CUBIC s-ARC TRANSITIVE CAYLEY GRAPHS

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ABSTRACT. This paper gives a characterization of connected cubic s-transitive Cayley graphs. It is shown that, for $s \geq 3$, every connected cubic s-transitive Cayley graph is a normal cover of one of 13 graphs: three 3-transitive graphs, four 4-transitive graphs and six 5-transitive graphs. Moreover, the argument in this paper also gives another proof for a well-known result which says that all connected cubic arc-transitive Cayley graphs of finite non-abelian simple groups are normal except two 5-transitive Cayley graphs of the alternating group A_{47}.

KEYWORDS. Cayley graph, s-arc-transitive, core-free, normal quotient.

1. INTRODUCTION

All graphs in this paper are assumed to be finite, simple and undirected.

Let Γ be a graph with vertex set $V(\Gamma)$, edge set $E(\Gamma)$ and full automorphism group $\mathsf{Aut}(\Gamma)$. Let X be a subgroup of $\mathsf{Aut}(\Gamma)$ (written as $X \leq \mathsf{Aut}(\Gamma)$). Then Γ is said to be X*-vertex-transitive* or X*-edge-transitive* if X acts transitively on $V(\Gamma)$ or on $E(\Gamma)$, respectively. Let s be a positive integer. An $(s+1)$-sequence $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices of Γ is called an s*-arc* if $\{\alpha_{i-1}, \alpha_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$. The graph Γ is called (X, s)*-arc-transitive* if Γ has at least one s-arc and X is transitive on the vertices and on the s-arcs of Γ; and Γ is said to be (X, s)*-transitive* if it is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive. In particular, a 1-arc is simply called an *arc*, and an $(X, 1)$-arc-transitive graph is said to be X*-arc-transitive* or X*-symmetric*. An arc-transitive graph Γ is said to be (X, s)*-regular* if it is (X, s)-arc-transitive and, for any two s-arcs of Γ, there is a unique automorphism of Γ mapping one arc to the other one. In the case where $X = \mathsf{Aut}(\Gamma)$, an (X, s)-arc-transitive ((X, s)-transitive, (X, s)-regular and X-symmetric, respectively) graph is simply called an s*-arc-transitive* (s*-transitive*, s*-regular* and *symmetric*, respectively) graph.

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Tutte [24, 25] proved that every finite connected cubic symmetric graph is s-regular for some $s \leq 5$. Since Tutte's seminal work, the study of s-arc-transitive graphs, aiming at constructing and characterizing such graphs, has received considerable attention in the literature, see [12, 13, 14, 10, 26, 2, 4, 5, 23, 6, 11, 17, 18, 20, 19, 28, 29] for example, and now there is an extensive body of knowledge on such graphs. In this paper, we investigate the cubic symmetric Cayley graphs.

Let G be a group and S a subset of G such that $S = S^{-1}:=\{g^{-1} \mid g \in S\}$ and S does not contain the identity element 1 of G. The *Cayley graph* $\mathsf{Cay}(G, S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Then a Cayley graph $\mathsf{Cay}(G, S)$ has valency $|S|$, and it is connected if and only if $\langle S\rangle = G$. Further, each $g \in G$ gives an automorphism $g : G \to G,\ x \mapsto xg$ of $\mathsf{Cay}(G, S)$. Thus G can be viewed as a regular subgroup of $\mathsf{Aut}(\mathsf{Cay}(G, S))$. A Cayley graph $\mathsf{Cay}(G, S)$ is said to be *normal* (with respect to G) if G is normal in $\mathsf{Aut}(\mathsf{Cay}(G, S))$; and $\mathsf{Cay}(G, S)$ is said to be *core-free* (with respect to G) if G is core-free in some $X \leq \mathsf{Aut}(\mathsf{Cay}(G, S))$, that is, $\mathsf{Core}_X(G) := \cap_{x\in X} G^x = 1$.

The main motivation for this paper arises from one result of Li [19] which says that for $s \in \{2, 3, 4, 5, 7\}$ and $k \geq 3$ there are only finite number of core-free s-transitive Cayley graphs of valency k, and that, with the exceptions $s = 2$ and $(s, k) = (3, 7)$, every s-transitive Cayley graph is a normal cover (see Section 3 for the definition) of a core-free one. In this paper, we shall give a characterization of cubic s-transitive Cayley graphs; in particular, determine all connected core-free cubic s-transitive Cayley graphs up to isomorphism, and then prove the following results.

Theorem 1.1. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected core-free (with respect to G) cubic s-transitive Cayley graph. Then $\Gamma \cong \mathsf{Cay}(G_{s,\imath}, S_{s,\imath})$ for $2 \leq s \leq 5$ and $1 \leq \imath \leq \ell_s$, where $\ell_2 = 2$, $\ell_3 = 3$, $\ell_4 = 4$, $\ell_5 = 6$, $G_{s,\imath} = \langle S_{s,\imath}\rangle$ and $S_{s,\imath}$ is given as in Subsections 4.1, 4.2, 4.3 and 4.4 while $s = 2$, 3, 4 and 5, respectively. Further, s, $\mathsf{Aut}(\Gamma)$ and G are listed in Table 1.*

Theorem 1.2. *Let Γ be a connected cubic s-transitive Cayley graph. Then*

(1) *$s \leq 2$ and $\mathsf{Aut}(\Gamma)$ contains a semi-regular normal subgroup which has at most two orbits on $V(\Gamma)$; or*
(2) *$\mathsf{Aut}(\Gamma)$ contains a regular subgroup which has a quotient group isomorphic to one of the groups listed in the third column of Table 1.*

2. A reduction to the core-free case

Let Γ be a connected X-vertex-transitive and X-edge-transitive graph with $X \leq \mathsf{Aut}(\Gamma)$. Denote by $\mathsf{val}(\Gamma)$ the valency of Γ. Let N be an intransitive normal subgroup of X and $\mathcal{B}$ be the set of N-orbits on $V(\Gamma)$. The *normal quotient* Γ_N of Γ induced by N is the graph with vertex set $\mathcal{B}$ such

s	$\mathsf{Aut}(\varGamma)$	G	Remark
2	$S_4 \times \mathbb{Z}_2$	D_8	Cube
2	S_4	$\mathbb{Z}_4$	K_4
3	$S_3 \wr \mathbb{Z}_2$	$\mathbb{Z}_6$ or D_6	$\mathsf{K}_{3,3}$
3	$\mathbb{Z}_2^4 \rtimes (S_3 \wr \mathbb{Z}_2)$	$\mathbb{Z}_4 \times S_4$ or $\mathbb{Z}_2^4 \rtimes S_3$	
3	$\mathrm{PGL}_2(11)$	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_{10}$	
4	$\mathrm{PGL}_2(7)$	D_{14}	Heawood's graph
4	$\mathrm{PGL}_2(23)$	$\mathbb{Z}_{23} \rtimes \mathbb{Z}_{22}$	
4	$\mathbb{Z}_3^7 \rtimes \mathrm{PGL}_2(7)$	$\mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_6)$	
4	S_{24}	S_{23}	
5	$N^2 \rtimes \mathbb{Z}_2^2$	$(\mathbb{Z}_7 \times N) \rtimes \mathbb{Z}_2$	$N = \mathrm{PSL}(2,7)$
5	$N^2 \rtimes \mathbb{Z}_2^2$	$(A_{23} \times N) \rtimes \mathbb{Z}_2$	$N = A_{24}$
5	$N^2 \rtimes \mathbb{Z}_2^2$	$(\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11} \times N) \rtimes \mathbb{Z}_2$	$N = \mathrm{PSL}(2,23)$
5	$N^2 \rtimes \mathbb{Z}_2^2$	$(\mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times N) \rtimes \mathbb{Z}_2$	$N = \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2,7)$
5	A_{48}	A_{47}	two graphs

TABLE 1. Core-free cubic s-transitive Cayley graphs.

that $B_1, B_2 \in \mathcal{B}$ are adjacent in $\varGamma_N$ if and only if some vertex $u \in B_1$ is adjacent in $\varGamma$ to some vertex $v \in B_2$. Since Γ is connected and X-edge-transitive, we conclude that $\varGamma_N$ is X/N-edge-transitive, each $B \in \mathcal{B}$ is an independent subset of $\varGamma$ and, for an edge $\{B_1, B_2\} \in E(\varGamma_N)$, the subgraph $\varGamma[B_1, B_2]$ of Γ induced by $B_1 \cup B_2$ is a regular bipartite graph which is independent of the choice of $\{B_1, B_2\}$ up to isomorphism. In particular, $\mathsf{val}(\varGamma) = \mathsf{val}(\Gamma_N)\mathsf{val}(\varGamma[B_1, B_2])$. If $\mathsf{val}(\varGamma) = \mathsf{val}(\Gamma_N)$, then $\varGamma$ is called a *normal cover* of Γ_N. It was proved by Praeger[23] that $\varGamma_N$ is $(X/N, s)$-arc-transitive if $\varGamma$ is (X, s)-arc-transitive, and that $\varGamma$ is a normal cover of $\varGamma_N$ if $s \geq 2$ and $|\mathcal{B}| \geq 3$. In general, if $\varGamma$ is a normal cover of $\varGamma_N$ then N acts regularly on each N-orbit, X/N is isomorphic to a subgroup of $\mathsf{Aut}(\varGamma_N)$ and $\varGamma_N$ is $(X/N, s)$-arc-transitive if and only if $\varGamma$ is (X, s)-arc-transitive.

In the following, we assume that $\varGamma = \mathsf{Cay}(G, S)$ is a connected X-edge-transitive Cayley graph with $G \leq X \leq \mathsf{Aut}(\varGamma)$. Set $\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}$. Let N be the maximal one among normal subgroups of X contained in G, that is, $N = \mathrm{Core}_X(G)$ is the core of G in X. Then either $|G : N| \leq 2$ or N has at least three orbits on $V(\varGamma)$. If $N = G$, then $X \leq G{\rtimes}\mathsf{Aut}(G, S)$ by [27]; if N is intransitive on $V(\varGamma)$, then every N-orbit is an independent set of $\varGamma$ since $\varGamma$ is connected and X-edge-transitive.

Assume that $|G : N| = 2$. Then N has exactly two orbits on $V(\varGamma)$ and $\varGamma$ is a bipartite graph; in this case $\varGamma$ is so called a *bi-normal Cayley graph* [19]. Further, $\varGamma$ is in fact a *bi-Cayley graph* [21] of N, say $\varGamma = \mathsf{BCay}(N, D)$, where $D \subseteq N$ and contains the identity of N with $\langle D \rangle = N$. Moreover, by [21], the arc-stabilizer X_{uv} is contained in $\mathsf{Aut}(N, D)$ for some arc (u, v) of $\varGamma$.

Now assume that N has at least three orbits on $V(\Gamma)$, and it is easily shown that G/N acts regularly on $V(\Gamma_N)$. Then Γ_N is a Cayley graph of the quotient G/N, and X/N acts transitively on the edges of Γ_N. Further either $\mathsf{val}(\Gamma)>\mathsf{val}(\Gamma_N)$ and Γ is not $(X,2)$-arc-transitive, or $\mathsf{val}(\Gamma) = \mathsf{val}(\Gamma_N)$, $X/N \lesssim \mathsf{Aut}(\Gamma_N)$ and Γ is a normal cover of Γ_N. In addition, if Γ is a normal cover of Γ_N then Γ_N is core-free with respect to G/N.

In summary we get a reduction for edge-transitive Cayley graphs.

Proposition 2.1. *Let $\Gamma = \mathsf{Cay}(G,S)$ be a connected X-edge-transitive Cayley graph with $G \le X \le \mathsf{Aut}(\Gamma)$ and let $N = \mathrm{Core}_X(G)$.*

1. *If $G = N$ then $X \le G \rtimes \mathsf{Aut}(G,S)$ and $X_1 \le \mathsf{Aut}(G,S)$.*
2. *If $|G:N| = 2$, then there exists $D \subseteq N$ with $1 \in D$, $\langle D\rangle = N$ and $X_{uv} \le \mathsf{Aut}(N,D)$ for an arc (u,v) of Γ.*
3. *If N has at least three orbits on $V(\Gamma)$, then Γ_N is an X/N-edge-transitive Cayley graph of G/N and either*
   (a) *$\mathsf{val}(\Gamma_N) < \mathsf{val}(\Gamma)$ and Γ is not $(X,2)$-arc-transitive; or*
   (b) *Γ is a normal cover of Γ_N, $G/N \le X/N \lesssim \mathsf{Aut}(\Gamma_N)$ and Γ_N is core-free with respect to G/N.*

Remark 2.1. (i) If we assume Γ with some further limits, then several cases in Proposition 2.1 are not necessary to happen. For example, (2) can not happen when $|V(\Gamma)|$ is odd, and (3.a) can not occur when Γ is either 2-arc-transitive or of prime valency.
(ii) In case (3.b), if $N = 1$ then, by considering the right multiplication action of X on the right cosets of G in X, we may view X as a subgroup of the symmetric group S_n for some n, which contains a regular subgroup (of S_n) isomorphic to a stabilizer of X acting on $V(\Gamma)$; and in this way, G is a stabilizer of X acting on $\{1,2,\cdots,n\}$. Replacing by a conjugation of G in X, we may assume G fixes 1.

Corollary 2.2. *Let $\Gamma = \mathsf{Cay}(G,S)$ be a connected cubic (X,s)-transitive Cayley graph with $G \le X \le \mathsf{Aut}(\Gamma)$ and let $N = \mathrm{Core}_X(G)$. Then either*

1. *$|G:N| \le 2$, and $s \le 2$ in this case; or*
2. *$|G:N| > 2$, $s \ge 2$, Γ_N is a core-free $(X/N,s)$-transitive Cayley graph of G/N, and Γ is a normal cover of Γ_N.*

Proof. Assume $|G:N| \le 2$. Then, by Proposition 2.1, either $X_1 \le \mathsf{Aut}(G,S) \lesssim \mathrm{S}_3$ or $X_{uv} \le \mathsf{Aut}(N,D) \cong \mathbb{Z}_2$ for an arc (u,v) of Γ. Each of these two cases implies that Γ is not $(X,3)$-arc-transitive, and so $s \le 2$. Thus, by Proposition 2.1, it suffices to show that $|G:N| > 2$ yields $s \ge 2$. Suppose to the contrary that $|G:N| > 2$ and $s = 1$. Then Γ is X-arc-regular and $X_1 \cong \mathbb{Z}_3$. By Remark 2.1 and Proposition 2.1 (3), $\bar{G} := G/N$ is a core-free subgroup of $\bar{X} := X/N = \bar{G}\bar{X}_1$, where $\bar{X}_1 = X_1N/N$. Further, $|\bar{X}_1| = |X_1| = 3$ and $|\bar{X}| = |\bar{G}||\bar{X}_1|$. Consider the right multiplication action of $\bar{X}$ on the right cosets of $\bar{G}$ in $\bar{X}$. Then $\bar{X}$ has a faithful permutation representation of degree $|\bar{X}_1| = 3$, and so $X/N = \bar{X} \lesssim \mathrm{S}_3$. Thus $G/N \lesssim \mathbb{Z}_2$, a contradiction. Hence $s \geq 2$. ∎

3. Construction of core-free Cayley graphs

Let X be an arbitrary finite group with a core-free subgroup H and let $D \subseteq X \setminus H$ with $D^{-1} = D$. The *coset graph* $\mathsf{Cos}(X, H, D)$, and denoted by $\mathsf{Cos}(X, H, z)$ for a singleton $D = \{z\}$ or a binary set $D = \{z, z^{-1}\}$, is the graph with vertex set $[X : H] := \{Hx \mid x \in X\}$ such that Hx and Hy are adjacent if and only if $yx^{-1} \in HDH$. Consider the action of X on $[X : H]$ by right multiplication on right cosets. Then this action is faithful and preserves the adjacency of the coset graph. Thus we identify X with a subgroup of $\mathsf{Aut}(\mathsf{Cos}(X, H, D))$. Further, we have the following basic facts.

Proposition 3.1. *Let $\mathsf{Cos}(X, H, D)$ be defined as above.*

1. *$\mathsf{Cos}(X, H, D)$ is connected if and only if $X = \langle H, D\rangle$;*
2. *$\mathsf{Cos}(X, H, D)$ is X-edge-transitive if and only if $HDH = H\{z, z^{-1}\}H$ for some $z \in X$;*
3. *The valency of $\mathsf{Cos}(X, H, z)$ is either $|H|/|H\cap H^z|$ if $HzH = Hz^{-1}H$, or $2|H|/|H \cap H^z|$ otherwise;*
4. *$\mathsf{Cos}(X, H, z)$ is X-arc-transitive if and only if $HzH = Hz^{-1}H$.*
5. *If X has a subgroup G acting regularly on the vertices of $\mathsf{Cos}(X, H, D)$, then $\mathsf{Cos}(X, H, D) \cong \mathsf{Cay}(G, S)$, where $S = G \cap HDH$.*

Proof. (1), (2), (3) and (4) are well-known, see [20] for example. Assume that X contains a regular subgroup G acting on $[X : H]$. Then $X = GH$ and $G \cap H = 1$, hence every right coset of H in X can be uniquely written as Hg for $g \in G$. Set $S = G \cap HDH$. Then for any $g_1, g_2 \in G$, the pair (Hg_1, Hg_2) is an arc of $\mathsf{Cos}(X, H, D)$ if and only if $g_2g_1^{-1} \in G \cap HDH = S$. Thus $\mathsf{Cos}(X, H, D) \cong \mathsf{Cay}(G, S)$, and (5) holds. ∎

Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph and $G \leq X \leq \mathsf{Aut}(\Gamma)$. Let $H = X_1$ be the stabilizer of $1 \in V(\Gamma)$ in X. Define $\rho : V(\Gamma) \to [X : H]$; $g \mapsto Hg$. It follows from $X = GH$ and $G \cap H = 1$ that ρ is a bijection. Further, it is easily shown that ρ is an isomorphism from Γ to $\mathsf{Cos}(X, H, S)$. Assume further that $\Gamma = \mathsf{Cay}(G, S)$ is X-arc-transitive. Then $\mathsf{Cos}(X, H, S)$ is X-arc-transitive. It follows that $HSH = HzH$ and $HzH = Hz^{-1}H$ for any $z \in S$. Then $\Gamma \cong \mathsf{Cos}(X, H, z)$ for any $z \in S$. Note that each involution z (if exists) in S normalizes $H \cap H^z$, the arc-stabilizer of $(1, z)$ in X. Since H is core-free in X, we have following simple result.

Proposition 3.2. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected X-arc-transitive Cayley graph with $G \leq X \leq \mathsf{Aut}(\Gamma)$. Let H be the stabilizer of $1 \in V(\Gamma)$ in X. If S contains an involution z, then $z \in G\cap \mathrm{N}_X(H \cap H^z) \setminus (\cup_{1\neq K\trianglelefteq H}\mathrm{N}_X(K))$, $\Gamma \cong \mathsf{Cos}(X, H, z)$, $\langle z, H\rangle = X$ and $G = \langle (G \cap HzH)\rangle$.*

The above argument and Remark 2.1 allow us to construct theoretically all possible connected core-free edge-transitive Cayley graphs with a given stabilizer isomorphic to a regular subgroup H of S_n. One may take $\tau \in \mathrm{S}_n \setminus (\cup_{1\neq K\trianglelefteq H}\mathrm{N}_{\mathrm{S}_n}(K))$ with $1^\tau = 1$. Then $\mathsf{Cos}(X,H,\tau) \cong \mathsf{Cay}(G,S)$ is a connected core-free X-edge-transitive Cayley graph with respect to G, where $X = \langle \tau, H\rangle$, $G = \{\sigma \in X \mid 1^\sigma = 1\}$ and $S = \{\sigma \in H\tau H \mid 1^\sigma = 1\}$. Note that all isomorphic regular subgroups of S_n are conjugate in S_n (see [29], for example). Thus, up to isomorphism, $\mathsf{Cos}(X,H,\tau)$ is independent of the choice of H. Note that $\mathsf{Cos}(X,H,\tau) \cong \mathsf{Cos}(X^\sigma,H,\tau^\sigma)$ for any $\sigma \in \mathrm{N}_{\mathrm{S}_n}(H)$. By Proposition 3.2, we may construct, up to isomorphism, the connected core-free arc-transitive Cayley graphs $\mathsf{Cay}(G,S)$ with a given vertex-stabilizer H of order n, a given arc-stabilizer P and S containing an involution by finding all possible such involutions as follows:

Step 1 Determine $I := \{\tau \in \mathrm{N}_{\mathrm{S}_n}(P) \setminus \cup_{1\neq K\trianglelefteq H}\mathrm{N}_{\mathrm{S}_n}(K) \mid \tau^2 = 1,\ 1^\tau = 1\}$.
Step 2 Determine the set $I(n,H)$ of involutions in I which are not conjugate to each other under $\mathrm{N}_{\mathrm{S}_n}(H)$;
Step 3 For $\tau \in I(n,H)$, determine $X = \langle \tau, H\rangle$, $G = \{\sigma \in X \mid 1^\sigma = 1\}$ and $S = \{\sigma \in H\tau H \mid 1^\sigma = 1\}$.

Remark 3.1. It is easy to know P has $|H : P|$ orbits on $\Omega = \{1, 2, \cdots, n\}$, which give an $\mathrm{N}_{\mathrm{S}_n}(P)$-invariant partition of Ω. Then, with the assumption that $1^\tau = 1$, τ fixes set-wise the P-orbit which contains 1.

4. Core-free cubic s-transitive Cayley graphs

In this section, we construct all possible core-free cubic s-transitive Cayley graphs up to isomorphism. Hereafter, we use σ^Δ to denote the restriction of σ on Δ, for $\sigma \in \mathrm{S}_n$ which fixes a subset Δ of $\Omega = \{1, 2, \cdots, n\}$ set-wise.

Let Γ be a core-free cubic (X, s)-transitive Cayley graph. Then $s \geq 2$ by Corollary 2.2. Note that, for a Cayley graph $\mathsf{Cay}(G,S)$ of odd valency, S must contain an involution. By Proposition 3.2, write $\mathit{\Gamma} = \mathsf{Cos}(X,H,\tau)$, where $H \leq \mathrm{S}_n$, $\tau \in I(n,H)$ and $n = |H|$. Then s, H, n and $P := H \cap H^\tau$ are listed in Table 2. (See [2, 18c] for example.) Note that P is a Sylow 2-subgroup of H and that $\mathit{\Gamma} = \mathsf{Cos}(X,H,\tau) \cong \mathsf{Cos}(X,H,\tau^\sigma)$ for any $\sigma \in H$. Thus, in practice, we may take a given regular subgroup H of S_n and a given Sylow 2-subgroup P of H. Since H acts regularly on $\Omega = \{1, 2, \cdots, n\}$ and

s	2	3	4	5
H	S_3	D_{12}	S_4	$\mathrm{S}_4 \times \mathbb{Z}_2$
n	6	12	24	48
P	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_8	$\mathrm{D}_8 \times \mathbb{Z}_2$

Table 2. Vertex-stabilizers of cubic s-transitive graphs.

$|H : P| = 3$, we know that P is semiregular on Ω and so has exactly three orbits, say Σ_1, Σ_2 and Σ_3. By Remark 3.1, we may assume that $1^\tau = 1 \in \Sigma_1 = \Sigma_1^\tau$, and τ either fixes or interchanges Σ_2 and Σ_3 set-wise.

4.1. **$s = 2$.** In this case, $H \cong \mathrm{S}_3$, $P \cong \mathbb{Z}_2$ and $X \leq \mathrm{S}_6$. Let $H = \langle \alpha, \beta\rangle$ and $P = \langle \beta\rangle$ where $\alpha = (1\ 2\ 3)(4\ 5\ 6)$ and $\beta = (1\ 5)(2\ 4)(3\ 6)$. Set $\Sigma_1 = \{1, 5\}$, $\Sigma_2 = \{2, 4\}$ and $\Sigma_3 = \{3, 6\}$. Since $\tau \in I(6, H)$, we have $\beta^\tau = \beta$ but $\langle\alpha\rangle^\tau \neq \langle\alpha\rangle$. Recalling that $\Sigma_1 = \Sigma_1^\tau$ and $1^\tau = 1$, it follows that τ is one of $(2\,4)$, $(3\,6)$, $(2\,4)(3\,6)$ and $(2\,6)(3\,4)$. It is easy to check that the first two permutations are conjugate under $\mathrm{N}_{\mathrm{S}_6}(H)$. Thus we assume that τ is one of

$$\tau_{2,1} = (2\,4),\ \tau_{2,1'} = (2\,4)(3\,6),\ \tau_{2,2} = (2\,6)(3\,4).$$

Set $X_{2,\imath} = \langle\tau_{2,\imath}, H\rangle$ and $\Gamma_{2,\imath} = \mathsf{Cos}(X_{2,\imath}, H, \tau_{2,\imath})$ for $\imath = 1, 1', 2$. Let $G_{2,\imath} = \{\sigma \in X_{2,\imath} \mid 1^\sigma = 1\}$ and $S_{2,\imath} = G_{2,\imath} \cap H\tau_{2,\imath}H$. Then $\Gamma_{2,\imath} \cong \mathsf{Cay}(G_{2,\imath}, S_{2,\imath})$, $\imath = 1, 1', 2$. By calculation, we get

$$\begin{aligned} S_{2,1} &= \{(2\,4), (3\,5), (2\,5)(3\,4)\}, & G_{2,1} &= \langle(2\,5\,4\,3), (2\,4)\rangle \cong \mathrm{D}_8,\\ S_{2,1'} &= \{(2\,6), (3\,4), (2\,4)(3\,6)\}, & G_{2,1'} &= \langle(2\,4\,6\,3), (2\,6)\rangle \cong \mathrm{D}_8,\\ S_{2,2} &= \{(2\,6)(4\,3), (2\,3\,6\,4), (2\,4\,6\,3)\}, & G_{2,2} &= \langle(2\,3\,6\,4)\rangle \cong \mathbb{Z}_4. \end{aligned}$$

Let $\rho = (2\,3)(5\,6)$. Then $G_{2,1}^\rho = G_{2,1'}$ and $S_{2,1}^\rho = S_{2,1'}$. Hence $\Gamma_{2,1} \cong \mathsf{Cay}(G_{2,1}, S_{2,1}) \cong \mathsf{Cay}(G_{2,1'}, S_{2,1'}) \cong \Gamma_{2,1'}$. In fact $\Gamma_{2,1}$ is the 3-dimensional cube Q_3 and $\Gamma_{2,2}$ is the complete graph K_4 on four vertices. Thus $\mathsf{Aut}(\Gamma_{2,1}) = X_{2,1} \cong \mathrm{S}_4 \times \mathbb{Z}_2$ and $\mathsf{Aut}(\Gamma_{2,2}) = X_{2,2} \cong \mathrm{S}_4$. In summary, we have

Lemma 4.1.1. *$\Gamma_{2,1} \cong \Gamma_{2,1'} \cong \mathrm{Q}_3$, $\Gamma_{2,2} \cong \mathsf{K}_4$, $G_{2,1} \cong G_{2,1'} \cong \mathrm{D}_8$, $G_{2,2} \cong \mathbb{Z}_4$, $\mathsf{Aut}(\Gamma_{2,1}) = X_{2,1} \cong \mathrm{S}_4 \times \mathbb{Z}_2$ and $\mathsf{Aut}(\Gamma_{2,2}) = X_{2,2} \cong \mathrm{S}_4$.*

4.2. **$s = 3$.** In this case, $H \cong \mathrm{D}_{12}$ and $X \leq \mathrm{S}_{12}$. We may take $H = \langle\alpha, \beta\rangle$ and $P = \langle\alpha^3\rangle \times \langle\beta\rangle$, where $\alpha = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$ and $\beta = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7)$. Set $\Sigma_1 = \{1, 4, 9, 12\}$, $\Sigma_2 = \{2, 5, 8, 11\}$ and $\Sigma_3 = \{3, 6, 7, 10\}$. It is easy to find all non-trivial normal subgroups of H as follows: $\langle\alpha\rangle$, $\langle\alpha^2\rangle$, $\langle\alpha^3\rangle$, $\langle\alpha^2, \beta\rangle$, $\langle\alpha^2, \alpha\beta\rangle$ and H itself. Noting that $\langle\alpha\rangle$ is a characteristic subgroup of H, it follows that $\cup_{1\neq K\trianglelefteq H}\mathrm{N}_{\mathrm{S}_{12}}(K) = \mathrm{N}_{\mathrm{S}_{12}}(\langle\alpha^2\rangle) \cup \mathrm{N}_{\mathrm{S}_{12}}(\langle\alpha^3\rangle) = \mathrm{N}_{\mathrm{S}_{12}}(\langle\alpha^2\rangle) \cup \mathrm{C}_{\mathrm{S}_{12}}(\langle\alpha^3\rangle)$.

Since $\tau \in I(12, H)$, τ normalizes $P = \{\alpha^3, \beta, \alpha^3\beta, 1\}$ and $\tau \notin \mathrm{N}_{\mathrm{S}_{12}}(\langle\alpha^2\rangle) \cup \mathrm{C}_{\mathrm{S}_{12}}(\langle\alpha^3\rangle)$. In particular, $(\alpha^3)^\tau \neq \alpha^3$. It follows that τ fixes, by conjugation, one of β and $\alpha^3\beta$, and interchanges the other one and α^3. Let $\delta = (9\,12)(8\,11)(7\,10)$. Then $\alpha^\delta = \alpha$ and $(\alpha^3\beta)^\delta = \beta$; and so $\delta \in \mathrm{N}_{\mathrm{S}_{12}}(H) \cap \mathrm{N}_{\mathrm{S}_{12}}(P)$. By replacing τ with τ^δ if necessary, we may assume that $\beta^\tau = \beta$ and $(\alpha^3)^\tau = \alpha^3\beta$. Recall the assumption that $\Sigma_1 = \Sigma_1^\tau$ and $1^\tau = 1$ before Subsection 4.1. Then $\beta^\tau = \beta$ yields $\tau^{\Sigma_1} = 1$ or $(4\ 9)$.

Assume first that τ interchanges Σ_2 and Σ_3. Then, by $\beta^\tau = \beta$, we have $(2\ 11)^\tau(5\ 8)^\tau = (\beta^{\Sigma_2})^\tau = \beta^{\Sigma_3} = (3\ 10)(6\ 7)$. Since

$$\begin{aligned} \alpha^3 &= (1\,4)(2\,5)(3\,6)(7\,10)(8\,11)(9\,12),\\ (\alpha^3)^\tau = \alpha^3\beta &= (1\,9)(2\,8)(3\,7)(4\,12)(5\,11)(6\,10), \end{aligned}$$

we have $(2\,5)^\tau(8\,11)^\tau = (3\,7)(6\,10)$. Checking case by case implies that τ is one of the following four permutations:

$$\tau_{3,1} = (4\,9)(2\,7)(6\,11)(3\,5)(8\,10),\ \tau_{3,2} = (4\,9)(2\,6)(7\,11)(3\,8)(5\,10),$$
$$\tau_{3,3} = (4\,9)(2\,3)(10\,11)(5\,7)(6\,8),\ \tau_{3,3'} = (4\,9)(2\,10)(3\,11)(5\,6)(7\,8).$$

Let $\gamma = (2\,6)(3\,5)(7\,11)(8\,10)$. Then $\gamma \in \mathrm{N}_{\mathrm{S}_{12}}(H)$ and $\tau_{3,3}^{\gamma} = \tau_{3,3'}$. Thus we may assume that τ is one of $\tau_{3,1}$, $\tau_{3,2}$ and $\tau_{3,3}$ in this case.

Now let τ fix every Σ_i set-wise. By $\beta^\tau = \beta$ and $(\alpha^3)^\tau = \alpha^3\beta$, we have

$$(1\,12)^\tau(4\,9)^\tau = (1\,12)(4\,9),\ (1\,4)^\tau(9\,12)^\tau = (1\,9)(4\,12),$$
$$(2\,11)^\tau(5\,8)^\tau = (2\,11)(5\,8),\ (2\,5)^\tau(8\,11)^\tau = (2\,8)(5\,11),$$
$$(3\,10)^\tau(6\,7)^\tau = (3\,10)(6\,7),\ (3\,6)^\tau(7\,10)^\tau = (3\,7)(6\,10).$$

It follows from $1^\tau = 1$ that τ is one of the following four permutations:

$$(4\,9)(2\,11)(6\,7),\ (4\,9)(2\,11)(3\,10),\ (4\,9)(5\,8)(3\,10),\ (4\,9)(5\,8)(6\,7).$$

It is not difficult to show that the last three involutions above are conjugate under $\mathrm{N}_{\mathrm{S}_{12}}(H)$. Thus, in this case, we may assume that τ is one of

$$\tau_{3,1'} = (4\,9)(2\,11)(6\,7),\ \tau_{3,2'} = (4\,9)(5\,8)(6\,7).$$

Set $X_{3,\imath} = \langle \tau_{3,\imath}, H\rangle$ and $\varGamma_{3,\imath} = \mathsf{Cos}(X_{3,\imath}, H, \tau_{3,\imath})$ for $\imath = 1, 1', 2, 2', 3$. Let $G_{3,\imath} = \{\sigma \in X_{3,\imath} \mid 1^\sigma = 1\}$ and $S_{3,\imath} = G_{3,\imath} \cap H\tau_{3,\imath}H$. Then $\varGamma_{3,\imath} \cong \mathsf{Cay}(G_{3,\imath}, S_{3,\imath})$ and $G_{3,\imath} = \langle S_{3,\imath}\rangle$ for $\imath = 1, 1', 2, 2', 3$, where

$$\begin{aligned}
S_{3,1} &= \{\tau_{3,1}, \sigma_{3,1}, \sigma_{3,1}^{-1}\}, & \sigma_{3,1} &= (2\,11\,4\,7\,6\,9)(3\,5)(8\,10),\\
S_{3,1'} &= \{\tau_{3,1'}, \sigma_{3,1'}, \tau_{3,1'}\sigma_{3,1'}\tau_{3,1'}\}, & \sigma_{3,1'} &= (2\,7)(4\,11)(6\,9),\\
S_{3,2} &= \{\tau_{3,2}, \sigma_{3,2}, \sigma_{3,2}^{-1}\}, & \sigma_{3,2} &= (2\,6\,9)(3\,5\,8\,10)(4\,7\,11),\\
S_{3,2'} &= \{\tau_{3,2'}, \sigma_{3,2'}, \alpha\sigma_{3,2'}\alpha^{-1}\}, & \sigma_{3,2'} &= (3\,8)(4\,7)(5\,12) = \alpha\tau_{3,2'}\alpha^{-1},\\
S_{3,3} &= \{\tau_{3,3}, \sigma_{3,3}, \sigma_{3,3}^{-1}\}, & \sigma_{3,3} &= (2\,8\,10\,11\,4\,7\,3\,12\,5\,6).
\end{aligned}$$

It is easy to show that $G_{3,1} \cong \mathbb{Z}_6$, $G_{3,1'} \cong \mathrm{D}_6$, $\varGamma_{3,1} \cong \varGamma_{3,1'} \cong \mathsf{K}_{3,3}$ and $\mathsf{Aut}(\varGamma_{3,1}) = X_{3,1} \cong X_{3,1'} \cong \mathrm{S}_3 \wr \mathbb{Z}_2$. Note that $G_{3,3}$ is a 2-transitive permutation group of degree 11 (on $\Omega \setminus \{1\}$). Thus $X_{3,3}$ is a 3-transitive permutation group of degree 12. Let $\sigma = \tau_{3,3}\sigma_{3,3}\tau_{3,3}\sigma_{3,3}^{-1}$. Then $\sigma = (2\,3\,5\,6\,10\,9\,12\,4\,11\,8\,7)$, $\sigma^{\tau_{3,3}} = \sigma^{-1}$ and $\sigma^{\sigma_{3,3}} = \sigma^8$. Thus $\mathbb{Z}_{11} \cong \langle\sigma\rangle \lhd G_{3,3}$. Then $G_{3,3} \cong \mathbb{Z}_{11} \rtimes \mathbb{Z}_{10}$, and hence $X_{3,3}$ is sharply 3-transitive on Ω. Then $X_{3,3} \cong \mathrm{PGL}(2, 11)$ by [15, XI.2.6]. Thus we have the following lemma.

Lemma 4.2.1. *$\varGamma_{3,1} \cong \varGamma_{3,1'} \cong \mathsf{K}_{3,3}$, $G_{3,1} \cong \mathbb{Z}_6$, $G_{3,1'} \cong \mathrm{D}_6$, $\mathsf{Aut}(\varGamma_{3,1}) = X_{3,1} \cong X_{3,1'} \cong \mathrm{S}_3 \wr \mathbb{Z}_2$, $G_{3,3} \cong \mathbb{Z}_{11} \rtimes \mathbb{Z}_{10}$ and $X_{3,3} \cong \mathrm{PGL}(2, 11)$.*

In the following we shall determine $X_{3,2}$, $X_{3,2'}$, $G_{3,2}$ and $G_{3,2'}$.

Lemma 4.2.2. *$G_{3,2} \cong \mathbb{Z}_4 \times \mathrm{S}_4$ and $G_{3,2'} \cong \mathbb{Z}_2^4 \rtimes \mathrm{S}_3$.*

Proof. Let $\eta = \sigma_{3,2}^4$ and $\rho = \sigma_{3,2}^6\tau_{3,2}$. We have $\eta = (2\,6\,9)(4\,7\,11)$, $\rho = (2\,6)(4\,9)(7\,11)$ and $\eta\rho = (4\,11\,9\,6)$. Further

$$\langle \eta, \rho\rangle = \langle (\eta\rho)^2, \eta, \rho^{(\eta\rho)^2}\rangle = \langle (\eta\rho)^2, ((\eta\rho)^2)^{\eta}\rangle \rtimes \langle \eta, \rho^{(\eta\rho)^2}\rangle \cong \mathrm{S}_4,$$
$$G_{3,2} = \langle \tau_{3,2}, \sigma_{3,2}\rangle = \langle \sigma_{3,2}^3, \sigma_{3,2}^4, \sigma_{3,2}^6\tau_{3,2}\rangle = \langle \sigma_{3,2}^3\rangle \times \langle \eta, \rho\rangle \cong \mathbb{Z}_4 \times \mathrm{S}_4.$$

Let $\delta_{3,2'} = \alpha\sigma_{3,2'}\alpha^{-1}$. Then $\delta_{3,2'} = (2\,7)(3\,12)(4\,11)$. Set $M = \langle \sigma_{3,2'}^{\sigma} \mid \sigma \in G_{3,2'}\rangle$ and $B = \langle \tau_{3,2'}, \delta_{3,2'}^{\tau_{3,2'}\sigma_{3,2'}}\rangle$. Then $M \trianglelefteq G_{3,2'}$, and $B \cong \mathrm{S}_3$ by calculation. Let $\pi_1 = \sigma_{3,2'}^{\tau_{3,2'}}$, $\pi_2 = \sigma_{3,2'}^{\delta_{3,2'}}$ and $\pi_3 = \sigma_{3,2'}^{\tau_{3,2'}\delta_{3,2'}}$. It is easily shown that $\langle \sigma_{3,2'}, \pi_1, \pi_2, \pi_3\rangle \cong \mathbb{Z}_2^4$ and that $\sigma_{3,2'}$, $\tau_{3,2'}$ and $\delta_{3,2'}$ normalize $\langle \sigma_{3,2'}, \pi_1, \pi_2, \pi_3\rangle$. Then $M = \langle \sigma_{3,2'}, \pi_1, \pi_2, \pi_3\rangle \cong \mathbb{Z}_2^4$. Noting that $M\cap B \trianglelefteq B$ and each normal subgroup of B has order 1, 3 or 6, it follows that $M \cap B = 1$. Hence $G_{3,2'} = \langle \tau_{3,2'}, \sigma_{3,2'}, \delta_{3,2'}\rangle = MB = M \rtimes B \cong \mathbb{Z}_2^4 \rtimes \mathrm{S}_3$. ∎

Lemma 4.2.3. *$X_{3,2} \cong X_{3,2'} \cong \mathbb{Z}_2^4 \rtimes (\mathrm{S}_3 \wr \mathbb{Z}_2)$ and $\Gamma_{3,2} \cong \Gamma_{3,2'}$.*

Proof. By calculation, $\beta = (\alpha^3\tau_{3,2})^2 = (\alpha^3\tau_{3,2'})^2$. Thus $X_{3,2} = \langle \alpha, \tau_{3,2}\rangle$ and $X_{3,2'} = \langle \alpha, \tau_{3,2'}\rangle$.

Let $\mu = \alpha^5(\tau_{3,2}\alpha)^2(\alpha\tau_{3,2})^3\alpha^2\tau_{3,2}\alpha^2$. Then $\mu = (3\,8)(5\,10)$, $\tau_{3,2}\mu = \mu\tau_{3,2}$, $\mu\beta = \beta\mu$ and $\alpha\mu = (1\,2\,8\,9\,5\,6)(3\,4\,10\,11\,12\,7)$. Set $N = \langle \mu^{\sigma} \mid \sigma \in X_{3,2}\rangle = \langle \mu^{\alpha^i} \mid 1 \le i \le 12\rangle$. Then $N \lhd X_{3,2}$ and $N = \langle \mu, \mu^{\alpha}, \mu^{\alpha^2}, \mu^{\alpha^3}\rangle \cong \mathbb{Z}_2^4$. Let $\nu = (\alpha^2\tau_{3,2})^4$ and $\omega = \alpha\tau_{3,2}\alpha^4(\tau_{3,2}\alpha)^2\alpha(\tau_{3,2}\alpha)^4$. Then $\nu = (1\,8\,5)(3\,10\,12)$, $\omega = (2\,7)(4\,6)(9\,11)$ and $\tau_{3,2} = (\alpha\mu)^3\nu\alpha\mu\omega\alpha\nu\alpha$. Thus

$$X_{3,2} = \langle \alpha, \tau_{3,2}\rangle = \langle \mu, \alpha\mu, \nu, \omega\rangle = N\langle \alpha\mu, \nu, \omega\rangle,$$
$$L := \langle \alpha\mu, \nu, \omega\rangle = \langle (\alpha\mu)^2, (\alpha\mu)^3, \nu, \omega, \omega^{\alpha\mu}\rangle = \langle (\alpha\mu)^2\nu, (\alpha\mu)^3, \nu, \omega, \omega^{\alpha\mu}\rangle$$
$$= (\langle \nu, \omega^{\alpha\mu}\rangle \times \langle (\alpha\mu)^2\nu^{-1}, \omega\rangle) \rtimes \langle (\alpha\mu)^3\rangle \cong \mathrm{S}_3 \wr \mathbb{Z}_2.$$

Since $|N||L|/|N\cap L| = |X_{3,2}| = |G_{3,2}||H| = |\mathbb{Z}_4 \times \mathrm{S}_4||\mathrm{D}_{12}| = 1152$, we have $N \cap L = 1$. Thus $X_{3,2} = N \rtimes L \cong \mathbb{Z}_2^4 \rtimes (\mathrm{S}_3 \wr \mathbb{Z}_2)$.

The above argument for $X_{3,2}$ also holds for $X_{3,2'}$ by replacing $\tau_{3,2}$ with $\tau_{3,2'}$. It follows that $\alpha \mapsto \alpha$; $\tau_{3,2} \mapsto \tau_{3,2'}$ gives an isomorphism ϕ from $X_{3,2}$ to $X_{3,2'}$. Then $X_{3,2} \cong X_{3,2'} \cong \mathbb{Z}_2^4 \rtimes (\mathrm{S}_3 \wr \mathbb{Z}_2)$. Since $\beta = (\alpha^3\tau_{3,2})^2 = (\alpha^3\tau_{3,2'})^2$, we know that $\beta^{\phi} = \beta$, and $H^{\phi} = H$. It is easy to verify that ϕ induces an isomorphism from $\Gamma_{3,2} = \mathsf{Cos}(X_{3,2}, H, \tau_{3,2})$ to $\Gamma_{3,2'} = \mathsf{Cos}(X_{3,2'}, H, \tau_{3,2'})$. ∎

4.3. $s = 4$**.** In this case, $H \cong \mathrm{S}_4$, $P \cong \mathrm{D}_8$ and $X \le \mathrm{S}_{24}$. We may take $H = \langle \alpha, \beta\rangle$ and $P = \langle \alpha, \gamma\rangle$, where $\gamma = (\alpha^2)^{\beta}$ and

$$\alpha = (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)(13\,14\,15\,16)(17\,18\,19\,20)(21\,22\,23\,24),$$
$$\beta = (1\,18)(2\,11)(3\,6)(4\,15)(5\,16)(7\,10)(8\,21)(9\,22)(12\,17)(13\,24)(14\,19)(20\,23),$$
$$\gamma = (1\,23)(2\,22)(3\,21)(4\,24)(5\,19)(6\,18)(7\,17)(8\,20)(9\,13)(10\,16)(11\,15)(12\,14).$$

Then the three orbits of P on Ω are $\Sigma_1 = \{1, 2, 3, 4, 21, 22, 23, 24\}$, $\Sigma_2 = \{5, 6, 7, 8, 17, 18, 19, 20\}$ and $\Sigma_3 = \{9, 10, 11, 12, 13, 14, 15, 16\}$. It is easy to know that H has totally three non-trivial normal subgroups: $K = \langle\alpha^2, \gamma\rangle \cong \mathbb{Z}_2^2$, $\langle\alpha^2, \gamma, \alpha\beta\rangle \cong \mathrm{A}_4$ and H itself. Noting that K is a characteristic subgroup of H, we have $\cup_{1\neq M\trianglelefteq H}\mathrm{N}_{\mathrm{S}_{24}}(M) = \mathrm{N}_{\mathrm{S}_{24}}(K)$.

Assume $\tau \in I(24, H)$. Then $\tau \in \mathrm{N}_{\mathrm{S}_{24}}(P) \setminus \mathrm{N}_{\mathrm{S}_{24}}(K)$. Noting that $\langle\alpha^2\rangle$ is the center of P, it follows that τ normalizes $\langle\alpha^2\rangle$, and so $(\alpha^2)^\tau = \alpha^2$. Since $K = \{1, \alpha^2, \gamma, \alpha^2\gamma\}$ and P contains totally 5 involutions, say, α^2, γ, $\alpha\gamma$, $\alpha^2\gamma$ and $\alpha^3\gamma$, we have $\{\gamma, \alpha^2\gamma\}^\tau = \{\alpha\gamma, \alpha^3\gamma\}$. Recall the assumption that $\Sigma_1 = \Sigma_1^\tau$ and $1^\tau = 1$ before Subsection 4.1. We have

$$\gamma^{\Sigma_1} = (1\,23)(2\,22)(3\,21)(4\,24), \quad (\alpha^2\gamma)^{\Sigma_1} = (1\,21)(2\,24)(3\,23)(4\,22),$$
$$(\alpha\gamma)^{\Sigma_1} = (1\,22)(2\,21)(3\,24)(4\,23), \quad (\alpha^3\gamma)^{\Sigma_1} = (1\,24)(2\,23)(3\,22)(4\,21).$$

Then $\{21, 23\}^\tau = \{22, 24\}$, and hence τ^{Σ_1} is one of $(2\,4)(21\,22)(23\,24)$ and $(2\,4)(21\,24)(22\,23)$. Thus, either $\gamma^\tau = \alpha^3\gamma$ and $(\alpha^2\gamma)^\tau = \alpha\gamma$ for $\tau^{\Sigma_1} = (2\,4)(21\,22)(23\,24)$, or $\gamma^\tau = \alpha\gamma$ and $(\alpha^2\gamma)^\tau = \alpha^3\gamma$ for $\tau^{\Sigma_1} = (2\,4)(21\,24)(22\,23)$.

Assume that τ interchanges Σ_2 and Σ_3. Set $\Delta = \Sigma_2 \cup \Sigma_3$ and consider the restrictions of γ, $\alpha^2\gamma$, $\alpha\gamma$ and $\alpha^3\gamma$ on Δ. Then

$$\gamma^\Delta = (5\,19)(6\,18)(7\,17)(8\,20)(9\,13)(10\,16)(11\,15)(12\,14),$$
$$(\alpha^2\gamma)^\Delta = (5\,17)(6\,20)(7\,19)(8\,18)(9\,15)(10\,14)(11\,13)(12\,16),$$
$$(\alpha\gamma)^\Delta = (5\,18)(6\,17)(7\,20)(8\,19)(9\,16)(10\,15)(11\,14)(12\,13),$$
$$(\alpha^3\gamma)^\Delta = (5\,20)(6\,19)(7\,18)(8\,17)(9\,14)(10\,13)(11\,16)(12\,15).$$

Considering all possible images of 5 under τ, it follows from $\{\gamma, \alpha^2\gamma\}^\tau = \{\alpha\gamma, \alpha^3\gamma\}$ that one of the following eight cases occurs:

$$5^\tau = 9,\ \{17, 19\}^\tau = \{14, 16\};\ 5^\tau = 10,\ \{17, 19\}^\tau = \{13, 15\};$$
$$5^\tau = 11,\ \{17, 19\}^\tau = \{14, 16\};\ 5^\tau = 12,\ \{17, 19\}^\tau = \{13, 15\};$$
$$5^\tau = 13,\ \{17, 19\}^\tau = \{10, 12\};\ 5^\tau = 14,\ \{17, 19\}^\tau = \{9, 11\};$$
$$5^\tau = 15,\ \{17, 19\}^\tau = \{10, 12\};\ 5^\tau = 16,\ \{17, 19\}^\tau = \{9, 11\}.$$

It is easy to check that there are exactly two possible τ's arising from each of the above eight cases. Then we get sixteen permutations, which are conjugate under $\mathrm{N}_{\mathrm{S}_{24}}(H)$ to one of the following two permutations:

$$\tau_{4,2} = (2\,4)(5\,10)(6\,9)(7\,12)(8\,11)(13\,19)(14\,18)(15\,17)(16\,20)(21\,22)(23\,24),$$
$$\tau_{4,3} = (2\,4)(5\,9)(6\,12)(7\,11)(8\,10)(13\,18)(14\,17)(15\,20)(16\,19)(21\,24)(22\,23).$$

Now assume that τ fixes every Σ_i set-wise. Consider the possible images of 5 and of 9 under τ. Then $5^\tau \in \{5, 6, 7, 8\}$ and $9^\tau \in \{9, 10, 11, 12\}$. If $\tau^{\Sigma_1} = (2\,4)(21\,22)(23\,24)$, then $\gamma^\tau = \alpha^3\gamma$ and $(\alpha^2\gamma)^\tau = \alpha\gamma$, and we get sixteen permutations. If $\tau^{\Sigma_1} = (2\,4)(21\,24)(22\,23)$, then $\gamma^\tau = \alpha\gamma$ and $(\alpha^2\gamma)^\tau = \alpha^3\gamma$, and we get another sixteen permutations. Further, these 32 permutations are conjugate under $\mathrm{N}_{\mathrm{S}_{24}}(H)$ to one of the following two permutations:

$$\tau_{4,1} = (2\,4)(5\,6)(7\,8)(9\,10)(11\,12)(14\,16)(18\,20)(21\,22)(23\,24),$$
$$\tau_{4,4} = (2\,4)(5\,6)(7\,8)(9\,10)(11\,12)(13\,15)(17\,19)(21\,24)(22\,23).$$

Set $X_{4,\imath} = \langle \tau_{4,\imath}, \alpha, \beta\rangle$ and $\Gamma_{4,\imath} = \mathsf{Cos}(X_{4,\imath}, H, \tau_{4,\imath})$ for $\imath = 1,2,3,4$. Let $G_{4,\imath} = \{\sigma \in X_{4,\imath} \mid 1^\sigma = 1\}$ and $S_{4,\imath} = G_{4,\imath} \cap H\tau_{4,\imath}H$. Then $\Gamma_{4,\imath} \cong \mathsf{Cay}(G_{4,\imath}, S_{4,\imath})$ for $1 \le \imath \le 4$. By calculation, we have

$$S_{4,\imath} = \{\tau_{4,\imath}, \sigma_{4,\imath}, \delta_{4,\imath}\},\ G_{4,\imath} = \langle \tau_{4,\imath}, \sigma_{4,\imath}, \delta_{4,\imath}\rangle \text{ for } 1 \le \imath \le 4,$$

where $\delta_{4,2} = \sigma_{4,2}^{-1}$, $\delta_{4,3} = \sigma_{4,3}^{-1}$ and

$$\begin{aligned}
\sigma_{4,1} &= (2\,24)(3\,18)(4\,13)(5\,10)(6\,20)(8\,23)(11\,22)(12\,16)(14\,17),\\
\delta_{4,1} &= (2\,7)(3\,10)(4\,24)(6\,18)(8\,13)(9\,20)(12\,14)(16\,21)(17\,22),\\
\sigma_{4,2} &= (2\,4\,7\,15\,19\,11\,22\,17\,8\,3\,16\,6\,12\,18\,21\,23\,10\,9\,5\,20\,14\,13),\\
\sigma_{4,3} &= (2\,4\,7\,18\,21\,23\,10\,8\,3\,16\,15\,19\,6\,12\,11\,22\,17\,13)(5\,9)(14\,20),\\
\sigma_{4,4} &= (2\,24)(3\,8)(4\,11)(5\,10)(6\,20)(7\,19)(13\,22)(14\,17)(18\,23),\\
\delta_{4,4} &= (2\,17)(3\,16)(4\,24)(7\,22)(8\,13)(9\,20)(10\,21)(11\,15)(12\,14).
\end{aligned}$$

It is easy to know $G_{4,1} \cong \mathrm{D}_{14}$. By [22], we have the following lemma.

Lemma 4.3.1. *$G_{4,1}\cong\mathrm{D}_{14}$, $X_{4,1}=\mathsf{Aut}(\Gamma_{4,1})\cong\mathrm{PGL}(2,7)$ and $\mathsf{Cay}(G_{4,1}, S_{4,1})$ is isomorphic to the point-line incidence graph of the seven-point plane.*

Lemma 4.3.2. *$X_{4,2} \cong \mathrm{PGL}(2,23)$ and $G_{4,2} \cong \mathbb{Z}_{23} \rtimes \mathbb{Z}_{22}$.*

Proof. Let $\sigma = \tau_{4,2}\sigma_{4,2}^{11}$. Then σ is a 23-cycle, $\sigma^{\tau_{4,2}} = \sigma^{-1}$ and $\sigma^{\sigma_{4,2}} = \sigma^{19}$. It follows that $G_{4,2}$ is a 2-transitive permutation group on $\Omega \setminus \{1\}$ and $G_{4,2}$ contains a normal regular subgroup $\langle\sigma\rangle \cong \mathbb{Z}_{23}$. Therefore, $G_{4,2} \cong \mathbb{Z}_{23} \rtimes \mathbb{Z}_{22}$. It implies that $X_{4,2} = HG_{4,2}$ is a sharply 3-transitive permutation group of degree 24. Then $X_{4,2} \cong \mathrm{PGL}(2,23)$ by [15, XI.2.6]. ∎

Lemma 4.3.3. *$X_{4,3} \cong \mathbb{Z}_3^7 \rtimes \mathrm{PGL}(2,7)$ and $G_{4,3} \cong \mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_6)$.*

Proof. Let $\pi = \tau_{4,3}\sigma_{4,3}$. Set $\mu = \sigma_{4,3}^2\pi\sigma_{4,3}^{10}\pi^2\sigma_{4,3}^2\pi$, $\nu = \sigma_{4,3}^2\pi^2\sigma_{4,3}^4\pi\sigma_{4,3}^7$ and $\omega = \pi^2\sigma_{4,3}^3(\pi\sigma_{4,3})^3\pi$. Then $\mu = (2\,6\,10)(14\,20\,24)$,

$$\begin{aligned}
\nu &= (2\,20\,15\,11\,12\,18)(3\,8\,16\,10\,14\,17)(4\,22\,6\,24\,21\,7)(5\,9)(13\,23),\\
\omega &= (2\,22\,15\,7\,24\,13\,12)(3\,14\,19\,8\,10\,16\,17)(4\,6\,18\,21\,11\,20\,23),
\end{aligned}$$

$\omega^\nu = \omega^3$, $\tau_{4,3} = \nu^2\omega\nu$ and $\sigma_{4,3} = \mu^2\nu\mu\nu^4\mu^2\nu^2\omega^2\mu^2$. Thus $\langle\omega\rangle \lhd \langle\nu,\omega\rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_6$, and $G_{4,3} = \langle\tau_{4,3}, \sigma_{4,3}\rangle = \langle\mu,\nu,\omega\rangle = M\langle\omega,\nu\rangle$, where $M = \langle\mu^\sigma \mid \sigma \in \langle\omega,\nu\rangle\rangle \lhd G_{4,3}$. By calculation, we have $M = \langle\mu, \mu^{\nu^2}, \mu^{\nu^3}, \mu^{\nu^4}, \mu^{\nu^5}, \mu^{\omega^5}\rangle \cong \mathbb{Z}_3^6$. Noting that $\langle\omega,\nu\rangle$ has no nontrivial normal subgroups of order a power of 3, it yields $M \cap \langle\omega,\nu\rangle = 1$. Thus $G_{4,3} = M \rtimes \langle\omega,\nu\rangle \cong \mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_6)$.

Let μ, ν and ω be as above. Then $\mu = ((\tau_{4,3}\beta)^8((\tau_{4,3}\beta)^8)^\alpha)^{\alpha\beta\alpha}$. Set $N = \langle\mu, \mu^\alpha, \mu^\beta, \mu^{\tau_{4,3}}, \mu^{\alpha^2}, \mu^{\alpha^3}, \mu^{\alpha\beta}\rangle$. It is easily shown that $N \cong \mathbb{Z}_3^7$, and further that, for each ε of the seven generators of N, the conjugations of ε by α, β and $\tau_{4,3}$ are contained in N. It implies that $N = \langle\mu^\sigma \mid \sigma \in X_{4,3}\rangle \lhd X_{4,3}$ and $M < N$. Suppose that $\nu^2 \in N$. Then $N = M \times \langle\nu^2\rangle \lhd G_{4,3}$. It follows that $\langle\nu^2\rangle \lhd \langle\nu,\omega\rangle$. Noting that $\langle\omega\rangle \lhd \langle\nu,\omega\rangle$, it implies that ν^2 centralizes ω. But $\omega^{\nu^2} = \omega^9 = \omega^2$, which is a contradiction. Thus $\nu^2 \notin N$.

Consider the normal quotient $(\Gamma_{4,3})_N$ of $\Gamma_{4,3}$ induced by N. Then $(\Gamma_{4,3})_N$ is a cubic $(X_{4,3}/N, 4)$-transitive graph on 14 vertices. It follows from [22] that $(\Gamma_{4,3})_N$ is (isomorphic to) the point-line incidence graph of the seven-point plane. Thus we conclude that $X_{4,3}/N \cong \mathrm{PGL}(2,7)$. In particular, $|X_{4,3}| = 2^4 \cdot 3^8 \cdot 7$, and $N\langle \nu^2 \rangle$ is a Sylow 3-subgroup of $X_{4,3}$. Noting that $N \cap \langle \nu^2 \rangle = 1$, it follows from Gaschütz' Theorem (see [1, (10.4)] for example) that there is $L \leq X_{4,3}$ with $X_{4,3} = NL$ and $N \cap L = 1$. Thus $L \cong X_{4,3}/N \cong \mathrm{PGL}(2,7)$ and $X_{4,3} = N \rtimes L \cong \mathbb{Z}_3^7 \rtimes \mathrm{PGL}(2,7)$. ∎

Lemma 4.3.4. *$X_{4,4} = \mathrm{S}_{24}$ and $G_{4,4} \cong \mathrm{S}_{23}$.*

Proof. Recall that $G_{4,4} = \langle \tau_{4,4}, \sigma_{4,4}, \delta_{4,4} \rangle$ is the stabilizer of 1 in $X_{4,4}$ acting on Ω. It is easy to see that $G_{4,4}$ is transitive on $\Omega \setminus \{1\}$. Then $X_{4,4}$ is a 2-transitive, and hence primitive on Ω. Let $\rho = \tau_{4,4}^{\alpha}\beta\sigma_{4,4}$. Then $\rho \in X_{4,4}$ and $X_{4,4}$ contains a 7-cycle $\rho^{24} = (5\,14\,6\,9\,24\,21\,10)$. Noting that $\sigma_{4,4}$ is an odd permutation, $X_{4,4} = \mathrm{S}_{24}$ by [9, Theorem 3.3E], and so $G_{4,4} \cong \mathrm{S}_{23}$. ∎

4.4. $s = 5$**.** For the completeness, this paper involves the following content constructing six known 5-transitive Cayley graphs (see [7] for example).

In this case $H \cong \mathrm{S}_4 \times \mathbb{Z}_2$, $P \cong \mathrm{D}_8 \times \mathbb{Z}_2$ and $X \leq \mathrm{S}_{48}$. Since all isomorphic regular groups on $\Omega = \{1, 2, \cdots, 48\}$ are conjugate in S_{48}, we may take $H = \langle \alpha, \beta, \gamma \rangle \times \langle \delta \rangle$ and $P = \langle \alpha, \beta, \delta \rangle$, where $\alpha^2 = \beta^\gamma \beta$ and

$\alpha=$ (1 2 3 4)(5 6 7 8)(9 10 11 12)(13 14 15 16)(17 18 19 20)(21 22 23 24)
(25 26 27 28)(29 30 31 32)(33 34 35 36)(37 38 39 40)
(41 42 43 44)(45 46 47 48),
$\beta=$ (1 8)(2 7)(3 6)(4 5)(9 16)(10 15)(11 14)(12 13)(17 24)(18 23)(19 22)
(20 21)(25 32)(26 31)(27 30)(28 29)(33 40)(34 39)(35 38)
(36 37)(41 48)(42 47)(43 46)(44 45),
$\gamma=$ (1 17 33)(2 39 20)(3 24 38)(4 34 23)(5 37 21)(6 19 40)(7 36 18)
(8 22 35)(9 25 41)(10 47 28)(11 32 46)(12 42 31)
(13 45 29)(14 27 48)(15 44 26)(16 30 43),
$\delta=$ (1 9)(2 10)(3 11)(4 12)(5 13)(6 14)(7 15)(8 16)(17 25)(18 26)(19 27)
(20 28)(21 29)(22 30)(23 31)(24 32)(33 41)(34 42)(35 43)
(36 44)(37 45)(38 46)(39 47)(40 48).

Then P has three orbits on $\Omega = \{1, 2, \cdots, 48\}$, say, $\Sigma_i = \{16(i-1)+j \mid 1 \leq j \leq 16\}$, where $i = 1, 2$ and 3. It is easy to know that H has totally eight non-trivial normal subgroups, say $\langle \delta \rangle$, $\langle \alpha^2, \beta \rangle$, $\langle \alpha^2, \beta, \delta \rangle$, $\langle \beta, \gamma \rangle$, $\langle \beta, \gamma, \delta \rangle$, $\langle \alpha, \beta, \gamma \rangle$, $\langle \alpha\delta, \beta, \gamma \rangle$ and H itself, which are isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, A_4, $\mathrm{A}_4 \times \mathbb{Z}_2$, S_4, S_4 and $\mathrm{S}_4 \times \mathbb{Z}_2$, respectively. Note that $\langle \delta \rangle$ is a characteristic subgroup of H and $\langle a^2, \beta \rangle$ is a characteristic subgroup of $\langle \alpha, \beta, \gamma \rangle$ and of $\langle \alpha\delta, \beta, \gamma \rangle$. It yields $\cup_{1 \neq K \lhd H} \mathrm{N}_{\mathrm{S}_{48}}(K) = \mathrm{N}_{\mathrm{S}_{48}}(\langle \delta \rangle) \cup \mathrm{N}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta \rangle) \cup \mathrm{N}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta, \delta \rangle)$.

Let $\tau \in I(48, H)$. Then $\tau \in \mathrm{N}_{\mathrm{S}_{48}}(P) \setminus (\mathrm{N}_{\mathrm{S}_{48}}(\langle \delta \rangle) \cup \mathrm{N}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta \rangle) \cup \mathrm{N}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta, \delta \rangle))$. Since τ normalizes P, we know that τ normalizes the

Frattini subgroup $\Phi(P) = \langle \alpha^2 \rangle$ and the center $Z(P) = \{1, \alpha^2, \delta, \alpha^2\delta\}$ of P. It follows that $(\alpha^2)^\tau = \alpha^2$, $\delta^\tau = \alpha^2\delta$, and hence $\beta^\tau \notin \langle \alpha^2, \beta, \delta \rangle$ as $\tau \notin \mathrm{N}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta, \delta \rangle)$. Considering the involutions in P, we have $\beta^\tau \in \{\alpha\beta, \alpha^3\beta, \alpha\beta\delta, \alpha^3\beta\delta\}$. Let

$$\begin{aligned} \iota_1 = \ & (2\,4)(5\,7)(10\,12)(13\,15)(17\,19)(22\,24)(25\,27)(30\,32)(33\,38) \\ & (34\,37)(35\,40)(36\,39)(41\,46)(42\,45)(43\,48)(44\,47), \\ \iota_2 = \ & (2\,10)(4\,12)(5\,13)(7\,15)(18\,26)(20\,28)(21\,29)(23\,31)(34\,42) \\ & (36\,44)(37\,45)(39\,47). \end{aligned}$$

Then $\iota_1, \iota_2 \in \mathrm{N}_{\mathrm{S}_{48}}(H) \cap \mathrm{N}_{\mathrm{S}_{48}}(P) \cap \mathrm{C}_{\mathrm{S}_{48}}(\langle \alpha^2, \beta, \delta \rangle)$, $(\alpha\beta)^{\iota_1} = \alpha^3\beta$, $(\alpha\beta\delta)^{\iota_1} = \alpha^3\beta\delta$ and $(\alpha\beta)^{\iota_2} = \alpha\beta\delta$. Further, both ι_1 and ι_2 fix every P-orbit set-wise. Thus, replacing τ with τ^{ι_1}, τ^{ι_2} or $\tau^{\iota_2\iota_1}$ if necessary, we may assume $\beta^\tau = \alpha\beta$. Then $\beta = \beta^{\tau^2} = \alpha^\tau\beta^\tau = a^\tau\alpha\beta$, and hence $\alpha^\tau = \alpha^{-1}$.

Recall the assumption that $\Sigma_1 = \Sigma_1^\tau$ and $1^\tau = 1$ before Subsection 4.1. Then $(\alpha^2)^\tau = \alpha^2$ yields $3^\tau = 3$, $\delta^\tau = \alpha^2\delta$ yields $9^\tau = 11$ and $\beta^\tau = \alpha\beta$ yields $8^\tau = 7$. It follows that $5^\tau = 6$, $4^\tau = 2$, $16^\tau = 13$, $14^\tau = 15$, $10^\tau = 10$ and $12^\tau = 12$. Thus $\tau^{\Sigma_1} = (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)$.

Note that $Z(P)$ has eight orbits on $\Omega \setminus \Sigma_1$ as follows:

$$\begin{aligned} &\Sigma_{21} = \{17, 19, 25, 27\},\ \Sigma_{22} = \{18, 20, 26, 28\}, \\ &\Sigma_{23} = \{21, 23, 29, 31\},\ \Sigma_{24} = \{22, 24, 30, 32\}, \\ &\Sigma_{31} = \{33, 35, 41, 43\},\ \Sigma_{32} = \{34, 36, 42, 44\}, \\ &\Sigma_{33} = \{37, 39, 45, 47\},\ \Sigma_{34} = \{38, 40, 46, 48\}, \end{aligned}$$

which form a τ-invariant partition of $\Sigma_2 \cup \Sigma_3$. Further, we have

$$\Sigma_{i1}^\beta = \Sigma_{i4},\ \Sigma_{i2}^\beta = \Sigma_{i3},\ \Sigma_{i1}^{\alpha\beta} = \Sigma_{i3},\ \Sigma_{i2}^{\alpha\beta} = \Sigma_{i4},\ \text{for } i = 2,\ 3.$$

Assume that τ fixes every Σ_i set-wise. It follows from $\beta^\tau = \alpha\beta$ that one of the following four cases occurs:

$$\begin{aligned} &\Sigma_{21}^\tau = \Sigma_{21},\ \Sigma_{22}^\tau = \Sigma_{22},\ \Sigma_{23}^\tau = \Sigma_{24},\ \Sigma_{31}^\tau = \Sigma_{31},\ \Sigma_{32}^\tau = \Sigma_{32},\ \Sigma_{33}^\tau = \Sigma_{34}; \\ &\Sigma_{21}^\tau = \Sigma_{21},\ \Sigma_{22}^\tau = \Sigma_{22},\ \Sigma_{23}^\tau = \Sigma_{24},\ \Sigma_{33}^\tau = \Sigma_{33},\ \Sigma_{34}^\tau = \Sigma_{34},\ \Sigma_{31}^\tau = \Sigma_{32}; \\ &\Sigma_{23}^\tau = \Sigma_{23},\ \Sigma_{24}^\tau = \Sigma_{24},\ \Sigma_{21}^\tau = \Sigma_{22},\ \Sigma_{31}^\tau = \Sigma_{31},\ \Sigma_{32}^\tau = \Sigma_{32},\ \Sigma_{33}^\tau = \Sigma_{34}; \\ &\Sigma_{23}^\tau = \Sigma_{23},\ \Sigma_{24}^\tau = \Sigma_{24},\ \Sigma_{21}^\tau = \Sigma_{22},\ \Sigma_{33}^\tau = \Sigma_{33},\ \Sigma_{34}^\tau = \Sigma_{34},\ \Sigma_{31}^\tau = \Sigma_{32}. \end{aligned}$$

Combining with $\delta^\tau = \alpha^2\delta$, each case gives 4 choices of $\tau^{\Sigma_2 \cup \Sigma_3}$. Thus we get 16 possible τ's, which are conjugate under $\mathrm{N}_{\mathrm{S}_{48}}(H)$ to one of the following two permutations:

$$\begin{aligned} \tau_{5,1} = \ & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,20)(18\,19)(21\,23)(25\,26)(27\,28) \\ & (30\,32)(33\,36)(34\,35)(37\,39)(41\,42)(43\,44)(46\,48),\ \text{or} \\ \tau_{5,2} = \ & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,19)(21\,24)(22\,23)(26\,28)(29\,30) \\ & (31\,32)(33\,35)(37\,40)(38\,39)(42\,44)(45\,46)(47\,48). \end{aligned}$$

Now assume that $\Sigma_2^\tau = \Sigma_3$. Then one of the following four cases holds:

$$\begin{array}{l}\Sigma_{21}^\tau = \Sigma_{31},\ \Sigma_{22}^\tau = \Sigma_{32},\ \Sigma_{23}^\tau = \Sigma_{34},\ \Sigma_{24}^\tau = \Sigma_{33};\\ \Sigma_{21}^\tau = \Sigma_{32},\ \Sigma_{22}^\tau = \Sigma_{31},\ \Sigma_{23}^\tau = \Sigma_{33},\ \Sigma_{24}^\tau = \Sigma_{34};\\ \Sigma_{21}^\tau = \Sigma_{33},\ \Sigma_{22}^\tau = \Sigma_{34},\ \Sigma_{23}^\tau = \Sigma_{32},\ \Sigma_{24}^\tau = \Sigma_{31};\\ \Sigma_{21}^\tau = \Sigma_{34},\ \Sigma_{22}^\tau = \Sigma_{33},\ \Sigma_{23}^\tau = \Sigma_{31},\ \Sigma_{24}^\tau = \Sigma_{32}.\end{array}$$

Further, each case gives four choices of $\tau^{\Sigma_2 \cup \Sigma_3}$, and then we get 16 possible τ's, which are conjugate under $\mathrm{N}_{\mathrm{S}_{48}}(H)$ to one of the following permutations:

$$\begin{array}{rl}\tau_{5,3} = & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,35)(18\,34)(19\,33)(20\,36)\\ & (21\,40)(22\,39)(23\,38)(24\,37)(25\,41)(26\,44)\\ & (27\,43)(28\,42)(29\,46)(30\,45)(31\,48)(32\,47),\\ \tau_{5,4} = & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,34)(18\,33)(19\,36)(20\,35)\\ & (21\,37)(22\,40)(23\,39)(24\,38)(25\,44)(26\,43)\\ & (27\,42)(28\,41)(29\,47)(30\,46)(31\,45)(32\,48),\\ \tau_{5,5} = & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,45)(18\,48)(19\,47)(20\,46)\\ & (21\,42)(22\,41)(23\,44)(24\,43)(25\,39)(26\,38)\\ & (27\,37)(28\,40)(29\,36)(30\,35)(31\,34)(32\,33),\\ \tau_{5,6} = & (2\,4)(5\,6)(7\,8)(9\,11)(13\,16)(14\,15)(17\,46)(18\,45)(19\,48)(20\,47)\\ & (21\,41)(22\,44)(23\,43)(24\,42)(25\,40)(26\,39)\\ & (27\,38)(28\,37)(29\,35)(30\,34)(31\,33)(32\,36).\end{array}$$

Set $X_{5,\imath} = \langle \alpha, \beta, \delta, \gamma, \tau_{5,\imath} \rangle$, $\Gamma_{5,\imath} = \mathsf{Cos}(X_{5,\imath}, H, \tau_{5,\imath})$, $G_{5,\imath} = \{\sigma \in X_{5,\imath} \mid 1^\sigma = 1\}$ and $S_{5,\imath} = \{\sigma \in H\tau_{5,\imath}H \mid 1^\sigma = 1\}$, $\imath = 1, 2, 3, 4, 5, 6$. Then $\Gamma_{5,\imath} \cong \mathsf{Cay}(G_{5,\imath}, S_{5,\imath})$. By calculation, $S_{5,\imath} = \{\tau_{5,\imath}, \sigma_{5,\imath}, \delta_{5,\imath}\}$ and $G_{5,\imath} = \langle \tau_{5,\imath}, \sigma_{5,\imath}, \delta_{5,\imath} \rangle$ for $1 \le \imath \le 6$, where $\delta_{5,\jmath} = \sigma_{5,\jmath}^{-1}$ for $\jmath \ge 3$, and

$$\begin{array}{rl}\sigma_{5,1} = & (2\,24)(3\,37)(4\,7)(5\,19)(8\,34)(9\,14)(10\,27)(11\,42)(13\,32)(16\,45)(18\,21)\\ & (20\,33)(23\,38)(25\,30)(28\,46)(31\,41)(36\,39)(43\,48) = \gamma\alpha^2\tau_{5,1}\beta\gamma\alpha,\\ \delta_{5,1} = & (2\,7)(3\,20)(4\,35)(5\,38)(6\,21)(9\,16)(11\,29)(12\,46)(13\,43)(14\,28)(17\,39)\\ & (18\,23)(24\,34)(25\,42)(27\,30)(32\,47)(36\,37)(41\,48) = \alpha\beta\gamma\tau_{5,1}\gamma,\\ \sigma_{5,2} = & (2\,7)(3\,21)(4\,38)(5\,35)(6\,20)(9\,16)(11\,28)(12\,43)(13\,46)(14\,29)(17\,24)\\ & (18\,23)(19\,36)(22\,37)(27\,45)(30\,44)(41\,48)(42\,47) = \alpha\beta\gamma\tau_{5,2}\alpha\gamma,\\ \delta_{5,2} = & (2\,19)(3\,34)(4\,7)(5\,24)(8\,37)(9\,14)(10\,32)(11\,45)(13\,27)(16\,42)(18\,40)\\ & (21\,35)(25\,30)(26\,43)(28\,31)(29\,48)(33\,38)(36\,39) = \alpha^2\delta\gamma\tau_{5,2}\gamma\alpha\delta,\\ \sigma_{5,3} = & (2\,4\,19\,18\,36\,40\,22\,21\,8\,6\,34\,23\,39\,20\,3\,37\,35\,5\,24\,33\,17\,38)(9\,14\,45\,48\,25\\ & 41\,30\,26\,47\,31\,44\,43\,10\,15\,12\,32\,46\,13\,27\,29\,11\,42\,28\,16) = \delta\gamma^2\tau_{5,3}\gamma^2\delta,\\ \sigma_{5,4} = & (2\,4\,24\,20\,8\,6\,37\,21\,3\,34\,33\,17\,23\,39\,38\,5\,19\,35)(9\,14\,42\,46\,10\,15\,12\,27\,48\\ & 25\,28\,11\,45\,26\,47\,41\,30\,43\,13\,32\,31\,44\,29\,16)(18\,36)(22\,40) = \alpha\gamma\tau_{5,4}\gamma\alpha,\\ \sigma_{5,5} = & (2\,5\,4\,40\,25\,10\,12\,41\,36\,23\,30\,15\,48\,24\,38\,44\,26\,34\,3\,20\,27\,46\,37\,6\,8\,21\,42\\ & 14\,9\,16\,28\,22)(7\,33\,45\,11\,29\,39\,18\,47\,31\,19\,35\,32\,43\,17) = \beta\gamma\tau_{5,5}\gamma\delta,\\ \sigma_{5,6} = & (2\,5\,4\,33\,27\,46\,19\,35\,42\,11\,28\,37\,3\,21\,25\,15\,41\,22\,7\,40\,47\,31\,36\,23\,45\,14\,9\\ & 16\,29\,24\,38\,30\,10\,12\,48\,34\,6\,8\,20\,44\,26\,17)(18\,32\,43\,39) = \delta\alpha\beta\gamma^2\tau_{5,6}\gamma^2\alpha^3.\end{array}$$

In the following we determine $X_{5,\imath}$ and $G_{5,\imath}$. Noting that α, β, δ, γ and $\tau_{5,\imath}$ are all even permutations, we have $G_{5,\imath} \le X_{5,\imath} \le \mathrm{A}_{48}$ for $1 \le \imath \le 6$.

Lemma 4.4.1. *$G_{5,1} \cong (\mathbb{Z}_7 \times \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2$ and $X_{5,1} \cong (\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2^2$.*

Proof. Let $\mu = (\delta_{5,1}^{\tau_{5,1}}\sigma_{5,1})^3$. Then

$$\mu = (2\,4\,35\,7\,24\,8\,34)(3\,33\,20\,37\,39\,17\,36)(5\,23\,21\,6\,18\,38\,19),$$

and $\mu^{\tau_{5,1}} = \mu^{-1}$, $\mu^{\sigma_{5,1}} = \mu^{-1}$, $\mu^{\delta_{5,1}} = \mu^{-1}$. Then $\langle\mu\rangle \lhd G_{5,1}$. Further, $\delta_{5,1} = ((\sigma_{5,1}\delta_{5,1})^5\tau_{5,1})^2(\sigma_{5,1}\delta_{5,1})^2\tau_{5,1}$. Thus

$$G_{5,1} = \langle\tau_{5,1}, \sigma_{5,1}, \delta_{5,1}\rangle = \langle\mu, \mu\sigma_{5,1}\delta_{5,1}, \tau_{5,1}\rangle = \langle\mu\rangle\langle\mu\sigma_{5,1}\delta_{5,1}, \tau_{5,1}\rangle.$$

Let $\nu = \mu\sigma_{5,1}\delta_{5,1}$, $\omega = \tau_{5,1}\tau_{5,1}^{\nu}$, $N = \langle\nu, \omega\rangle$ and $L = \langle\nu, \omega, \tau_{5,1}\rangle$. Then

$$\begin{aligned}\nu &= (9\,28\,12\,46\,14\,16\,45)(10\,30\,42\,29\,11\,25\,27)(13\,47\,32\,43\,41\,31\,48),\\ \omega &= (9\,11)(10\,12)(13\,15)(14\,16)(25\,27)(26\,28)(29\,31)(30\,32)(41\,43)\\ &\quad (42\,44)(45\,47)(46\,48).\end{aligned}$$

Further, $\nu^{\tau_{5,1}} = \nu\omega$, $\tau_{5,1}$ centralizes ω and μ centralizes N; in particular, $L = N \rtimes \langle\tau_{5,1}\rangle$ and hence $G_{5,1} = (\langle\mu\rangle \times N) \rtimes \langle\tau_{5,1}\rangle$. Note that $N = \langle\nu^4, \omega\rangle$ has the same presentation as $\mathrm{PSL}(2,7)$. Then $N \cong \mathrm{PSL}(2,7)$ (see [8] for example), and hence $G_{5,1} \cong (\mathbb{Z}_7 \times \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2$.

Set $M = \langle N, N^{\delta}\rangle$. Then $M = \langle\nu, \omega, \nu^{\delta}, \omega^{\delta}\rangle = N \times N^{\delta}$ and $|X_{5,1} : M| = |X_{5,1}|/|M| = |G_{5,1}||H|/|M| = 4$. Considering the transitive permutation representation of $X_{5,1}$ on the right cosets of M, we have $X_{5,1}/\mathrm{Core}_{X_{5,1}}(M) \lesssim \mathrm{S}_4$. It follows that $M \lhd X_{5,1}$. It is easy to know that M has exactly two orbits, say $\Delta = \{i + 16j \mid 1 \le i \le 8,\ j = 0, 1, 2\}$ and $\Theta = \Omega \setminus \Delta$. Further, $\Delta^{\delta} = \Theta$; in particular, $\delta \notin M$. Consider the restrictions M^{Δ} and M^{Θ} of M on Δ and Θ, respectively. It follows that $M^{\Delta} = N^{\delta} \le \mathrm{Alt}(\Delta)$ and $M^{\Theta} = N \le \mathrm{Alt}(\Theta)$. Let $\rho = \tau_{5,1}^{\nu}$. Then $\nu^{\rho} = \omega\nu$, $\omega^{\rho} = \omega$ and $\delta\rho = \rho\delta$. By calculation, $\rho^{\Delta} = (2\,4)(5\,6)(7\,8)(17\,20)(18\,19)(21\,23)(33\,36)(34\,35)(37\,39)$ and $\rho^{\Theta} = (10\,12)(13\,14)(15\,16)(25\,28)(26\,27)(29\,31)(41\,44)(42\,43)(45\,47)$ are odd permutations. Then $\rho \notin M$, $\langle N, \rho\rangle = N\langle\rho\rangle \cong \mathrm{PGL}(2,7)$, $\langle N^{\delta}, \rho\rangle = N^{\delta}\langle\rho\rangle \cong \mathrm{PGL}(2,7)$ and $X_{5,1} = M \rtimes \langle\rho, \delta\rangle \cong (\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2^2$. ∎

Lemma 4.4.2. *$G_{5,2} \cong (\mathrm{A}_{23} \times \mathrm{A}_{24}) \rtimes \mathbb{Z}_2$ and $X_{5,2} \cong (\mathrm{A}_{24} \times \mathrm{A}_{24}) \rtimes \mathbb{Z}_2^2$.*

Proof. Let $\mu = \sigma_{5,2}\tau_{5,2}$ and $\nu = \delta_{5,2}\tau_{5,2}$. Then $\mu^{\tau_{5,2}} = \mu^{-1}$, $\nu^{\tau_{5,2}} = \nu^{-1}$ and $L := \langle\mu, \nu\rangle \lhd G_{5,2} = \langle\mu, \nu\rangle\langle\tau_{5,2}\rangle$, where

$$\begin{aligned}\mu = {} & (2\,8\,7\,4\,39\,38)(3\,24\,19\,36\,17\,21)(5\,33\,35\,6\,20)(9\,13\,45\,27\,46\,16\,11\,26\,28)\\ & (12\,43)(14\,30\,42\,48\,41\,47\,44\,29\,15)(18\,22\,40\,37\,23)(31\,32),\\ \nu = {} & (2\,17\,19\,4\,8\,40\,18\,37\,7)(3\,34)(5\,21\,33\,39\,36\,38\,35\,24\,6)(9\,15\,14\,11\,46\,45)\\ & (10\,31\,26\,43\,28\,32)(13\,27\,16\,44\,42)(22\,23)(25\,29\,47\,48\,30).\end{aligned}$$

It is easy to know that L has two orbits, say $\Delta_1 = \Delta \setminus \{1\}$ and Θ on $\Omega \setminus \{1\}$, where Δ and Θ are given as in Lemma 4.4.1. Consider the restrictions of μ and ν on Δ_1 and Θ. We know that μ^{Δ_1} and ν^{Δ_1} are even permutations (on

Δ_1), μ^Θ and ν^Θ are even permutations (on Θ). It implies $L \le L^{\Delta_1} \times L^\Theta \le \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Theta) \cong \mathrm{A}_{23} \times \mathrm{A}_{24}$. By calculation,

$$\begin{aligned}
&\mu^{\Delta_1}\nu^{\Delta_1} = (2\,40\,7\,8)(3\,6\,20\,21\,34)(4\,36\,19\,38\,17\,33\,24)(5\,39\,35)(18\,23\,37\,22),\\
&\mu^{\Delta_1}\nu^{\Delta_1}\mu^{\Delta_1} = (2\,37\,40\,4\,17\,35\,33\,19)(3\,20)(5\,38\,21\,34\,24\,39\,6),\\
&(\mu^{\Delta_1}\nu^{\Delta_1})^4 = (3\,34\,21\,20\,6)(4\,17\,36\,33\,19\,24\,38)(5\,39\,35),\\
&((\mu\nu\mu)^8\nu)^{36} = (5\,35\,24\,36\,38\,33\,39)(13\,27\,16\,44\,42).
\end{aligned}$$

It follows that L^{Δ_1} is 2-transitive on Δ_1 and contains a 3-cycle $(5\,39\,35)$. Then $L^{\Delta_1} = \mathrm{Alt}(\Delta_1) \cong \mathrm{A}_{23}$ by [9, Thorem 3.3A]. A similar argument yields $L^\Theta = \mathrm{Alt}(\Theta) \cong \mathrm{A}_{24}$. Further, L contains a 7-cycle $\iota = (5\,35\,24\,36\,38\,33\,39)$ and a 5-cycle $\kappa = (13\,27\,16\,44\,42)$. Since $\iota \in L^{\Delta_1}$ and $\kappa \in L^\Theta$, we have $\iota^\sigma = \iota^{\sigma^{\Delta_1}}$ and $\kappa^\sigma = \kappa^{\sigma^\Theta}$ for any $\sigma \in L$. Take $\epsilon = (5\,35\,24)(33\,38)(36\,39) \in L^{\Delta_1}$ and $\varepsilon = (13\,16\,44)$. Then $\iota\iota^\epsilon = (5\,24\,35) \in L$ and $\kappa\kappa^\varepsilon = (13\,44\,16) \in L$. Consider the conjugations of $(5\,24\,35)$ and $(13\,44\,16)$ under L^{Δ_1} and L^Θ, respectively. We conclude that L contains all 3-cycles of L^{Δ_1} and of L^Θ. Then $L^{\Delta_1} \le L$ and $L^\Theta \le L$, so $L = L^{\Delta_1} \times L^\Theta = \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Theta) \cong \mathrm{A}_{23} \times \mathrm{A}_{24}$. Note that $\tau_{5,2}^{\Delta_1}$ and $\tau_{5,2}^{\Theta}$ are odd permutations. Then $\tau_{5,2} \notin L$. Thus $G_{5,2} = L\langle\tau_{5,2}\rangle = L \rtimes \langle\tau_{5,2}\rangle \cong (\mathrm{A}_{23} \times \mathrm{A}_{24}) \rtimes \mathbb{Z}_2$.

Set $N = \langle\mu^\Theta, \nu^\Theta\rangle$ and $M = \langle N, N^\delta\rangle = N \times N^\delta$. A similar argument as in the proof of Lemma 4.4.1 leads to $|X_{5,2} : M| = 4$ and $M \lhd X_{5,2}$. Let $o = (10\,12)(25\,27)$, $\pi = (5\,6)(7\,8)(17\,19)(21\,24)(22\,23)(33\,35)(37\,40)(38\,39)$ and $\varpi = (9\,11)(13\,16)(14\,15)(25\,27)(26\,28)(29\,30)(31\,32)(42\,44)(45\,46)(47\,48)$. We have $\pi \in M^\Delta = N^\delta$ and $o, \varpi \in M^\Theta = N$, and so $\rho := (2\,4)(10\,12) = \tau_{5,2}o\pi\varpi \in X_{5,2}$. It is easy to see that $\rho, \delta \notin M$ and $\rho\delta = \delta\rho$. Then $X_{5,2} = M \rtimes \langle\rho, \delta\rangle \cong (\mathrm{A}_{24} \times \mathrm{A}_{24}) \rtimes \mathbb{Z}_2^2$. ∎

Lemma 4.4.3. $G_{5,3} \cong (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11} \times \mathrm{PSL}(2,23)) \rtimes \mathbb{Z}_2$ *and* $X_{5,3} \cong (\mathrm{PSL}(2,23) \times \mathrm{PSL}(2,23)) \rtimes \mathbb{Z}_2^2$.

Proof. Let $\omega = (\tau_{5,3}\tau_{5,3}^{\sigma_{5,3}})^{12}$, $\mu = (\tau_{5,3}\tau_{5,3}^{\sigma_{5,3}})^{23}$, $\upsilon = ((\tau_{5,3}\sigma_{5,3})^6(\tau_{5,3}\tau_{5,3}^{\sigma_{5,3}})^{23})^{12}$, $\nu = ((\tau_{5,3}\sigma_{5,3})^6(\tau_{5,3}\tau_{5,3}^{\sigma_{5,3}})^{23})^{11}$ and $\rho = \omega^5\tau_{5,3}$. By calculation, we have

$$\begin{aligned}
&\omega = (2\,6\,19\,38\,35\,36\,18\,21\,24\,3\,37\,40\,34\,20\,17\,23\,33\,5\,4\,7\,39\,22\,8),\\
&\upsilon = (2\,3\,19\,37\,17\,33\,5\,18\,34\,23\,36)(6\,22\,24\,20\,35\,40\,38\,8\,39\,7\,21),\\
&\mu = (9\,43\,32\,47\,27\,11\,16\,42\,15\,14\,28\,13)(10\,46\,48\,44\,41\,45\,12\,30\,25\,26\,31\,29),\\
&\nu = (9\,10\,27\,32\,16\,25\,11\,43\,15\,45\,41\,12)(13\,28\,30\,48\,31\,42\,26\,46\,29\,47\,44\,14),\\
&\rho = (2\,20)(3\,35)(5\,7)(6\,34)(8\,17)(18\,21)(19\,40)(22\,23)(24\,36)(33\,39)(37\,38)\\
&(9\,11)(13\,16)(14\,15)(25\,41)(26\,44)(27\,43)(28\,42)(29\,46)(30\,45)(31\,48)(32\,47),\\
&G_{5,3} = \langle\tau_{5,3}, \sigma_{5,3}\rangle = \langle\tau_{5,3}, \tau_{5,3}\sigma_{5,3}, \tau_{5,3}\tau_{5,3}^{\sigma_{5,3}}\rangle = \langle\rho, (\tau_{5,3}\sigma_{5,3})^6, \mu, \omega\rangle\\
&\qquad = \langle\rho, (\tau_{5,3}\sigma_{5,3})^6\mu, \mu, \omega\rangle = \langle\rho, \nu, \upsilon, \mu, \omega\rangle.
\end{aligned}$$

Further, $\omega^\upsilon = \omega^{12}$, $\omega^\rho = \omega^{-1}$, $\upsilon^\rho = \upsilon$, $\mu^\rho = \mu^{-1}$ and $\nu^\rho = \mu^9\nu(\mu^2\nu^2)^2\mu\nu\mu$. Set $L = \langle\omega, \upsilon\rangle$ and $N = \langle\mu, \nu\rangle$. Then $L\langle\rho\rangle \cong \mathbb{Z}_{23} \rtimes \mathbb{Z}_{22}$ and $LN = L \times N \lhd G_{5,3}$. Note that LN has exactly two orbits on $\Omega \setminus \{1\}$ given as in the proof of Lemma 4.4.2, say Δ_1 and Θ. Considering the restrictions of

ρ, L and N on Δ_1 and Θ, we have $\rho \notin LN$. Thus $G_{5,3} = (L \times N) \rtimes \langle \rho \rangle$. Let $\pi = (\mu\nu)^2\nu^4\mu^4$ and $\varpi = \mu^8\nu^2\mu^4\nu^4\mu^2$. Then $\mu = \pi^{17}\varpi\pi^7\varpi\pi^2\varpi\pi^3\varpi$ and $\nu = \pi^{20}\varpi\pi^9\varpi\pi$, and hence $N = \langle \pi, \varpi \rangle$. Further, calculation shows that $\pi^{23} = (\pi^4\varpi\pi^{12}\varpi)^2 = (\pi\varpi)^3 = \varpi^2 = 1$. Then $N \cong \mathrm{PSL}(2, 23)$ and $N\langle \rho \rangle \cong \mathrm{PGL}(2, 23)$. Thus $G_{5,3} \cong (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11} \times \mathrm{PSL}(2, 23)) \rtimes \mathbb{Z}_2$.

Let $M = \langle N, N^\delta \rangle$. Then $\delta \notin M$ and $M = N \times N^\delta$ has index 4 in $X_{5,3}$, and then $M \lhd X_{5,3}$. Consider the restrictions of M on $\Delta = \Delta_1 \cup \{1\}$ and on Θ. We conclude that all elements of M^Δ and M^Θ are even permutations. It implies that $\rho \notin M$. Note that $\langle \rho, \delta \rangle \cong \mathrm{D}_{92}$ and $|M \cap \langle \rho, \delta \rangle| = 23$. It follows that $X_{5,3} = M\langle \rho, \delta \rangle = M \rtimes \langle (\rho\delta)^{23}, \delta \rangle \cong (\mathrm{PSL}(2, 23) \times \mathrm{PSL}(2, 23)) \rtimes \mathbb{Z}_2^2$. ∎

Lemma 4.4.4. $G_{5,4} \cong (\mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2, 7)) \rtimes \mathbb{Z}_2$ *and* $X_{5,4} \cong (\mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2, 7) \times \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2, 7)) \rtimes \mathbb{Z}_2^2$.

Proof. Let $\zeta = \tau_{5,4}\sigma_{5,4}$ and $\xi = \tau_{5,4}\tau_{5,4}^{\sigma_{5,4}}$. Then, by calculation, we have

$$\begin{aligned}\zeta &= (2\,24\,5\,37\,3\,34\,23\,38\,20)(6\,19\,18\,17\,33\,36\,35\,8\,7)\\ &\quad(9\,45\,44\,28\,30\,10\,15\,42\,48\,31\,26\,13)(11\,14\,12\,27\,46\,43\,47\,16\,32\,25\,29\,41),\\ \xi &= (2\,24\,39\,33\,35\,5\,7)(3\,21\,19\,17\,34\,36\,37)(4\,8\,6\,20\,18\,23\,38)(9\,30\,48)\\ &\quad(10\,43\,44\,31\,14\,15\,45\,25\,26)(11\,32\,46)(12\,42\,27)(13\,41\,29)(16\,47\,28).\end{aligned}$$

Then $G_{5,4} = \langle \tau_{5,4}, \sigma_{5,4} \rangle = \langle \tau_{5,4}, \tau_{5,4}\sigma_{5,4}, \tau_{5,4}\tau_{5,4}^{\sigma_{5,4}} \rangle = \langle \tau_{5,4}, \zeta, \xi \rangle$. Further, $\xi^{\tau_{5,4}} = \xi^{-1}$ and $\zeta^{\tau_{5,4}} = \zeta\xi^{-1}$. Set $L = \langle \zeta, \xi \rangle$. Then $L \lhd G_{5,4}$. Since both ζ and ξ fix 22 and 40, we have $\tau_{5,4} \notin L$. Thus $G_{5,4} = L \rtimes \langle \tau_{5,4}, \rangle$. Let $\upsilon = (\xi^2\zeta\xi)^4$, $\omega = \xi^9$, $\mu = (\xi^2\zeta\xi)^9$, $\nu = \xi^7$, $K = \langle \upsilon, \omega \rangle$ and $N = \langle \mu, \nu \rangle$. Then

$$\begin{aligned}L &= \langle \zeta, \xi \rangle = \langle \xi^2\zeta\xi, \xi \rangle = \langle \upsilon, \omega, \mu, \nu \rangle = \langle \upsilon, \omega \rangle \times \langle \mu, \nu \rangle = K \times N,\\ \upsilon &= (2\,8\,38\,23\,19\,3\,37\,33\,24)(4\,6\,20\,39\,35\,5\,21\,17\,34),\\ \omega &= (2\,39\,35\,7\,24\,33\,5)(3\,19\,34\,37\,21\,17\,36)(4\,6\,18\,38\,8\,20\,23),\\ \mu &= (9\,14\,31\,27)(10\,16\,48\,43)(11\,44\,42\,12)(13\,29\,32\,15)(25\,45\,41\,30)(26\,28\,47\,46),\\ \nu &= (9\,30\,48)(10\,25\,15\,31\,43\,26\,45\,14\,44)(11\,32\,46)(12\,42\,27)(13\,41\,29)(16\,47\,28).\end{aligned}$$

Let $\eta = \upsilon^7\omega^{-1}\upsilon^3\omega^2\upsilon^3\omega$ and $\epsilon = \upsilon^3$. Then $\epsilon^\eta = \epsilon^{\omega^2}$, $\omega^\eta = \omega^4$ and $\epsilon\epsilon^\omega\epsilon^{\omega^2}\epsilon^{\omega^3}\epsilon^{\omega^4}\epsilon^{\omega^5}\epsilon^{\omega^6} = 1$. It follows that $B := \langle \epsilon^\sigma \bigm| \sigma \in L \rangle \cong \mathbb{Z}_3^6$, $Q := \langle \omega, \eta \rangle \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. Noting that Q has no normal subgroups of order 3, we have $B \cap Q = 1$. Thus $K = \langle \upsilon, \omega \rangle = \langle \upsilon^7, \upsilon^3, \omega \rangle = \langle \upsilon^7\omega^{-1}\upsilon^3\omega^2\upsilon^3\omega, \upsilon^3, \omega \rangle = \langle \epsilon, \eta, \omega \rangle = B \rtimes Q \cong \mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$.

Let $\varepsilon = \nu^3$, $\pi = (\nu^{-1}\nu^\mu)^3$ and $o = (\varepsilon^2)^\mu\pi\varepsilon^2\pi^{-1}\nu\pi^{-1}$. Then

$$\begin{aligned}\varepsilon &= (10\,31\,45)(14\,25\,43)(15\,26\,44),\\ \pi &= (9\,31\,13\,47\,25\,32\,15)(10\,42\,29\,14\,11\,44\,48)(12\,43\,46\,26\,30\,45\,28),\\ o &= (9\,15)(10\,29)(11\,14)(12\,45)(13\,27)(16\,42)(25\,32)(26\,30)(28\,41)\\ &\quad(31\,47)(43\,46)(44\,48).\end{aligned}$$

Then $\pi^7 = o^2 = (\pi^4 o)^4 = (\pi o)^3 = 1$, $\mu = (\pi^{-1}\varepsilon)^2\varepsilon\pi^5(\varepsilon\pi^{-1})^2\varepsilon\pi^2 o\pi^4 o$ and $\nu = \varepsilon^{\pi^{-1}}\varepsilon^\mu o\pi$. It follows that $\langle \pi, o \rangle \cong \mathrm{PSL}(2, 7)$ and $N = \langle \varepsilon^\sigma \bigm| \sigma \in N \rangle\langle \pi, o \rangle = \langle \varepsilon, \varepsilon^\pi, \varepsilon^{\pi^2}, \varepsilon^{\pi^3}, \varepsilon^{\pi^4}, \varepsilon^{\pi^5}, \varepsilon^\mu \rangle \rtimes \langle \pi, o \rangle \cong \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2, 7)$.

The above argument yields $G_{5,4} \cong (\mathbb{Z}_3^6 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2$. Set $M = \langle N, N^\delta \rangle$. Then $\delta \notin M$, $M = N \times N^\delta$ and $|X_{5,4} : M| = 4$. Considering the transitive permutation representation of $X_{5,4}$ on the right cosets of M, we have $X_{5,4}/\mathrm{Core}_{X_{5,4}}(M) \lesssim \mathrm{S}_4$. It is easily shown that $M = \mathrm{Core}_{X_{5,4}}(M) \lhd X_{5,4}$. Let $\rho = \sigma_{5,4}\delta\sigma_{5,4}^{-1}$. Then $\rho\delta = \delta\rho$, and $\rho \notin M$ by considering the restrictions of M on its orbits on Ω. Thus $X_{5,4} = M \rtimes \langle \rho, \delta \rangle \cong (\mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2,7) \times \mathbb{Z}_3^7 \rtimes \mathrm{PSL}(2,7)) \rtimes \mathbb{Z}_2^2$. ∎

Lemma 4.4.5. $G_{5,5} = G_{5,6} \cong \mathrm{A}_{47}$ *and* $X_{5,5} = X_{5,6} = \mathrm{A}_{48}$.

Proof. Let $\imath = 5$ or 6. Consider the actions of $G_{5,\imath}$ and of $\langle \sigma_{5,\imath}^{-1}\sigma_{5,\imath}^{\tau_{5,\imath}}, (\sigma_{5,\imath}^2\tau_{5,\imath})^2 \rangle$ on $\Omega \setminus \{1\}$. Then $G_{5,\imath}$ is a 2-transitive permutation group of degree 47. Since all generators of $G_{5,\imath}$ are even permutations (on $\Omega \setminus \{1\}$), we have $G_{5,\imath} \leq \mathrm{Alt}(\Omega \setminus \{1\})$. Note that $(\tau_{5,5}\sigma_{5,5}^7)^{36}$ is a 5-cycle and $(\tau_{5,6}\sigma_{5,6}^9)^{32}$ is a 7-cycle. It follows from [9, Theorem 3.3E] that $G_{5,\imath} = \mathrm{Alt}(\Omega \setminus \{1\}) \cong \mathrm{A}_{47}$, and hence $X_{5,5} = X_{5,6} = \mathrm{A}_{48}$. ∎

4.5. **Conclusions.** Now we prove Theorem 1.1 and 1.2.

Proof of Theorem 1.1. Let Γ be a connected core-free cubic (X, s)-transitive Cayley graph. Then $s \geq 2$ by Corollary 2.2. The argument in Subsection 4.1 to 4.4 says that Γ is isomorphic to one of $\Gamma_{s,\imath}$ and $\Gamma_{t,\jmath_1} \ncong \Gamma_{t,\jmath_2}$, where $2 \leq s, t \leq 5$, $t \neq 5$, $1 \leq \imath \leq \ell_s$, $1 \leq \jmath_1$, $\jmath_2 \leq \ell_t$, $\jmath_1 \neq \jmath_2$, $\ell_2 = 2$, $\ell_3 = 3$, $\ell_4 = 4$ and $\ell_5 = 6$.

We claim that $\Gamma_{s,\jmath}$ is not t-transitive for $s < t$. Suppose to the contrary that $\Gamma_{s,\jmath}$ is $(X_\jmath, t)$-transitive for some $G_{s,\jmath} \leq X_\jmath \leq \mathsf{Aut}(\Gamma_{s,\jmath})$. By Corollary 2.2, the quotient $(\Gamma_{s,\jmath})_N$ induced by $N = \mathrm{Core}_{X_\jmath}(G_{s,\jmath})$ is isomorphic to some $\Gamma_{t,\imath}$, in particular, $G_{t,\imath} \cong G_{s,\jmath}/N$, which is impossible. It follows that $\mathsf{Aut}(\Gamma_{s,\jmath}) = X_{s,\jmath}$ for $2 \leq s \leq 5$ and $1 \leq \jmath \leq \ell_s$, and $\Gamma_{s,\jmath} \ncong \Gamma_{t,\imath}$ for possible $s < t$, $\jmath$ and $\imath$. Thus it suffices to show that $\Gamma_{5,5} \ncong \Gamma_{5,6}$ in the following.

Recall that $\Gamma_{5,\imath} = \mathsf{Cos}(X_{5,\imath}, H, \tau_{5,\imath})$ and $\mathsf{Aut}(\Gamma_{5,\imath}) = X_{5,\imath} = \mathrm{A}_{48}$, where $H \cong \mathrm{S}_4 \times \mathbb{Z}_2$ is a regular subgroup of A_{48} under the natural action. Suppose that $\Gamma_{5,5} \cong \Gamma_{5,6}$. Then, by [20, Lemma 2.3], there is some $\sigma \in \mathsf{Aut}(\mathrm{A}_{48}) = \mathrm{S}_{48}$ with $H\tau_{5,5}^\sigma H = H\tau_{5,6}H$ such that $H\tau \mapsto H\tau^\sigma$ gives an isomorphism from $\Gamma_{5,5}$ to $\Gamma_{5,6}$. Consider the neighborhood of H (as a vertex) in $\Gamma_{5,\imath}$. Then $\{H\tau_{5,5}^\sigma, H\sigma_{5,5}^\sigma, H(\sigma_{5,5}^{-1})^\sigma\} = \{H\tau_{5,6}, H\sigma_{5,6}, H\sigma_{5,6}^{-1}\}$. In particular, one of cosets $H\tau_{5,5}$,$H\sigma_{5,5}$ and $H\sigma_{5,5}^{-1}$ must contain a permutation with the same order 84 of $\sigma_{5,6}$, which is impossible by calculation. Thus $\Gamma_{5,5} \ncong \Gamma_{5,6}$. ∎

Theorem 1.2 is a direct consequence of Corollary 2.2 and Theorem 1.1.

Finally, since a Cayley graph of a finite non-abelian simple group is either normal or core-free, our argument leads to the following well-known result which can be derived from [16, 28, 29].

Theorem 4.1. *Let Γ be a connected cubic arc-transitive Cayley graph of a finite non-abelian simple group T. Then either Γ is normal with respect to T, or Γ is isomorphic to one of $\Gamma_{5,5}$ and $\Gamma_{5,6}$.*

Note: All calculational results in this paper were also confirmed by GAP.

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