Then we have $V(P_3) \subseteq D_i$. Set

$$D_i^1 = \{w \in D_1 \mid \exists \text{ both an odd alternating trail } R_1 \text{ and an even alternating trail } R_2 \text{ traversing } P_1 \text{ from } x \text{ to } w \text{ such that } V(R_1 \cup R_2) - V(P_1 - y) \subseteq D_i\}.$$

Now we choose a maximal subset D_i^2 of D_i^1 such that $V(P_3) \subseteq D_i^2$ and there are both an odd alternating trail T_1 and an even alternating trail T_2 traversing P_1 from x to w, where $V(T_1 \cup T_2) - (V(P_1) - y)) \subseteq D_i^2$, for every $w \in D_i^2$.

With a similar proof as in that of Lemma 2.2, we have $D_i^1 = D_i = D_i^2$. Let $x_3y_3 \in E(G) - yy_1$, where $x_3 \in D_i$ and $y_3 \in B^*$. By the above discussion, there exists an even alternating trail P_4 from y_1 to x_3 such that $V(P_4) - y_1 \subseteq D_i$. If $x_3y_3 \notin E(F)$, then $P_1 \cup P_4 \cup x_3y_3$ is an odd alternating trail from x to y_3, a contradiction. Now suppose $x_4y_4 \in E(F)$, where $x_4 \in D_i$ and $y_4 \in A^*$. Similarly, there exists an odd alternating trail P_5 from y_1 to x_4 and $V(P_5) - y_1 \subseteq D_i$, and then $P_1 \cup (P_5 - y_1) \cup x_4y_4$ is an even alternating trail from x to y_4, a contradiction. The proofs of other cases can be dealt similarly.

We complete the proof. $\square$

From Lemmas 2.2 and 2.5, we obtain the following.

Lemma 2.6 *For $i = 1, \ldots, \tau$, we have*

$$|E[D_i, B^*]| + \sum_{v \in D_i} f(v) \equiv 1 \ (mod\ 2),$$

and for every component R of $G[C^*]$, either $g(v) = f(v)$ for all $v \in R$ and

$$|E[R, B^*]| + \sum_{v \in R} f(v) \equiv 0 \ (mod\ 2),$$

or there exists a vertex $v \in V(R)$ such that $g(v) < f(v)$.

Now we present a constructive proof to Lovász's deficiency theorem.

Theorem 2.7 (Lovász, [6]) $def_F(G) = \tau + g(B^*) - \sum_{v \in B^*} d_{G-A^*}(v) - f(A^*)$.

Proof. Let τ_1 denote the number of components of $G[D^*]$ which satisfies $def_F(D_i) = 1$ for $i = 1, \ldots, \tau$. Let τ_{B^*} (or τ_{A^*}) be the number of components T of $G[D^*]$ such that F misses (or contains) an edge from T to B^* (or A^*). By Lemmas 2.3 and 2.5, we have $\tau = \tau_1 + \tau_{A^*} + \tau_{B^*}$. Note that $d_F(v) \leq g(v)$ for all $v \in B^*$ and $d_F(v) = f(v)$ for all $v \in A^*$. So

$$\begin{aligned} def_F(G) &= \tau_1 + g(B^*) - \sum_{v \in B^*} d_F(v) \\ &= \tau_1 + g(B^*) - (\sum_{v \in B^*} d_{G-A^*}(v) - \tau_{B^*}) - (f(A^*) - \tau_{A^*}) \\ &= \tau + g(B^*) - \sum_{v \in B^*} d_{G-A^*}(v) - f(A^*). \end{aligned}$$

$\square$

Summarizing all discussion above, we have the following.

Theorem 2.8 *Let F be any (g,f)-optimal subgraph of G. Then we have*

(i) $d_F(v) \in [g(v), f(v)]$ *for all* $v \in C^*$*;*

(ii) $d_F(v) \leq g(v)$ *for all* $v \in B^*$*;*

(iii) $d_F(v) \geq f(v)$ *for all* $v \in A^*$*;*

(iv) $f(v) - 1 \leq d_F(v) \leq f(v) + 1$ *for all* $v \in D^*$*.*

Remark: The above proof shows that F is a (g,f)-optimal subgraph if and only if it does not admit an augmenting trail. Since each search for an augmenting trail can be performed by breadth-first search in time $O(|E|)$ and the corresponding augmentation lowers the value $g(x)$ for at least one vertex x, so we have a very simple (g,f)-factor algorithm of time complexity $O(g(V)|E|)$. By the above discussion, we also give an algorithm to determine if a graph is f-factor-critical. In particular, we obtain a canonical decomposition which is equivalent with the Lovás decomposition. So we obtain Lovás decomposition via efficient algorithms.

From Theorem 2.7, we are able to derive characterizations of various factors as consequences.

Corollary 2.9 (Lovász, [6]) *A graph G has a (g,f)-factor if and only if*

$$\tau^* - g(T) - \sum_{v \in T} d_{G-S}(v) - f(S) \leq 0,$$

for any pair of disjoint subsets $S, T \subseteq V(G)$, where τ^* denotes the number of components C of $G-S-T$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(V(C),T) + \sum_{x \in V(C)} f(x) \equiv 1$ (mod 2).

If $g(x) < f(x)$ for all $x \in V(G)$, then $D^* = \emptyset$ and $\tau = 0$. So, by Theorem 2.7, we obtain the following corollary.

Corollary 2.10 ([1], [4], [6]) *A graph G contains a (g,f)-factor, where $g < f$, if and only if*

$$g(T) - \sum_{v \in T} d_{G-S}(v) - f(S) \leq 0,$$

for any pair of disjoint subsets $S, T \subseteq V(G)$.

Note that for every component of D_i of $G[D^*]$, D_i contains an odd cycle. So if G is a bipartite graph, then $D^* = \emptyset$ and $\tau = 0$.

Corollary 2.11 ([1], [4], [6]) *Let G be a bipartite graph. Then G contains a f-factor if and only if*

$$f(T) - f(S) - \sum_{v \in T} d_{G-S}(v) \leq 0,$$

for any pair of disjoint subsets $S, T \subseteq V(G)$.

Given two nonnegative integer-valued functions f and g on $V(G)$ and a vertex-subset $S \subseteq V$, let $C_1(S)$ denote the number of odd components C of $G[S]$ with $g(x) = f(x) = 1$ for every vertex $x \in V(C)$. Let $C_o(S)$ be the number of odd components of $G[S]$.

Theorem 2.12 *Let $G = (V, E)$ be a graph and f, g be two integer-valued functions defined $V(G)$ such that $0 \leq g(x) \leq 1 \leq f(x)$ for each $v \in V(G)$. Then the deficiency of (g, f)-optimal subgraphs of G is*

$$def(G) = \max\{C_1(G - S) - f(S) \mid \ S \subseteq V(G)\}.$$

Proof. Clearly, $def(G) \geq \max\{C_1(G - S) - f(S) \mid \ S \subseteq V(G)\}$. So we only need to show that there exists $T \subseteq V(G)$ such that $def(G) = C_1(G - T) - f(T)$.

Let F be an optimal (g, f)-subgraph such that $d_F(v) \leq f(v)$ for all $v \in V(G)$. Let D^*, A^*, B^*, C^* and B_0 be defined above. Then $E(B^*, B^* \cup C^*) \subseteq F$. Let $W = \{x \in V(G) \mid \ g(x) = 0\}$, by Theorem 2.8, we have $W \cap (D^* \cup B^*) = \emptyset$.

Claim 1. $E(C^*, B^*) = \emptyset$, and $G[B^*]$ consists of isolated vertices.

Otherwise, assume $e = xy \in E(B^*, C^* \cup B^*)$, where $x \in B^*$. Then $e \in F$. Since $d_F(x) \leq 1$, so $d_F(x) = 1$ and $x, y \notin B_0$. By definition of B^*, there exists an even F-alternating trail P from a vertex u of B_0 to x. Then $e \in P$, but $P - e$ is an odd F-alternating trail from u to y, a contradiction.

Claim 2. $E(D^*, B^*) = \emptyset$.

Otherwise, let $uv \in E(G)$ with $u \in D_i^*$ and $v \in B^*$. Firstly, considering $uv \notin E(F)$, by the definition of B_0, there exists an even alternating trail P from B_0 to u. Note that $d_F(u) \leq 1$. If $uv \notin E(P)$, then $P \cup uv$ is an odd alternating trail from B_0 to v, a contradiction. So $uv \in E(P)$ and it is from v to u in the trail. Then we have $d_F(u) \geq 2$, a contradiction since $d_F(u) \leq f(u) = g(u) = 1$. So we assume $uv \in E(F)$. Then an even alternating trail from B_0 to u must contains uv since $d_F(v) \leq 1$ and thus $P - uv$ is an odd alternating trail from B_0 to v, a contradiction.

So every component of $G[D^*]$ is an odd component. By Lemma 2.3, then $|V(D_i)|$ is odd, for $i = 1, \ldots, \tau$. Hence,

$$\begin{aligned} def_F(G) &= \tau + g(B^*) - \sum_{v \in B^*} d_{G-A^*}(v) - f(A^*) \\ &= \tau + |B^*| - f(A^*) \\ &= C_1(G - A^*) - f(A^*). \end{aligned}$$

We complete the proof. □

Corollary 2.13 ([5]) *Let G be a graph and f, g be two integer-valued function defined $V(G)$ such that $0 \le g(x) \le 1 \le f(x)$ for all $v \in V(G)$. Then G has a (g, f)-factor if and only if, for any subset $S \subseteq V$*

$$C_o(G - X) \le f(X).$$

Let $f \equiv g \equiv 1$ in Theorem 2.12, Berge's Formula is followed.

Corollary 2.14 (Berge's Formula, [2]) *Let G be a graph. The number of vertices missed by a maximum matching of G is*

$$def(G) = \max\{C_o(G - S) - |S| \mid \ S \subseteq V(G)\}.$$

Corollary 2.15 (Tutte's 1-Factor Theorem) *Let G be a graph. Then there exists a 1-factor if and only if, for any sebset $S \subseteq V(G)$*

$$C_o(G - S) \le |S|.$$

Corollary 2.16 (Lovász, [6]) *Let G be a graph and X be a subset of $V(G)$. Then G contains a matching covering all vertices in X if and only if for each $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ odd components which entirely in X.*

Proof. Let f, g be two integer-valued functions defined $V(G)$ such that $f(x) = 1$ for every vertex $x \in V(G)$, and

$$g(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Then G contains a matching covering all vertices in X if and only if G has a (g, f)-factor. Suppose that G contains no a (g, f)-factor. Let A^*, B^*, C^*, D^* be defined above. We have $(V(G) - X) \cap (D^* \cup B^*) = \emptyset$. By Theorem 2.12, the result is followed. □

A graph G is said to have the *odd cycle property* if every pair of odd cycles in G either has a vertex in common or are joined by an edge. Let $i(G)$ be the number of isolated vertices in G.

Corollary 2.17 *Let G be a connected graph possessing odd cycle property, and f be an integer-valued function. Then G contains a $(1, f)$-factor if and only if $i(G-S) \leq f(S) - \varepsilon_0$ for every $S \subseteq V(G)$, where $\varepsilon_0 = 1$ if $G - S$ contains an odd component C with $|C| \geq 3$ and $f(v) = 1$ for all $v \in V(C)$; otherwise, $\varepsilon_0 = 0$.*

Proof. Suppose that G contains no $(1, f)$-factor. Let A^*, B^*, C^*, D^* be defined above. By the proof of Theorem 2.12, $E(B^*, B^* \cup C^* \cup D^*) = \emptyset$ and B^* consists of isolated vertices. Moreover, D^* contains at most one component C, where $|C| \geq 3$ is odd and $f(v) = 1$ for every $v \in V(C)$. Denote the number of components of D^* by ε. So, by Theorem 2.7, we have

$$def(G) = |B^*| + \varepsilon - f(A^*) = i(G - A^*) + \varepsilon - f(A^*).$$

We complete the proof. $\square$

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