Rainbow number of matchings in regular bipartite graphs *

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Abstract

Given a graph G and a subgraph H of G, let $rb(G, H)$ be the minimum number r for which any edge-coloring of G with r colors has a rainbow subgraph H. The number $rb(G, H)$ is called the rainbow number of H with respect to G. Denote mK_2 a matching of size m and $B_{n,k}$ the set of all the k-regular bipartite graphs with bipartition (X, Y) such that $|X| = |Y| = n$ and $k \leq n$. Let k, m, n be given positive integers, where $k \geq 3$, $m \geq 2$ and $n > 3(m-1)$. We show that for every $G \in B_{n,k}$, $rb(G, mK_2) = k(m-2) + 2$. We also determine the rainbow numbers of matchings in paths and cycles.

Keywords: edge-colored graph, rainbow subgraph, rainbow number, matching, regular bipartite graph

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1 Introduction

We use Bondy and Murty [3] for terminology and notations not defined here and consider simple, finite graphs only.

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The Ramsey problem asks for the optimal total number of colors used on the edges of a graph without creating a monochromatic subgraph. In anti-Ramsey problems, we are interested in heterochromatic or rainbow subgraphs instead of monochromatic subgraphs in edge-colorings. Given a graph G and a subgraph H of G, if G is edge-colored and H contains no two edges of the same color, then H is called a rainbow subgraph of G and we say that G contains rainbow H. Let $f(G, H)$ denote the maximum number of colors in an edge-coloring of G with no rainbow H. Define $rb(G, H)$ the minimum number of colors such that any edge-coloring of G with at least $rb(G, H) = f(G, H)+1$ colors contains a rainbow subgraph H. $rb(G, H)$ is called the *rainbow number* of H with respect to G.

When $G = K_n$, $f(G, H)$ is called the *anti-Ramsey number* of H. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós in the 1970s. Let P_k and C_k denote the path and the cycle with k edges, respectively. Simonovits and Sós [10] determined $f(K_n, P_k)$ for large enough n. Erdős et al. [5] conjectured that for every fixed $k \geq 3$, $f(K_n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$, and proved it for $k = 3$ by showing that $f(K_n, C_3) = n - 1$. Alon [1] showed that $f(K_n, C_4) = \lfloor \frac{4n}{3} \rfloor - 1$, and the conjecture is thus proved for $k = 4$. Jiang and West [6] verified the conjecture for k at most 6. Recently the conjecture is proved for all $k \geq 3$ by Montellano-Ballesteros and Neumann-Lara [8]. Axenovich, Jiang and Kündgen [2] determined $f(K_{m,n}, C_{2k})$ for all $k \geq 2$.

In 2004, Schiermeyer [9] determined the rainbow numbers $rb(K_n, K_k)$ for all $n \geq k \geq 4$, and the rainbow numbers $rb(K_n, mK_2)$ for all $m \geq 2$ and $n \geq 3m + 3$, where mK_2 is a matching of size m. Li, Tu and Jin [7] proved that $rb(K_{m,n}, pK_2) = m(p - 2) + 2$ for all $m \geq n \geq p \geq 3$. Chen, Li and Tu [4] determined $rb(K_n, mK_2)$.

Let $B_{n,k}$ be the set of all the k-regular bipartite graphs with bipartition (X, Y) such that $|X| = |Y| = n$ and $k \leq n$. In this paper we give an upper and lower bound for $rb(G, mK_2)$, where $G \in B_{n,k}$. Let k, m, n be given positive integers, where $k \geq 3$, $m \geq 2$ and $n > 3(m - 1)$. We show that for every $G \in B_{n,k}$, $rb(G, mK_2) = k(m-2)+2$. We also determine the rainbow numbers of matchings in paths and cycles.

2 Rainbow number of matchings in regular bipartite graphs

Denote by mK_2 a matching of size m and $B_{n,k}$ the set of all the k-regular bipartite graphs with bipartition (X, Y) such that $|X| = |Y| = n$ and $k \leq n$. From a result of Li, Tu and Jin in [7] we know that if $n \geq 3$ and $2 \leq m \leq n$, then $rb(B_{n,n}, mK_2) = n(m-2)+2$. In this section we discuss the rainbow numbers of matchings in a k-regular bipartite graph $G \in B_{n,k}$.

A vertex cover of G is a set S of vertices such that S contains at least one end-vertex of every edge of G. For any $U \subset V(G)$, denote by $N_G(U)$ the neighborhood of U in G, we abbreviate it as $N(U)$ when there is no ambiguity.

Lemma 2.1. *[3] For any bipartite graph G, the size of a maximum matching equals the size of a minimum vertex cover. Let P be a minimum vertex cover of G, then every maximum matching of G saturates P.*

Let $ext(G, H)$ denote the maximum number of edges that G can have with no subgraph isomorphic to H.

Theorem 2.2. *For any subgraph H of a graph $G \in B_{n,k}$, if $|E(H)| > k(m-1)$ and $2 \leq m \leq n$, then $mK_2 \subset H$. That is*

$$ext(G, mK_2) = k(m-1).$$

Proof. By contradiction. Suppose H is a subgraph of $B_{n,k}$ with $|E(H)| > k(m-1)$ and contains no mK_2. Then H is bipartite and the maximum degree of the vertices in H is k. By Lemma 2.1 H has a vertex cover of size at most $m-1$, which can cover at most $(m-1)k$ edges, contrary to $|E(H)| > k(m-1)$. □

Theorem 2.3. *If $G \in B_{n,k}$ and $1 \leq m \leq n$, then*

$$k(m-2)+2 \leq rb(G, mK_2) \leq k(m-1)+1.$$

Proof. The upper bound is obvious from Theorem 2.3. For the lower bound, let $G = (X, Y)$ and $Y_1 \subset Y$ with $|Y_1| = m-2$, color the $k(m-2)$ edges between Y_1 and X with $k(m-2)$ distinct colors and the remaining edges with one extra color. It is easy to check that $k(m-2)+1$ colors are used and there is no rainbow mK_2 in such a coloring. □

The following lemma may already exist. However, we cannot find it in the literature. For the convenience of the reader, we give a full proof of it.

Lemma 2.4. *Let G be a bipartite graph. Then there exists a maximum matching that saturates all the vertices of maximum degree.*

Proof. Let Δ denote the maximum degree of G. Among all maximum matchings of G, let M be one that saturates the largest number of vertices of degree Δ. Suppose some vertex v of degree Δ is not saturated by M, we derive a contradiction. Let (X, Y) be a bipartition of G. Without loss of generality, suppose v is in X. Let S denote the set of all the vertices in X reachable from v by an M-alternating path and T the set of vertices in Y reachable from v by an M-alternating path. If some vertex w in S has degree less than Δ, then let M' be obtained from M by switching the M-edges along an M-alternating path from v to w. We can check that M' is a maximum matching in which v is saturated instead of w, and M and M' saturate the set of vertices besides v and w. This contradicts our choice of M.

So all the vertices in S have degree Δ. Since M is a maximum matching in G, there is no M-augment path in G, all the vertices in $S \cup T$ are M-saturated and there exists a natural bijection between S and T through M (see for instance the proof of Hall's Theorem in [11]). So $|S| = |T|$. Furthermore $N(\{v\} \cup S) = T$. But there are $\Delta|S| + \Delta$ edges from $\{v\} \cup S$ to T while T can be incident to at most $\Delta|T|$ edges in G, a contradiction. □

Corollary 2.5. *If G is a bipartite graph with maximum degree k, $|E(G)| \geq k(m-2)+j$ with $1 \leq j \leq k$ and G has no matching of size m, then G contains j pairwise edge disjoint matchings $M_1, M_2, \cdots, M_j$ of size $m-1$. Furthermore, for any $1 \leq s \leq j$, the maximum degree of $G \backslash \cup_{i=1}^{s} M_i$ is $k - s$.*

Proof. We prove by induction on j.

If $j = 1$ and $|E(G)| \geq k(m-2)+1$, since G has no matching of size m, by Lemma 2.1 G contains a maximum matching M_1 of size $m-1$ which saturates all the vertices of degree k and the maximum degree of $G \backslash M_1$ is $k-1$. Suppose that when $j = t$ the result is true. Let $j = t+1$ and $|E(G)| \geq k(m-2)+t+1$. By the induction hypothesis G has t pairwise edge disjoint matchings $M_1, M_2, \cdots, M_t$ of size $m-1$ and the maximum degree in $G \backslash \cup_{i=1}^{t} M_i$ is $k-t$. Now there are $k(m-2)+t+1-t(m-1) = (k-t)(m-2)+1$ edges in $G \backslash \cup_{i=1}^{t} M_i$, by Lemma 2.1 and Lemma 2.4, there is a matching M_{t+1} of size $m-1$ which saturates all the vertices of degree $k-t$ in $G \backslash \cup_{i=1}^{t} M_i$ and this completes the proof. □

The following theorem shows that for given k and m, if n is large enough, $rb(B_{n,k}, mK_2)$ will always be equal to the lower bound $k(m-2)+2$.

Theorem 2.6. *For all $m \geq 2$, $k \geq 3$, $n > 3(m-1)$, if G is a k-regular bipartite graph with n vertices in each partite set, then*

$$rb(G, mK_2) = k(m-2)+2.$$

Proof. From Theorem 2.4 it suffices to show that for any $m \geq 2$, $k \geq 3$, if $n > 3(m-1)$, any coloring c of G with $k(m-2)+2$ colors contains a rainbow mK_2. By contradiction, suppose there is no rainbow mK_2 in G. Let H be a subgraph of G formed by taking one edge of each color from G. We have $|E(H)| = k(m-2)+2$ and there is no mK_2 in H. From Corollary 2.5, let M and M' be two edge-disjoint matchings of size $m-1$ in H.

Since M and M' are both maximum matchings in H, by Lemma 2.1 the edges in $M \cup M'$ are incident to at most $3(m-1)$ vertices, which can be incident to at most $3k(m-1)$ edges. If $n > 3(m-1)$, then $|E(G)| > 3k(m-1)$ and there is at least one edge, say e, in G that is independent of $E(M) \cup E(G')$. Without loss of generality, suppose $c(e) \in C(M)$, then $M' \cup \{e\}$ is a rainbow mK_2 in G. □

3 Rainbow numbers of matchings in paths and cycles

In this section we suppose $n \geq 3$. Let P_n be the path with n edges with $V(P_n) = \{x_0, x_1, \cdots, x_n\}$ and $E(P_n) = \{e_i | e_i = x_{i-1}x_i, 1 \leq i \leq n\}$, and let C_n be the cycle with n edges.

Theorem 3.1. *For any $1 \leq m \leq \lceil \frac{n}{2} \rceil$,*

$$2m-2 \leq rb(P_n, mK_2) \leq 2m-1.$$

Proof. For the upper bound, let c be any coloring of P_n with $2m-1$ colors, and G be the spanning subgraph formed by taking one edge of each color from P_n. Then G is a bipartite graph, and so the size of its maximum matchings equals the size of its minimum vertex covers. Since one vertex can cover at most two edges in G, the size of a minimum vertex cover of G is at least m, and so there is a matching of size m in G and hence there is a rainbow mK_2 in P_n.

To obtain the lower bound we need to show that there is a coloring c of P_n with $2m-3$ colors without rainbow mK_2. Let $c(e_i)=i$ for $i=1,\cdots,2m-4$ and color all the other edges with $2m-3$. It is easy to see that there is no rainbow mK_2 in such a coloring. $\square$

Let G be a graph, $x',x''\in V(G)$ with $N(x')\cap N(x'')=\emptyset$. Identify x' and x'' into one vertex x and let the resultant graph be H, that is $V(H)=V(G)\cup\{x\}\backslash\{x',x''\}$ and $E(H)=\{uv|uv\in E(G)\text{ and }\{u,v\}\cap\{x',x''\}=\emptyset\}\cup\{xu|x'u\in E(G)\text{ or }x''u\in E(G)\}$. Let $rb(H,mK_2)=p$ and c be any coloring of G with p colors. For each edge in G, color the corresponding edge in H with the same color. Then there is a rainbow mK_2 in H. Since the corresponding edge set in G of an independent edge set in H is still independent, we have a rainbow mK_2 in G, and so $rb(G,mK_2)\le rb(H,mK_2)$.

Notice that C_n can be obtained from P_n by identifying the two ends of P_n. Thus from above observation we have

Theorem 3.2. *$rb(P_n,mK_2)\le rb(C_n,mK_2)$.*

In Theorem 3.1, if we replace P_n by C_n and $m\le\lceil\frac{n}{2}\rceil$ by $m\le\lfloor\frac{n}{2}\rfloor$, then from Theorem 3.2 we get the following theorem.

Theorem 3.3. *For any $1\le m\le\lfloor\frac{n}{2}\rfloor$,*

$$2m-2\le rb(C_n,mK_2)\le 2m-1.$$

Theorem 3.4. *For any $2\le m\le\lceil\frac{n}{2}\rceil$,*

$$rb(P_n,mK_2)=\begin{cases}2m-1, & n\le 3m-3;\\ 2m-2, & n>3m-3.\end{cases}$$

Proof. For $n\le 3m-3$, since $2m-2\le rb(P_n,mK_2)\le 2m-1$, we can construct a coloring of P_n with $2m-2$ colors that contains no rainbow mK_2. In fact, let $p=n-(2m-2)$, and for $1\le i\le p$ let $c(e_{3i-2})=c(e_{3i})=2i$ and $c(e_{3i-1})=2i-1$, and for $1\le j\le n-3p$ let $c(e_{3p+j})=2p+j$. It is easy to check that for such a coloring, in any rainbow matching of P_n only one color of $2i-1$ and $2i$ $(1\le i\le m-1)$ may appear, and so there is no rainbow mK_2 in P_n.

For $n>3m-3$, let c be any coloring of P_n with $2m-2$ colors. We will prove that there is a rainbow mK_2 in P_n. By contradiction, suppose there is no rainbow mK_2 in P_n. Let G be the spanning subgraph of P_n formed by taking one edge of each color in P_n, $E(G) = \{e_{i_1}, e_{i_2}, \cdots, e_{i_{2m-2}}\}$, $1 \le i_1 < i_2 < \cdots < i_{2m-2} \le n$ with $c(e_{i_j}) = j,\ 1 \le j \le 2m-2$. There is no mK_2 in G. Notice that G is bipartite, and so the size of maximum matchings equals the size of minimum vertex covers. Since one vertex of G can cover at most two edges, there is a vertex cover of size $m-1$ in G, and so $e_{i_{2l-1}}$ is adjacent to $e_{i_{2l}},\ 1 \le l \le m-1$.

Claim 1. Every edge e in $P_n \backslash E(G)$ is adjacent to an edge in $E(G)$. Otherwise suppose there is an edge $e \in E(P_n)\backslash E(G)$ independent of $E(G)$. Notice that $M_1 = \{e_{i_1}, e_{i_3}, \cdots, e_{i_{2m-3}}\}$ and $M_2 = \{e_{i_2}, e_{i_4}, \cdots, e_{i_{2m-2}}\}$ are two disjoint matchings of size $m-1$ in G. Let $c(e) = c(e_{i_l})$, and without loss of generality, let $e_{i_l} \in M_1$. Then $M_2 \cup \{e\}$ is a rainbow mK_2 in P_n, a contradiction.

Claim 2. There is no subgraph isomorphic to P_3 in $P_n \backslash E(G)$. Otherwise the middle edge of P_3 is independent of $E(G)$, which is contrary to Claim 1.

From Claims 1 and 2 we know that every nontrivial component of $P_n \backslash E(G)$ is a single edge P_1 or a P_2. We consider three cases and each leads to a contradiction.

Case 1. All the nontrivial components of $P_n \backslash E(G)$ are single edges. From Claim 1 and $n > 3m-3$, we can deduce that $n = 3m-2$ and $E(G) = \{e_2, e_3, e_5, e_6, e_8, e_9, \cdots, e_{3m-4}, e_{3m-3}\}$ with $c(e_{3i-1}) = 2i-1,\ c(e_{3i}) = 2i,\ 1 \le i \le m-1$. Now $M_1^1 = \{e_{3i} | 1 \le i \le m-1\}$ and $M_2^1 = \{e_3\} \cup \{e_{3i-1} | 2 \le i \le m-1\}$ have only e_3 in common and both are independent of e_1. To avoid the existence of a rainbow mK_2 in P_n, we have $c(e_1) = c(e_3) = 2$. Similarly, $M_1^2 = \{e_1, e_6\} \cup \{e_{3i-1} | 3 \le i \le m-1\}$ and $M_2^2 = \{e_2, e_6\} \cup \{e_{3i} | 3 \le i \le m-1\}$ have only e_6 in common and both are independent of e_4, and $c(e_4) = c(e_6) = 4$. By the same method, we know that $c(e_{3i-2}) = c(e_{3i}) = 2i, 1 \le i \le m-1$. Then, $M_1^m = \{e_{3i-2} | 1 \le i \le m-1\}$ and $M_2^m = \{e_{3i-1} | 1 \le i \le m-1\}$ are disjoint and both are independent of e_{3m-2}. Whatever color e_{3m-2} receives, we will get a rainbow mK_2 in P_n, a contradiction.

Now at least one component of $P_n \backslash E(G)$ is isomorphic to P_2.

Case 2. At least one of the end edges of P_n is in $P_n \backslash E(G)$. Without loss of generality, let $E(G) = \{e_2, e_3, e_6, e_7, e_9, e_{10}, \cdots, e_{3m-3}, e_{3m-2}\}$ with $c(e_2) = 1$, $c(e_3) = 2$, $c(e_{3i}) = 2i-1$, $c(e_{3i+1}) = 2i$, $2 \leq i \leq m-1$. Since $M_1' = \{e_{3i} | 1 \leq i \leq m-1\}$ and $M_2' = \{e_3\} \cup \{e_{3i+1} | 2 \leq i \leq m-1\}$ have only e_3 in common and both are independent of e_1, $c(e_1) = c(e_3) = 2$. Now $M_1'' = \{e_1\} \cup \{e_{3i} | 2 \leq i \leq m-1\}$ and $M_2'' = \{e_2\} \cup \{e_{3i+1} | 2 \leq i \leq m-1\}$ are disjoint and both are independent of e_4. Whatever color e_4 receives, we will get a rainbow mK_2 in P_n.

Case 3. Since none of the end edges of P_n is in $P_n \backslash E(G)$, there are at least two components in $P_n \backslash E(G)$ isomorphic to P_2. Without loss of generality, let $E(G) = \{e_1, e_2, e_5, e_6, e_9, e_{10}, e_{12}, e_{13}, \cdots, e_{3m-3}, e_{3m-2}\}$ with $c(e_2) = 1$, $c(e_2) = 2$, $c(e_5) = 3$, $c(e_6) = 4$, $c(e_{3i}) = 2i-1$, $c(e_{3i+1}) = 2i$, $3 \leq i \leq m-1$. Since $M_1' = \{e_1, e_3\} \cup \{e_{3i} | 3 \leq i \leq m-1\}$ and $M_2' = \{e_1, e_4\} \cup \{e_{3i+1} | 2 \leq i \leq m-1\}$ have only e_1 in common and both are independent of e_3, we have $c(e_1) = c(e_3) = 1$. $M_1'' = \{e_1, e_6\} \cup \{e_{3i} | 3 \leq i \leq m-1\}$ and $M_2'' = \{e_2, e_6\} \cup \{e_{3i+1} | 2 \leq i \leq m-1\}$ have only e_6 in common and both are independent of e_4, we have $c(e_4) = c(e_6) = 4$. Now $M_1 = \{e_1, e_4\} \cup \{e_{3i} | 3 \leq i \leq m-1\}$ and $M_2 = \{e_2, e_5\} \cup \{e_{3i+1} | 2 \leq i \leq m-1\}$ are disjoint and both are independent of e_7. Whatever color e_7 receives, we will get a rainbow mK_2 in P_n. □

From Theorem 3.2 and Theorem 3.4, we have $rb(C_n, mK_2) = 2m-1$, $n \leq 3m-3$. For $n > 3m-3$, by a similar proof in Theorem 3.4, we have $rb(C_n, mK_2) = 2m-2$. Thus we have

Theorem 3.5. *For any* $m \leq \lfloor \frac{n}{2} \rfloor$,

$$rb(C_n, mK_2) = \begin{cases} 2m-1, & n \leq 3m-3; \\ 2m-2, & n > 3m-3. \end{cases}$$

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