All Connected Graphs with Maximum Degree at Most 3 whose Energies are Equal to the Number of Vertices

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Abstract

The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues. Let S_2 be the star of order 2 (or K_2) and Q be the graph obtained from S_2 by attaching two pendent edges to each of the end vertices of S_2. Majstorović et al. conjectured that S_2, Q and the complete bipartite graphs $K_{2,2}$ and $K_{3,3}$ are the only 4 connected graphs with maximum degree $\Delta \leq 3$ whose energies are equal to the number of vertices. This paper is devoted to giving a confirmative proof to the conjecture.

1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here. Let G be a simple graph with n vertices and m edges. The *cyclomatic number* of a connected graph G is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a *k-cyclic graph*. In particular, for $c(G) = 0, 1$ or 2 we call G a tree, unicyclic or bicyclic graph, respectively. Denote by Δ the maximum degree of a graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix $A(G)$ of G are said to be the eigenvalues of the graph G. The *energy* of G is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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For several classes of graphs it has been demonstrated that the energy exceeds the number of vertices (see, [6]). In 2007, Nikiforov [12] showed that for almost all graphs,

$$E = \left(\frac{4}{3\pi} + o(1)\right) n^{3/2}.$$

Thus the number of graphs G satisfying the condition $E(G) < n$ is relatively small. In [8], a connected graph G of order n is called *hypoenergetic* if $E(G) < n$. For hypoenergetic graphs with $\Delta \leq 3$, we have the following well known results.

Lemma 1.1. *[7] There exist only four hypoenergetic trees with* $\Delta \leq 3$*, dipicted in Figure 1.*

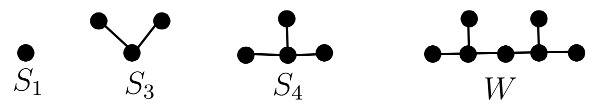

Figure 1: The hypoenergetic trees with maximum degree at most 3.

Lemma 1.2. *[13] Let* G *be a graph of order* n *with at least* n *edges and with no isolated vertices. If* G *is quadrangle-free and* $\Delta(G) \leq 3$*, then* $E(G) > n$*.*

The present authors first in [9] showed that complete bipartite graph $K_{2,3}$ is the only hypoenergetic graph among all unicyclic and bicyclic graphs with $\Delta \leq 3$, and then recently they obtained the following general result:

Lemma 1.3. *[10] Complete bipartite graph* $K_{2,3}$ *is the only hypoenergetic connected cycle-containing (or cyclic) graph with* $\Delta \leq 3$*.*

Therefore, all connected hypoenergetic graphs with maximum degree at most 3 have been characterized.

Lemma 1.4. *[10]* S_1, S_3, S_4, W *and* $K_{2,3}$ *are the only* 5 *hypoenergetic connected graphs with* $\Delta \leq 3$*.*

In [11] Majstorović et al. proposed the following conjecture, which is the second half of their Conjecture 3.7.

Conjecture 1.5. *[11] There are exactly four connected graphs* G *with order* n *and* $\Delta \leq 3$ *for which the equality* $E(G) = n$ *holds, which are dipicted in Figure 2.*

In this paper, we will prove this conjecture.

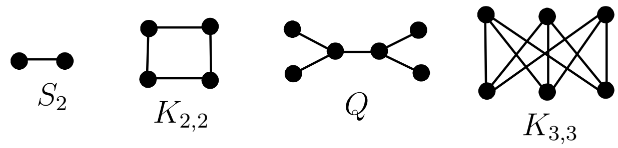

Figure 2: All connected graphs with maximum degree at most 3 and $E = n$.

2 Main results

The following results are needed in the sequel.

Lemma 2.1. *[5] If F is an edge cut of a graph G, then $E(G - F) \leq E(G)$, where $G - F$ is the subgraph obtained from G by deleting the edges in F.*

Lemma 2.2. *[5] Let $F = [S, V \setminus S]$ be an edge cut of a graph G with vertex set V, where S is a nonempty proper subset of V. Suppose that F is not empty and all edges in F are incident to one and only one vertex in S, i.e., the edges in F form a star. Then $E(G - F) < E(G)$.*

Lemma 2.3. *[1] The energy of a graph can not be an odd integer.*

In the following we first show that Conjecture 1.5 holds for trees, unicyclic and bicyclic graphs, respectively. Then we show that Conjecture 1.5 holds in general.

Let F be an edge cut of a connected graph F. If $G - F$ has exactly two components G_1 and G_2, then we denote $G - F = G_1 + G_2$ for convenience. The following lemma is needed.

Lemma 2.4. *Let F be an edge cut of a connected graph G of order n such that $G - F = G_1 + G_2$. If $E(G_1) \geq |V(G_1)|$, $E(G_2) \geq |V(G_2)|$ and either at least one of the above inequalities is strict or the edges in F form a star or both, then $E(G) > n$.*

Proof. If $E(G_1) > |V(G_1)|$ or $E(G_2) > |V(G_2)|$, then by Lemma 2.1, we have

$$E(G) \geq E(G - F) = E(G_1) + E(G_2) > |V(G_1)| + |V(G_2)| = n.$$

Otherwise by Lemma 2.2, we have

$$E(G) > E(G - F) = E(G_1) + E(G_2) \geq |V(G_1)| + |V(G_2)| = n,$$

which completes the proof. ∎

The result Lemma 2.4 is easy but useful in our proofs.

Theorem 2.5. *S_2 and Q are the only two trees T with order n and $\Delta \leq 3$ for which the equality $E(T) = n$ holds.*

Proof. Let T be a tree with n vertices and $\Delta \leq 3$. From Table 2 of [3], we know that S_2 and Q are the only two trees with $\Delta \leq 3$ and $n \leq 10$ for which the equality $E = n$ holds. By Lemma 2.3, we may assume that $n \geq 12$ is even. We will prove that $E(T) > n$.

We divide the trees with $\Delta \leq 3$ into two classes: **Class 1** contains the trees T that have an edge e, such that $T - e = T_1 + T_2$ and $T_1, T_2 \ncong S_1, S_3, S_4, W$. **Class 2** contains the trees T in which there exists no edge e, such that $T - e = T_1 + T_2$ and $T_1, T_2 \ncong S_1, S_3, S_4, W$, i.e., for any edge e of T at least one of components of $T - e$ is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$.

Case 1. T belongs to Class 1. Then there exists an edge e such that $T - e = T_1 + T_2$ and $T_1, T_2 \ncong S_1, S_3, S_4, W$. Hence by Lemmas 1.1 and 2.2, we have $E(T) > E(T-e) = E(T_1) + E(T_2) \geq |V(T_1)| + |V(T_2)| = n$, which completes the proof.

Case 2. T belongs to Class 2. Consider the center of T. There are two subcases: either T has a (unique) center edge e or a (unique) center vertex v.

Subcase 2.1. T has a center edge e. The two fragments attached to e will be denoted by T_1 and T_2, i.e., $T - e = T_1 + T_2$.

Without loss of generality, we assume that T_1 is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$.

If T_1 is isomorphic to a tree in $\{S_1, S_3, S_4\}$, then it is easy to see that $n \leq 11$, which is a contradiction.

If $T_1 \cong W$ and it is attached to the center edge e through the vertex of degree 2, then it is easy to see that T must be the tree as given in Figure 3 (a) or (b). By direct computing, we have that $E(T) = 12.61708 > 12 = n$ in the former case while $E(T) = 14.91128 > 14 = n$ in the latter case. If $T_1 \cong W$ and it is attached to the center edge e through a pendent vertex, see Figure 3 (c). Since T belongs to Class 2, deleting the edge f, we then have that $T_2 \cup e$ is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$, which contradicts to the fact that e is the center edge of T.

Subcase 2.2. T has a center vertex v. If v is of degree 2, then the two fragments attached to it will be denoted by T_1 and T_2. If v is of degree 3, then the three fragments attached to it will be denoted by T_1, T_2 and T_3.

Let v_i be the adjacent vertex of v in T_i. Denote $T - vv_1 = T_1 + T_2'$. Since T belongs to Class 2, either T_1 or T_2' is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$.

Subsubcase 2.2.1. T_2' is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$.

Clearly $T_2' \ncong S_1$. If $T_2' \cong S_3$ or S_4, then it is easy to see that $n \leq 7$, which is a contradiction. If $T_2' \cong W$ and v is of degree 3, then it is easy to see that $n \leq 10$, which is a contradiction. If $T_2' \cong W$ and v is of degree 2, i.e., $N(v) = \{v_1, v_2\}$. Consider $T - vv_2$, since T belongs to Class 2, we have that $T_1 \cup vv_1$ is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$. By the fact that v is the center of T, we have that $T_1 \cup vv_1 \cong W$, and so $n = 13$, which is a contradiction.

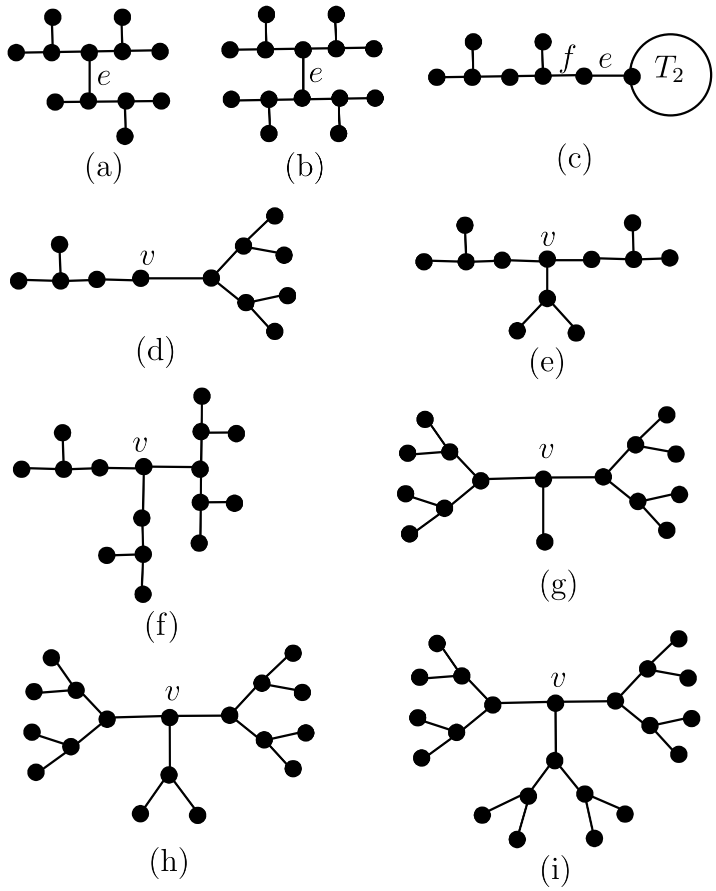

Figure 3: The graphs in the proof of Theorem 2.5.

Subsubcase 2.2.2. T_1 is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$.

If $T_1 \cong S_1$, then it is easy to see that $n \leq 4$, which is a contradiction.

If $T_1 \cong S_3$ and v_1 is of degree 2 in T_1, then it is easy to see that $n \leq 10$, which is a contradiction. If $T_1 \cong S_3$ and v_1 is a pendent vertex in T_1, denote by u the unique adjacent vertex of v_1 in T_1. Since T belongs to Class 2, deleting the edge uv_1, we then have that $T_2' \cup vv_1$ is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$, and so $n \leq 9$, which is a contradiction.

If $T_1 \cong S_4$ or $T_1 \cong W$ and v_1 is of degree 2 in T_1, then by the facts that T belongs to Class 2, v is the center of T and n is even, it is not hard to obtain that T_2, T_3 must be isomorphic to a tree in $\{S_1, S_3, S_4, W\}$, and at least one of T_2 and T_3 is isomorphic to a tree in $\{S_4, W\}$, and if T_2 (T_3, respectively) is isomorphic to W, then v_2 (v_3, respectively) is of degree 2 in T_2 (T_3, respectively). Hence there are 6 such trees, as given in Figure 3 (d), (e), (f), (g), (h) and (i). The energy of these trees are 12.72729 ($> 12 = n$), 12.65406 ($> 12 = n$), 16.81987 ($> 16 = n$), 16.77215 ($> 16 = n$), 19.18674 ($> 18 = n$) and 23.38426 ($> 22 = n$), respectively.

If $T_1 \cong W$ and v_1 is a pendent vertex in T_1, denote by u the unique adjacent vertex of v_1 in T_1. Since T belongs to Class 2, deleting the edge uv_1, we then have that $T'_2 \cup vv_1$ is isomorphic to a tree in $\{S_1, S_3, S_4, W\}$, which contradicts to the fact that v is the center vertex of T. The proof is thus complete. ∎

From Table 1 of [3], we know that $K_{2,2}$ is the only connected graph of order 4 with $\Delta \leq 3$ and $E = 4$. From Tables 1 and 2 of [4], we know that $K_{3,3}$ is the only connected cycle-containing graph of order 6 with $\Delta \leq 3$ and $E = 6$.

Theorem 2.6. *$K_{2,2}$ is the only unicyclic graph with $\Delta \leq 3$ for which the equality $E = n$ holds.*

Proof. Let $G \ncong K_{2,2}$ be a unicyclic graph of order n with $\Delta \leq 3$. It is sufficient to show that $E(G) > n$. By Lemmas 1.2 and 2.3, we can assume that $n \geq 8$ is even and G contains a quadrangle $C = x_1x_2x_3x_4x_1$. We distinguish the following four cases:
Case 1. There exists an edge e on C such that the end vertices of e are of degree 2.

Without loss of generality, we assume that $d(x_1) = d(x_4) = 2$. Let $F = \{x_1x_2, x_4x_3\}$, then $G - F = G_1 + G_2$, where $G_1 \cong S_2$ and G_2 is a tree of order at least 6 since $n \geq 8$. Since $\Delta(G) \leq 3$, G_2 can not be isomorphic to W or Q. Therefore we have $E(G_1) = |V(G_1)|$ and $E(G_2) > |V(G_2)|$ by Lemma 1.1 and Theorem 2.5. It follows from Lemma 2.4 that $E(G) > n$.
Case 2. There exist exactly two nonadjacent vertices x_i and x_j on C such that $d(x_i) = d(x_j) = 2$.

Without loss of generality, we assume that $d(x_2) = d(x_4) = 2$, $d(x_1) = d(x_3) = 3$. Let y_3 be the adjacent vertex of x_3 outside C. Then $G - x_3y_3 = G_1 + G_2$, where G_1 is a unicyclic graph and G_2 is a tree. Notice that $E(G_1) \geq |V(G_1)|$ by Lemma 1.3. If $G_2 \ncong S_1, S_3, S_4, W$, then we have $E(G_2) \geq |V(G_2)|$ by Lemma 1.1 and so $E(G) > E(G - x_3y_3) \geq n$ by Lemma 2.4. Therefore we only need to consider the following four subcases.
Subcase 2.1. $G_2 \cong S_1$. Let $F = \{x_2x_3, x_3x_4\}$, then $G - F = G'_1 + G'_2$, where $G'_2 \cong S_2$ and G'_1 is a tree of order at least 6 since $n \geq 8$. If $G'_1 \cong W$, then $n = 9$, which is a contradiction. Otherwise, it follows from Lemmas 1.1 and 2.4 that $E(G) > n$.
Subcase 2.2. $G_2 \cong S_3$. Then G must have the structure as given in Figure 4 (a) or (b). In the former case, $G - y_3z = G'_1 + G'_2$, where G'_1 is a unicyclic graph and $G'_2 \cong S_2$. It follows from Lemmas 1.4 and 2.4 that $E(G) > n$. In the latter case, $G - \{x_1x_2, x_4x_3\} = G'_1 + G'_2$, where G'_2 is the tree of order 5 containing x_3 and G'_1 is a tree of order at least 3. By Lemma 1.1 and Theorem 2.5, we have $E(G'_2) > |V(G'_2)|$. If $G'_1 \ncong S_3, S_4, W$, then we have $E(G) > n$ by Lemmas 1.1 and 2.4. Since $\Delta(G) \leq 3$, G'_1 can not be isomorphic to S_4 or W. If $G'_1 \cong S_3$, then G must be the graph as given in Figure 4 (c). By choosing the edge cut $\{x_1x_2, x_1x_4\}$, we can similarly obtain that $E(G) > n$.

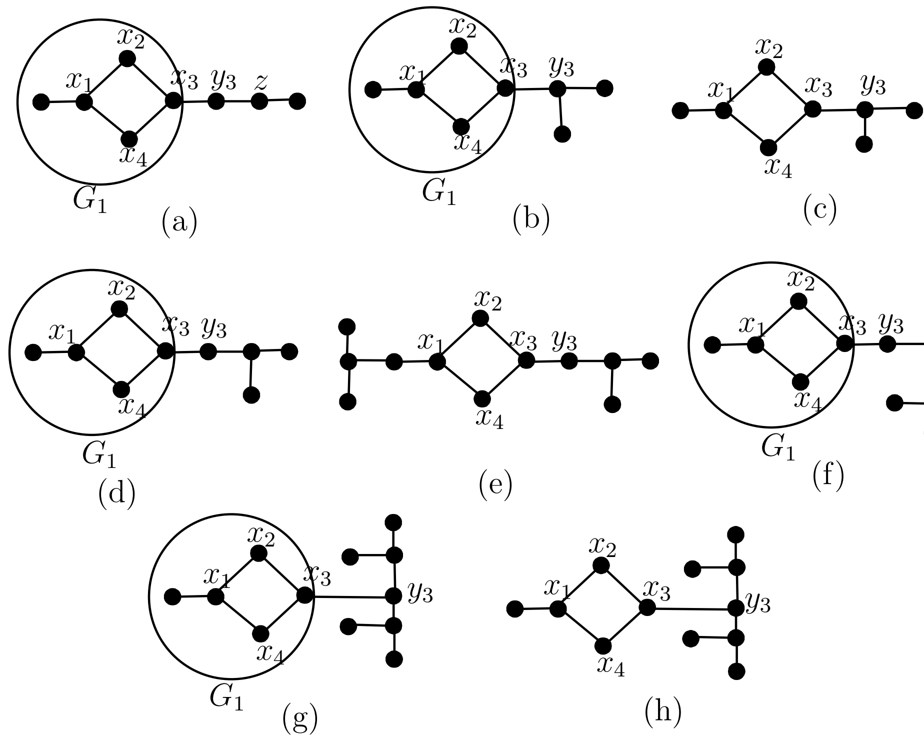

Figure 4: The graphs in the proof of Theorem 2.6.

Subcase 2.3. $G_2 \cong S_4$. Then G must have the structure as given in Figure 4 (d). Let $F = \{x_2x_3, x_3x_4\}$, then $G - F = G_1' + G_2'$, where G_2' is the tree of order 5 containing x_3 and G_1' is a tree of order at least 4. By Lemma 1.1 and Theorem 2.5, we have $E(G_2') > |V(G_2')|$. If $G_1' \ncong S_4, W$, then we have $E(G) > n$ by Lemmas 1.1 and 2.4. If $G_1' \cong S_4$, then $n = 9$, which is a contradiction. If $G_1' \cong W$, then G must be the graph as given in Figure 4 (e). By choosing the edge cut $\{x_1x_2, x_3x_4\}$, we can similarly obtain that $E(G) > n$.

Subcase 2.4. $G_2 \cong W$. Then G must have the structure as given in Figure 4 (f) or (g). In the former case, $G - y_3z = G_1' + G_2'$, where G_1' is a unicyclic graph and G_2' is a tree of order 6. It follows from Lemmas 1.4 and 2.4 that $E(G) > n$. In the latter case, $G - \{x_2x_3, x_3x_4\} = G_1' + G_2'$, where G_2' is the tree of order 8 containing x_3 and G_1' is a tree of order at least 4. If $G_1' \ncong S_4, W$, then we have $E(G) > n$ by Lemmas 1.1 and 2.4. If $G_1' \cong S_4$, then G must be the graph as given in Figure 4 (h). By choosing the edge cut $\{x_1x_2, x_1x_4\}$, we can similarly obtain that $E(G) > n$. If $G_1' \cong W$, then $n = 15$, which is a contradiction.

Case 3. There exists exactly one vertices x_i on C such that $d(x_i) = 2$.

Without loss of generality, we assume that $d(x_1) = 2$. Let $F = \{x_1x_4, x_2x_3\}$, then $G - F = G_1 + G_2$, where G_1 is the tree of order at least 3 containing x_1 and G_2 is a tree of order at least 4. Since $\Delta(G) \leq 3$, G_1, G_2 can not be isomorphic to S_4, W

or Q. If $G_1 \ncong S_3$, then we have $E(G) > n$ by Lemmas 1.1, 2.4 and Theorem 2.5. If $G_1 \cong S_3$, then $G - \{x_1x_2, x_2x_3\} = G_1' + G_2'$, where G_1' is the tree of order at least 5 containing x_1 and $G_2' \cong S_2$. If $G_1' \ncong W$, then we have $E(G) > n$ by Lemmas 1.1 and 2.4. If $G_1' \cong W$, then $n = 9$, which is a contradiction.

Case 4. $d(x_1) = d(x_2) = d(x_3) = d(x_4) = 3$.

Let $F = \{x_1x_4, x_2x_3\}$, then $G - F = G_1 + G_2$, where G_1 and G_2 are trees of order at least 4 and it is easy to see that G_1, G_2 can not be isomorphic to S_4, W or Q. So it follows from Lemmas 1.1, 2.4 and Theorem 2.5 that $E(G) > n$. The proof is thus complete. ∎

Theorem 2.7. *There does not exist any bicyclic graph with $\Delta \leq 3$ for which the equality $E = n$ holds.*

Proof. Let G be a bicyclic graph of order n with $\Delta \leq 3$. We know that $E(G) \neq n$ for $n = 4$ or 6. By Lemmas 1.2 and 2.3, we may assume that $n \geq 8$ is even and G contains a quadrangle. Then we will show that $E(G) > n$.

If the cycles in G are disjoint, then it is clear that there exists a path P connecting the two cycles in G. For any edge e on P, we have $G - e = G_1 + G_2$, where G_1 and G_2 are unicyclic graphs. By Lemma 1.3, we have $E(G_1) \geq |V(G_1)|$ and $E(G_2) \geq |V(G_2)|$. Therefore we have $E(G) > n$ by Lemma 2.4. Otherwise, the cycles in G have two or more common vertices. Then we can assume that G contains a subgraph as given in Figure 5 (a), where P_1, P_2, P_3 are paths in G. We distinguish the following three cases:

Case 1. At least one of P_1, P_2 and P_3, say P_2 has length not less than 3.

Let e_1 and e_2 be the edges on P_2 incident with u and v, respectively. Then $G - \{e_1, e_2\} = G_1 + G_2$, where G_1 is a unicyclic graph and G_2 is a tree of order at least 2. It follows from Lemma 1.3 that $E(G_1) \geq |V(G_1)|$. If $G_2 \ncong S_3, S_4, W, S_2, Q$, then we have $E(G_2) > |V(G_2)|$ by Lemma 1.1 and Theorem 2.5, and so $E(G) > n$ by Lemma 2.4. Hence we only need to consider the following five subcases.

Subcase 1.1. $G_2 \cong S_3$. Then G must have the structure as given in Figure 5 (b) or (c). In either case, $G - \{e_2, e_3\} = G_1' + G_2'$, where G_1' is a unicyclic graph and $G_2' \cong S_2$. Obviously, $G_1' \ncong K_{2,2}$. Then $E(G_1') > |V(G_1')|$ by Lemma 1.3 and Theorems 2.6. Since $E(G_2') = |V(G_2')|$, we have $E(G) > n$ by Lemma 2.4.

Subcase 1.2. $G_2 \cong S_4$. Then G must have the structure as given in Figure 5 (d). Obviously, $G - \{e_3, e_4\} = G_1' + G_2'$, where G_1' is a unicyclic graph which is not isomorphic to $K_{2,2}$ and $G_2' \cong S_2$. Similar to the proof of Subcase 1.1, we have $E(G) > n$.

Subcase 1.3. $G_2 \cong W$. Then G must have the structure as given in Figure 5 (e), (f) or (g). Obviously, $G - \{xy, yz\} = G_1' + G_2'$, where G_1' is a unicyclic graph which

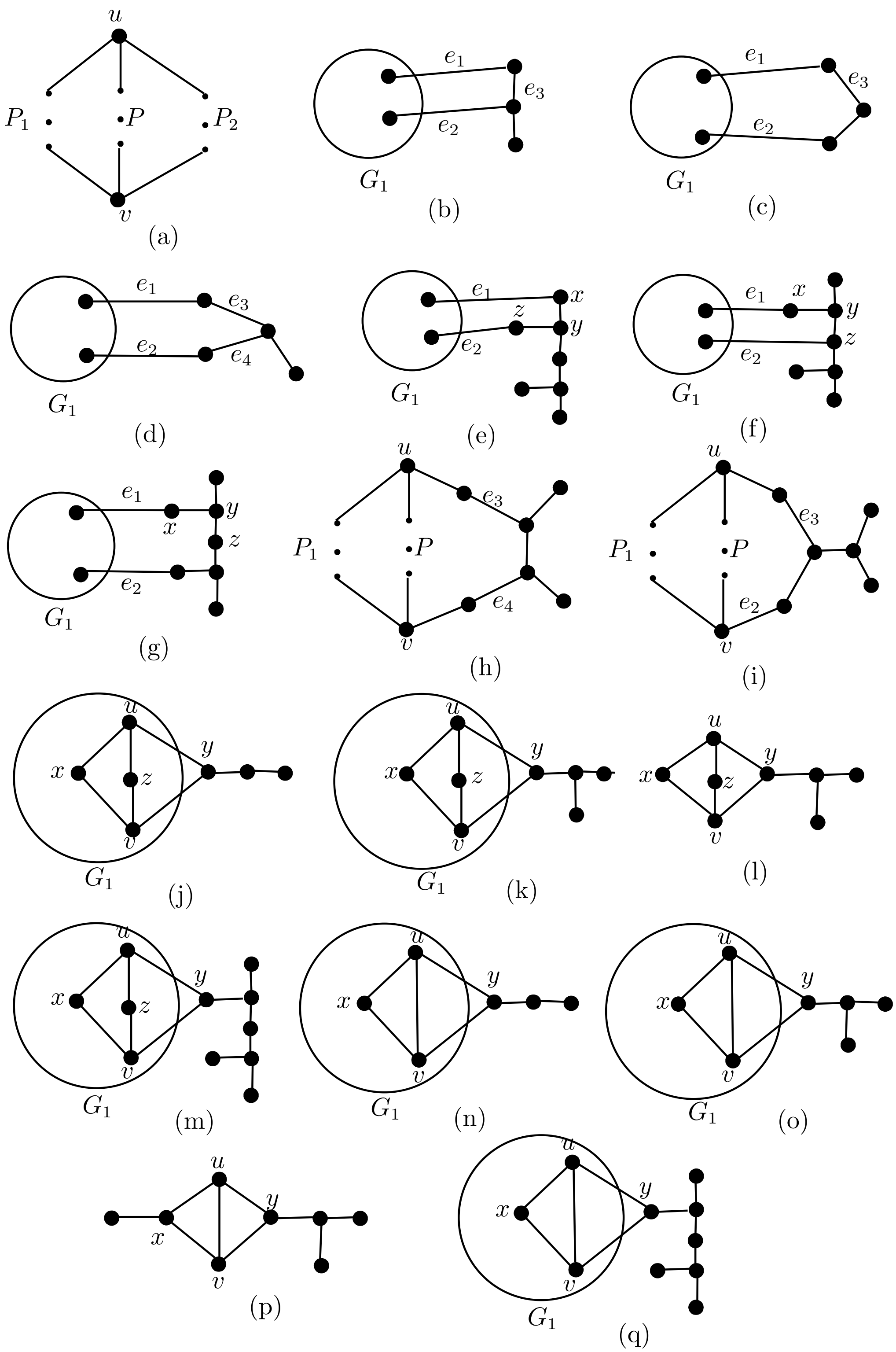

Figure 5: The graphs in the proof of Theorem 2.7.

is not isomorphic to $K_{2,2}$ and G_2' is a tree of order 5 or 2. Similarly, we can obtain that $E(G) > n$.

Subcase 1.4. $G_2 \cong S_2$. Since G_1 is a unicyclic graph, if $G_1 \ncong K_{2,2}$, then we can similarly obtain that $E(G) > n$. If $G_1 \cong K_{2,2}$, then $n = 6$, which is a contradiction.

Subcase 1.5. $G_2 \cong Q$. Then G must have the structure as given in Figure 5 (h) or (i). In the former case, $G - \{e_3, e_4\} = G_1' + G_2'$, where G_2' is a path of order 4 and G_1' is a unicyclic graph which is not isomorphic to $K_{2,2}$. Similarly, we can obtain that $E(G) > n$. In the latter case, $G - \{e_2, e_3\} = G_1' + G_2'$, where G_2' is a tree of order 5 and G_1' is a unicyclic graph which is not isomorphic to $K_{2,2}$. Similarly, we can obtain that $E(G) > n$.

Case 2. All the paths P_1, P_2 and P_3 have length 2.

We assume that $P_1 = uxv$, $P = uzv$ and $P_2 = uyv$. Let $F = \{uy, vy\}$, then $G-F = G_1+G_2$, where G_1 is a unicyclic graph and G_2 is a tree. It follows from Lemma 1.3 that $E(G_1) \geq |V(G_1)|$. If $G_2 \ncong S_1, S_3, S_4, W$, then we have $E(G_2) \geq |V(G_2)|$ by Lemma 1.1 and so $E(G) > n$ by Lemma 2.4. Hence we only need to consider the following four subcases.

Subcase 2.1. $G_2 \cong S_1$. Let $F' = \{uy, zv, xv\}$, then $G-F' = G_1'+G_2'$, where $G_2' \cong S_2$ and G_1' is a tree of order at least 6 since $n \geq 8$. It is easy to see that G_1' can not be isomorphic to Q or W. Therefore we have $E(G_1') > |V(G_1')|$ and $E(G_2') = |V(G_2')|$ by Lemma 1.1 and Theorem 2.5. It follows from Lemma 2.4 that $E(G) > n$.

Subcase 2.2. $G_2 \cong S_3$. Then G must have the structure as given in Figure 5 (j). Let $F' = \{uy, zv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the path of order 4 containing y and G_1' is a tree of order at least 4 since $n \geq 8$. Clearly, G_1' can not be isomorphic to S_4, Q or W. Similar to the proof of Subcase 2.1, we have $E(G) > n$.

Subcase 2.3. $G_2 \cong S_4$. Then G must have the structure as given in Figure 5 (k). Let $F' = \{uy, zv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the tree of order 5 containing y and G_1' is a tree of order at least 3. Clearly, G_1' can not be isomorphic to S_4 or W. If $G_1' \ncong S_3$, then we can similarly obtain that $E(G) > n$. If $G_1' \cong S_3$, then G must be the graph as given in Figure 5 (l). By choosing the edge cut $\{uy, uz, xv\}$, we can also obtain that $E(G) > n$.

Subcase 2.4. $G_2 \cong W$. Then G must have the structure as given in Figure 5 (m). Let $F' = \{uy, zv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the tree of order 8 containing y and G_1' is a tree of order at least 3. Clearly, G_1' can not be isomorphic to S_4 or W. If $G_1' \cong S_3$, then $n = 11$, which is a contradiction. If $G_1' \ncong S_3$, then we can similarly obtain that $E(G) > n$.

Case 3. One of the paths P_1, P_2 and P_3 has length 1, and the other two paths have length 2.

Without loss of generality, we assume that $P = uv$, $P_1 = uxv$ and $P_2 = uyv$. Let $F = \{uy, vy\}$, then $G - F = G_1 + G_2$, where G_1 is a unicyclic graph and G_2 is a tree. Similarly, if $G_2 \ncong S_1, S_3, S_4, W$, then we have $E(G) > n$. Hence we also need to consider the following four subcases.

Subcase 3.1. $G_2 \cong S_1$. Let $F' = \{uy, uv, xv\}$, then $G - F' = G_1' + G_2'$, where $G_2' \cong S_2$ and G_1' is a tree of order at least 6 since $n \geq 8$. Since $\Delta(G) \leq 3$, G_1' can not be isomorphic to Q or W. Similar to the proof of Subcase 2.1, we have $E(G) > n$.

Subcase 3.2. $G_2 \cong S_3$. Then G must have the structure as given in Figure 5 (n). Let $F' = \{uy, uv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the path of order 4 containing y and G_1' is a tree of order at least 4 since $n \geq 8$. Clearly, G_1' can not be isomorphic to S_4 or W. Similarly, we have $E(G) > n$.

Subcase 3.3. $G_2 \cong S_4$. Then G must have the structure as given in Figure 5 (o). Let $F' = \{uy, uv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the tree of order 5 containing y and G_1' is a tree of order at least 3. Clearly, G_1' can not be isomorphic to S_4 or W. If $G_1' \ncong S_3$, then we can similarly obtain that $E(G) > n$. If $G_1' \cong S_3$, then G must be the graph as given in Figure 5 (p). By choosing the edge cut $\{xu, xv\}$, we can similarly obtain that $E(G) > n$.

Subcase 3.4. $G_2 \cong W$. Then G must have the structure as given in Figure 5 (q). Let $F' = \{uy, uv, xv\}$, then $G - F' = G_1' + G_2'$, where G_2' is the tree of order 8 containing y and G_1' is a tree of order at least 2. Clearly, G_1' can not be isomorphic to S_4 or W. If $G_1' \cong S_3$, then $n = 11$, which is a contradiction. If $G_1' \ncong S_3$, then we can similarly obtain that $E(G) > n$. The proof is thus complete. ∎

Proof of Conjecture 1.5: Let G be a connected graph of order n with $\Delta \leq 3$. Clearly, if G is isomorphic to a graph in $\{S_2, Q, K_{2,2}, K_{3,3}\}$, then $E(G) = n$. We will prove that $E(G) \neq n$ if $G \ncong S_2$, Q, $K_{2,2}$ or $K_{3,3}$ by induction on the cyclomatic number $c(G)$. It follows from Theorems 2.5, 2.6 and 2.7 that the result holds for $c(G) \leq 2$. Let $k \geq 3$ be an integer. We assume that the result holds for $c(G) < k$. Now let G be a graph with $c(G) = k \geq 3$. We will show that $E(G) \neq n$.

By Lemma 2.3, the result holds if n is odd. By the fact that $K_{3,3}$ is the only connected cycle-containing graph of order 6 with $\Delta \leq 3$ and $E = 6$, we know that the result holds for $n \leq 6$. So in the following we assume that $n \geq 8$ is even. In our proof we will repeatedly make use of the following claim:

Claim 1. *Let F be an edge cut of G such that $G - F = G_1 + G_2$ with $c(G_1), c(G_2) < k$. If $G_1, G_2 \ncong S_1, S_3, S_4, W$ or $K_{2,3}$ and either the edges in F form a star or at least one of G_1 and G_2 is not isomorphic to S_2, Q or $K_{2,2}$, then we are done.*

Proof. By Lemma 1.4, we have $E(G_1) \geq |V(G_1)|$ and $E(G_2) \geq |V(G_2)|$. Clearly, $G_1, G_2 \ncong K_{3,3}$. If $G_i \ncong S_2, Q$ or $K_{2,2}$, then by induction hypothesis, we have $E(G_i) \neq |V(G_i)|$. Therefore we have $E(G) > n$ by Lemma 2.4. ∎

In what follows, we use $\hat{G}$ to denote the graph obtained from G by repeatedly deleting the pendent vertices. Clearly, $c(\hat{G}) = c(G)$. Denote by $\kappa'(\hat{G})$ the edge connectivity of $\hat{G}$. Since $\Delta(\hat{G}) \leq 3$, we have $1 \leq \kappa'(\hat{G}) \leq 3$. Therefore we only need to consider the following three cases.

Case 1. $\kappa'(\hat{G}) = 1$.

Let e be a cut edge of $\hat{G}$. Then $\hat{G} - e$ has exactly two components, say, H_1 and H_2. It is clear that $c(H_1) \geq 1$, $c(H_2) \geq 1$ and $c(H_1) + c(H_2) = k$. Consequently, $G - e$ has exactly two components G_1 and G_2 with $c(G_1) \geq 1$, $c(G_2) \geq 1$ and $c(G_1) + c(G_2) = k$, where H_i is a subgraph of G_i for $i = 1, 2$. If neither G_1 nor G_2 is isomorphic to $K_{2,3}$, then we are done by Claim 1. Otherwise, without loss of generality, we assume that $G_1 \cong K_{2,3}$. Then G must have the structure as given in Figure 6 (a). Now, let $F = \{e_1, e_2\}$. Then $G - F = G_1' + G_2'$, where $G_1' \cong K_{2,2}$ and $G_2' = G_2 \cup e$. Therefore we have that $c(G_2') = k - 2 \geq 1$ and $G_2' \ncong K_{2,2}, K_{2,3}$, and so we are done by Claim 1.

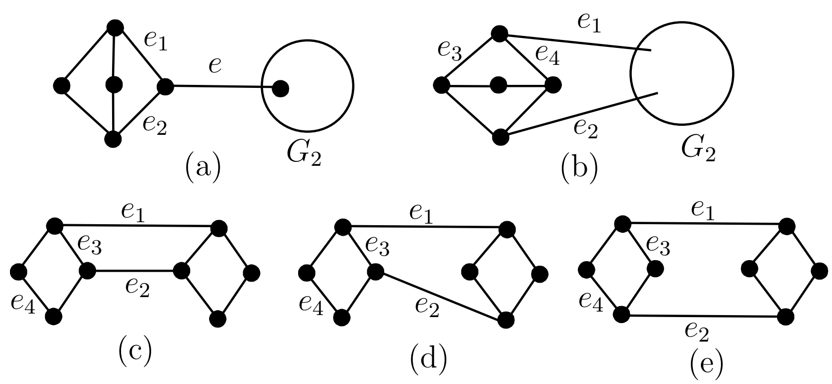

Figure 6: The graphs in the proof of Case 1 and Subcase 2.1 of Conjecture 1.5.

Case 2. $\kappa'(\hat{G}) = 2$.

Let $F = \{e_1, e_2\}$ be an edge cut of $\hat{G}$. Then $\hat{G} - F$ has exactly two components, say, H_1 and H_2. Clearly, $c(H_1) + c(H_2) = k - 1 \geq 2$.

Subcase 2.1. $c(H_1) \geq 1$ and $c(H_2) \geq 1$. Therefore, $G - F$ has exactly two components G_1 and G_2 with $c(G_1) \geq 1$, $c(G_2) \geq 1$ and $c(G_1) + c(G_2) = k - 1$, where H_i is a subgraph of G_i for $i = 1, 2$. If $G_1, G_2 \ncong K_{2,3}$ and at least one of G_1 and G_2 is not isomorphic to $K_{2,2}$, then we are done by Claim 1. If at least one of G_1 and G_2 is isomorphic to $K_{2,3}$, say $G_1 \cong K_{2,3}$. Then G must have the structure as given in Figure 6 (b). Now, let $F' = \{e_2, e_3, e_4\}$, then $G - F' = G_1' + G_2'$, where $G_1' \cong K_{2,2}$ and $G_2' = G_2 \cup e_1$. Therefore we have that $c(G_2') = k - 3$ and $G_2' \ncong K_{2,2}, K_{2,3}$, and so we are done by Claim 1. If $G_1, G_2 \cong K_{2,2}$, then G must be the graph as given in Figure 6 (c), (d) or (e). Let $F' = \{e_1, e_3, e_4\}$, then $G - F' = G_1' + G_2'$, where $G_1' \cong S_2$ and $G_2' \ncong K_{2,2}$ is a unicyclic graph. Hence we are done by Claim 1.

Subcase 2.2. One of H_1 and H_2, say H_2 is a tree. Therefore, $G - F$ has exactly two components G_1 and G_2 with $c(G_1) = k - 1$ and $c(G_2) = 0$, where H_i is a subgraph of

G_i for $i = 1, 2$. Since $k-1 \geq 2$, $G_1 \ncong S_2, Q, K_{2,2}$. If $G_1 \ncong K_{2,3}$ and $G_2 \ncong S_1, S_3, S_4, W$, then we are done by Claim 1. So we assume that this is not true. We only need to consider the following five subsubcases.

Subsubcase 2.2.1. $G_2 \cong S_1$. Let $V(G_2) = \{x\}$, $e_1 = xx_1$ and $e_2 = xx_2$. It is clear that $d_{G_1}(x_2) = 1$ or 2. If $d_{G_1}(x_2) = 1$, let $N_{G_1}(x_2) = \{y_1\}$ (see Figure 7 (a), where y_1 may be equal to x_1). Let $F' = \{e_1, x_2y_1\}$. Then $G - F' = G'_1 + G'_2$, where G'_1 is a graph obtained from G_1 by deleting a pendent vertex and $G'_2 \cong S_2$. Therefore, $c(G'_1) = k - 1 \geq 2$. If $G'_1 \ncong K_{2,3}$, then we are done by Claim 1. Otherwise, $n = 7$, which is a contradiction.

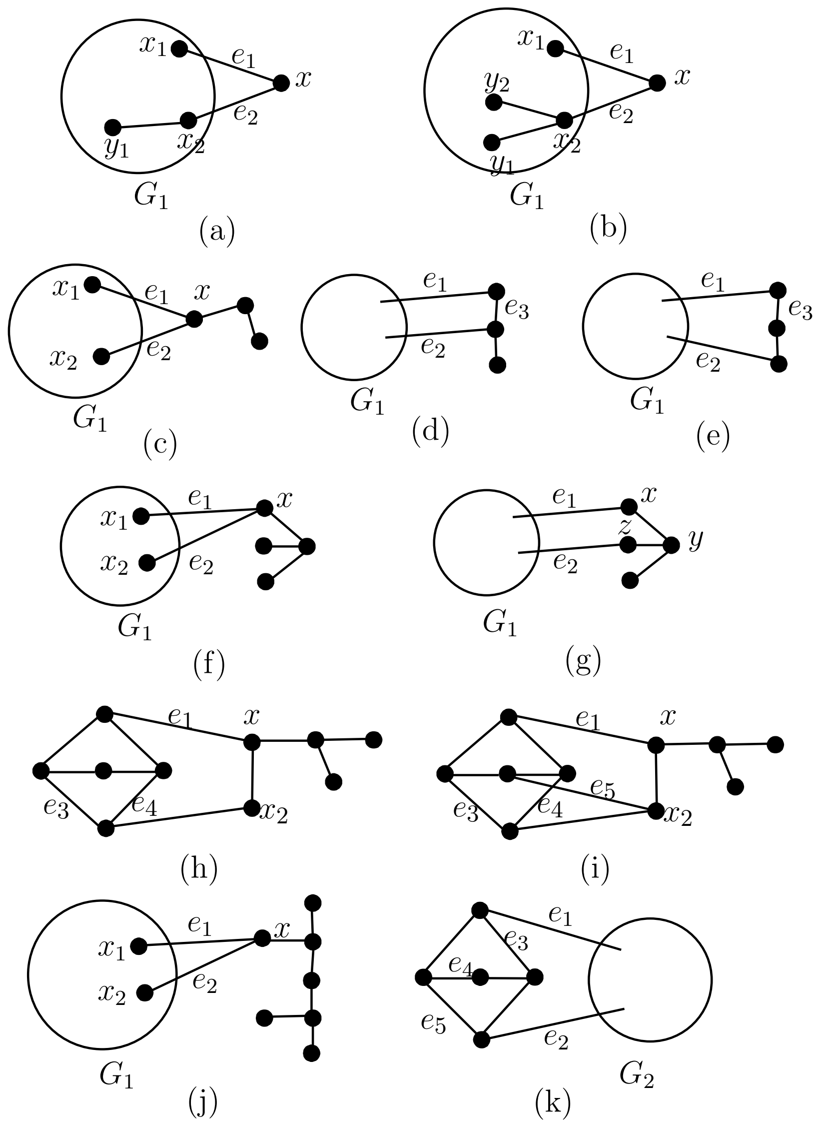

Figure 7: The graphs in the proof of Subcase 2.2 of Conjecture 1.5.

If $d_{G_1}(x_2) = 2$, let $N_{G_1}(x_2) = \{y_1, y_2\}$ (see Figure 7 (b), where one of y_1 and y_2 may be equal to x_1). Let $F' = \{e_1, x_2y_1, x_2y_2\}$. Then $G - F' = G'_1 + G'_2$, where G'_1 is a graph obtained from G_1 by deleting a vertex of degree 2 and $G_2' \cong S_2$. Therefore, $c(G_1') = k-2 \geq 1$. If $G_1' \ncong K_{2,2}, K_{2,3}$, then we are done by Claim 1. Otherwise, $n=6$ or 7, which is a contradiction.

Subsubcase 2.2.2. $G_2 \cong S_3$. If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Figure 7 (c). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F' = G_1' + G_2'$ satisfying that $c(G_1') = k-1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k-2$ if $d_{G_1}(x_2) = 2$ and G_2' is a path of order 4. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise $n = 9$, which is a contradiction.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Figure 7 (d) or (e). Let $F' = \{e_2, e_3\}$, then $G - F' = G_1' + G_2'$, where $G_1' = G_1 \cup e_1$ and $G_2' \cong S_2$. Therefore we have that $c(G_1') = k-1 \geq 2$ and $G_2' \ncong K_{2,3}$, and so we are done by Claim 1.

Subsubcase 2.2.3. $G_2 \cong S_4$. If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Figure 7 (f). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F' = G_1' + G_2'$ satisfying that $c(G_1') = k-1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k-2$ if $d_{G_1}(x_2) = 2$ and G_2' is a tree of order 5. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise G must be the graph as given in Figure 7 (h) or (i). In the former case let $F'' = \{e_1, e_3, e_4\}$ while in the latter case let $F'' = \{e_1, e_3, e_4, e_5\}$. Then $G - F'' = G_1'' + G_2''$, where $G_1'' \cong K_{2,2}$, G_2'' is a tree of order 6 and $G_2'' \ncong Q$. Therefore we are done by Claim 1.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Figure 7 (g). Let $F' = \{xy, yz\}$, then $G - F' = G_1' + G_2'$, where $G_1' = G_1 \cup \{e_1, e_2\}$ and $G_2' \cong S_2$. Therefore we have that $c(G_1') = k-1 \geq 2$ and $G_2' \ncong K_{2,3}$, and so we are done by Claim 1.

Subsubcase 2.2.4. $G_2 \cong W$. If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Figure 7 (j). Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F' = G_1' + G_2'$ satisfying that $c(G_1') = k-1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k-2$ if $d_{G_1}(x_2) = 2$ and G_2' is a tree of order 8. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise, $n = 13$, which is a contradiction.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Figure 5 (e), (f) or (g) (e_1, e_2 may be incident with a common vertex in G_1). Let $F' = \{xy, yz\}$, then $G - F' = G_1' + G_2'$, where G_2' is the tree of order 5 or 2 containing y. Clearly, $c(G_1') = k-1 \geq 2$ and $G_1' \ncong K_{2,3}$. Therefore we are done by Claim 1.

Subsubcase 2.2.5. $G_1 \cong K_{2,3}$ and $G_2 \ncong S_1, S_3, S_4, W$. It is easy to see that G must have the structure as given in Figure 7 (k). Let $F' = \{e_1, e_3, e_4, e_5\}$. Then $G - F' = G_1' + G_2'$, where $G_1' \cong S_2$ and G_2' is a tree of order at least 6 since $n \geq 8$. It is easy to see that G_2' can not be isomorphic to W or Q. Therefore we are done by Claim 1.

Case 3. $\kappa'(\hat{G}) = 3$.

Noticing that $\Delta(\hat{G}) \leq 3$ and $\Delta(G) \leq 3$, we obtain that $G = \hat{G}$ is a connected 3-regular graph. Hence we have $n + k - 1 = m = \frac{3}{2}n$, i.e., $n = 2k - 2$. Since $n \geq 8$, we have $k \geq 5$.

Let $F = \{e_1, e_2, e_3\}$ be an edge cut of G. Then $G - F$ has exactly two components, say, G_1 and G_2. Clearly, $c(G_1) + c(G_2) = k - 2 \geq 3$. Let $c(G_1) \geq c(G_2)$. If $c(G_2) \geq 3$, then we are done by Claim 1. Hence we only need to consider the following three subcases.

Subcase 3.1. $c(G_2) = 0$ and $c(G_1) = k - 2$. Let $|V(G_2)| = n_2$. Then we have $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2(n_2 - 1) + 3 = 2n_2 + 1$. Therefore, $n_2 = 1$, i.e., $G_2 = S_1$. Let $V(G_2) = \{x\}$, $e_1 = xx_1$, $e_2 = xx_2$ and $e_3 = xx_3$. Let $N_{G_1}(x_2) = \{y_1, y_2\}$ (see Figure 8 (a)). Let $F' = \{e_1, e_3, x_2y_1, x_2y_2\}$. Then $G - F' = G_1' + G_2'$, where $G_2' \cong S_2$ and G_1' is a graph obtained from G_1 by deleting a vertex of degree 2. Therefore, $c(G_1') = k - 3 \geq 2$. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. If $G_1' \cong K_{2,3}$, then $n = 7$, which is a contradiction.

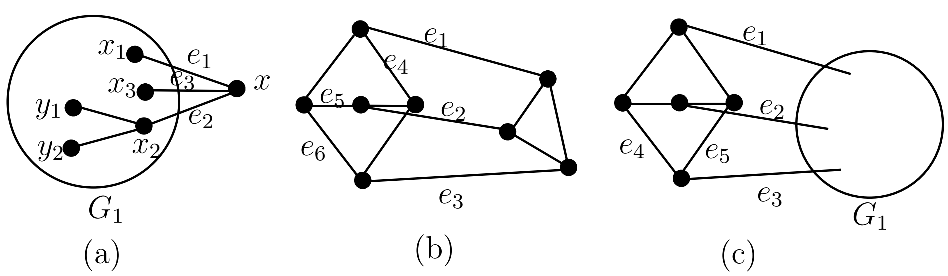

Figure 8: The graphs in the proof of Case 3 of Conjecture 1.5.

Subcase 3.2. $c(G_2) = 1$ and $c(G_1) = k - 3$. Let $|V(G_2)| = n_2$. Then we have $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2n_2 + 3$. Therefore, $n_2 = 3$, i.e., G_2 is a triangle. If $G_1 \ncong K_{2,3}$, then we are done by Claim 1. If $G_1 \cong K_{2,3}$, then G must be the graph as given in Figure 8 (b). Let $F' = \{e_1, e_4, e_5, e_6\}$. Then $G - F' = G_1' + G_2'$, where $G_1' \cong S_2$ and G_2' is a bicyclic graph which is not isomorphic to $K_{2,3}$. Then we are done by Claim 1.

Subcase 3.3. $c(G_2) = 2$ and $c(G_1) = k - 4$. Let $|V(G_2)| = n_2$. Then we have $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2(n_2 + 1) + 3 = 2n_2 + 5$. Therefore, $n_2 = 5$. If neither G_1 nor G_2 is isomorphic to $K_{2,3}$, then we are done by Claim 1. Otherwise, we assume that $G_2 \cong K_{2,3}$ (similar for $G_1 \cong K_{2,3}$). Then G must have the structure as given in Figure 8 (c). Let $F' = \{e_1, e_2, e_4, e_5\}$. Then $G - F' = G_1' + G_2'$, where $G_2' \cong K_{2,2}$ and G_1' is a $(k-4)$-cyclic graph which is not isomorphic to $K_{2,3}$. Then we are done by Claim 1. The proof is thus complete. ∎

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