Labeled Partitions with Colored Permutations

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Abstract. In this paper, we extend the notion of labeled partitions with ordinary permutations to colored permutations in the sense that the colors are endowed with a cyclic structure. We use labeled partitions with colored permutations to derive the generating function of the fmaj_k indices of colored permutations. The second result is a combinatorial treatment of a relation on the q-derangement numbers with respect to colored permutations which leads to the formula of Chow for signed permutations and the formula of Faliharimalala and Zeng [10] on colored permutations. The third result is an involution on permutations that implies the generating function formula for the signed q-counting of the major indices due to Gessel and Simon. This involution can be extended to signed permutations. In this way, we obtain a combinatorial interpretation of a formula of Adin, Gessel and Roichman.

Keywords: labeled partition, flag major index, colored permutation, q-derangement number

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1 Introduction

In this paper, we will be concerned with the wreath product $S_n^k = C_k \wr S_n$ of the symmetric group on $[n] = \{1, 2, \ldots, n\}$ and the cyclic group C_k on $\{0, 1, \ldots, k-1\}$ whose elements are considered being arranged on a cycle, see Adin and Roichman [2] and Wagner [16]. The elements in S_n^k are also called colored permutations [5]. The derangements with respect to the group S_n^k are studied by Faliharimalala and Zeng [9, 10].

We will extend the notion of labeled partitions with ordinary permutations to colored permutations. A k-colored permutation is written as $\pi(1)_{c_1}\, \pi(2)_{c_2} \,\cdots\, \pi(n)_{c_n}$, where $\pi(1)\,\pi(2)\,\cdots\,\pi(n)$ is a permutation on $[n]$ and $c_i \in \{0, 1, \ldots, k-1\}$. For example, $4_2\, 3_0\, 1_2\, 5_0\, 2_1$ is a colored permutation in S_5^3. We define a total order on the elements of S_n^k as follows

$$1_{k-1} < 2_{k-1} < \cdots < n_{k-1} < 1_{k-2} < 2_{k-2} < \cdots < n_{k-2} < \cdots < 1_0 < 2_0 < \cdots < n_0. \tag{1.1}$$

We now recall the following definitions:

$$\begin{aligned}
D(\sigma) &:= \{i \in [n-1] : \sigma(i) > \sigma(i+1)\},\\
\mathrm{maj}(\sigma) &:= \sum_{i \in D(\sigma)} i,\\
N_j(\sigma) &:= \#\{i \in [n] : \sigma(i) \text{ has subscript } j\}, \quad j = 1, \ldots, k-1,\\
\mathrm{fmaj_k}(\sigma) &:= k\mathrm{maj}(\sigma) + N_1(\sigma) + 2N_2(\sigma) + \cdots + (k-1)N_{k-1}(\sigma).
\end{aligned} \tag{1.2}$$

The set $D(\sigma)$ is called the descent set of $\sigma \in S_n^k$. It should be noted that Adin and Roichman [2] give the definition of flag major index of an element in S_n^k by the unique factorization into Coxeter elements, and they prove that fmaj_k has the above expression (1.2). In this paper, we will consider the formula (1.2) as the definition of the fmaj_k index. From this point of view, our approach may be regarded as purely combinatorial.

For $k = 1$, S_n^1 is usually written as S_n. For $k = 2$, S_n^2 becomes the group of signed permutations on $[n]$, often denoted by B_n, and the minus sign is often denoted by a bar. Moreover, the fmaj_k index reduces to the fmaj index for signed permutations as defined by

$$\mathrm{fmaj}(\pi) = 2\mathrm{maj}(\pi) + N(\pi),$$

where $N(\pi)$ denotes the number of negative elements of π and $\mathrm{maj}(\pi)$ is defined with respect to the following order

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n.$$

Using labeled partitions with colored permutations, we get the generating function of the $\mathrm{fmaj_k}$ indices on S_n^k,

$$\sum_{\pi \in S_n^k} q^{\mathrm{fmaj}_k(\pi)} = [k]_q [2k]_q \cdots [nk]_q, \tag{1.3}$$

where $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$. The above formula is a natural extension of the formulas for the generating functions for the major index and the fmaj index, see Faliharimalala and Zeng [10]. Recall that for the cases of ordinary permutations and signed permutations we have

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]! \tag{1.4}$$

and

$$\sum_{\pi \in B_n} q^{\mathrm{fmaj}(\pi)} = [2]_q [4]_q \cdots [2n]_q. \tag{1.5}$$

The second result is a combinatorial treatment of a relation on the q-derangement numbers $D_n^k(q)$ with respect to S_n^k. This relation implies the formula for $d_n^k(q)$ by the q-binomial inversion, as given by Faliharimalala and Zeng [10]. For $n \geq 1$, let

$$\mathscr{D}_n := \{\sigma \in S_n : \sigma(i) \neq i \text{ for all } i \in [n]\}$$

be the set of derangements on S_n. Gessel defined the q-derangement numbers by

$$d_n(q) := \sum_{\sigma \in \mathscr{D}_n} q^{\mathrm{maj}(\sigma)}$$

and proved that

$$d_n(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}, \tag{1.6}$$

where $[n]_q! = [1]_q[2]_q\cdots[n]_q$. Wachs [14] found a combinatorial proof of the above formula. Later, Chow [8] generalized Wachs's argument to type B derangements. Similarly, Chow defined

$$\mathscr{D}_n^B := \{\sigma \in B_n : \sigma(i) \neq i \text{ for all } i \in [n]\}$$

as the set of derangements in B_n and

$$d_n^B(q) := \sum_{\sigma\in\mathscr{D}_n^B} q^{\mathrm{fmaj}(\sigma)}.$$

Chow has shown that

$$d_n^B(q) = [2]_q[4]_q\cdots[2n]_q \sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q[4]_q\cdots[2k]_q}. \tag{1.7}$$

The notion of derangements of type B can be generalized to S_n^k, as given by Faliharimalala and Zeng [10]. We define

$$\mathscr{D}_n^k := \{\sigma \in S_n^k : \sigma(i) \neq i_0 \text{ for all } i \in [n]\}$$

and

$$d_n^k(q) := \sum_{\sigma\in\mathscr{D}_n^k} q^{\mathrm{fmaj}_k(\sigma)}.$$

Faliharimalala and Zeng have shown that

$$d_n^k(q) = [k]_q[2k]_q\cdots[nk]_q \sum_{j=0}^{n} \frac{(-1)^j q^{k\binom{j}{2}}}{[k]_q[2k]_q\cdots[jk]_q}. \tag{1.8}$$

The argument of Chow for $d_n^B(q)$ can be extended to the case of $d_n^k(q)$. Our proof is based on the structure of labeled partitions with colored permutations, which is an extension of the combinatorial approach of Chen and Xu [7] for ordinary permutations. We will present the proof for the case $k = 3$, which is essentially a proof for the general case.

The third result is concerned the following formula of Gessel and Simon [15] on the signed q-counting of permutations with respect to the major index:

$$\sum_{\pi\in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = [1]_q[2]_{-q}[3]_q[4]_{-q}\cdots[n]_{(-1)^{n-1}q}.$$

Note that a combinatorial proof of the above formula has been given by Wachs [15] based on permutations. We will present an involution on labeled partitions that leads to a combinatorial interpretation of the above formula. Moreover, our involution can be extended to signed permutations. This gives a combinatorial proof of the following formula of Adin-Gessel-Roichman [3] for the signed q-counting of signed permutations with respect to the fmaj index:

$$\sum_{\pi\in B_n} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)} = [2]_{-q}[4]_q\cdots[2n]_{(-1)^n q}.$$

2 Labeled Partitions and the fmaj_k Index

In this section, we introduce the notion of labeled partitions with colored permutations. Using labeled partitions, we give a combinatorial proof of the following formula for the generating function of the fmaj_k indices of colored permutations in S_n^k, given by Haglund, Loehr and Remmel [12], see also, Faliharimalala and Zeng [10].

Theorem 2.1. *We have*

$$\sum_{\pi \in S_n^k} q^{\mathrm{fmaj}_k(\pi)} = [k]_q[2k]_q \cdots [nk]_q.$$

Recall that given a colored permutation $\pi \in S_n^k$, $N_j(\pi)$ denotes the number of elements $\pi(i) \in \pi$ with subscript j, where $j = 1, 2, \ldots, k-1$. The fmaj_k index which is originally defined algebraically by Adin and Roichman has the following equivalent form

$$\mathrm{fmaj}_k(\pi) = k\mathrm{maj}(\pi) + N_1(\pi) + 2N_2(\pi) + \cdots + (k-1)N_{k-1}(\pi).$$

Clearly, Theorem 2.1 is a generalization of the formulas (1.4) and (1.5) for permutations and signed permutations. We now proceed to give a combinatorial proof of Theorem 2.1 by using labeled partitions with colored permutations.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an integer partition with at most n parts where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We adopt the notation in Andrews [4]. We write $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. A *labeled partition* associated with S_n^3 is defined as a pair (λ, π), where λ is a partition with at most n parts and $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is a colored permutation in S_n^3. We can also employ the two-row notation to represent a labeled partition

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

A labeled partition (λ, π) is said to be *standard* if $\pi(i) > \pi(i+1)$ implies $\lambda_i > \lambda_{i+1}$. It is easy to see that a labeled partition (λ, π) is standard if $\lambda_i = \lambda_{i+1}$ implies $\pi(i) < \pi(i+1)$.

Given a colored element w_i, we use $c(w_i)$ to denote the color or subscript i, and use $d(w_i)$ to denote the element w after removing the color i.

Let P_n^3 denote the set of partitions with at most n parts such that each part is divisible by 3. Given $\pi \in S_n^3$, we denote by Q_π the set of standard labeled partitions such that $\lambda_i - \mathrm{c}(\pi(i))$ is divisible by 3.

Lemma 2.2. *Given $\pi \in S_n^3$, there is a bijection $g_\pi \colon \lambda \to (\mu, \pi)$ from P_n^3 to Q_π such that $|\lambda| + \mathrm{fmaj}_3(\pi) = |\mu|$.*

Proof. We define μ as follows:

$$\mu = (\lambda_1 + 3a_1 + \mathrm{c}(\pi(1)), \lambda_2 + 3a_2 + \mathrm{c}(\pi(2)), \ldots, \lambda_n + 3a_n + \mathrm{c}(\pi(n))),$$

where a_i is the number of descents in $\pi(i)\pi(i+1)\cdots\pi(n)$. From the above definition, it is clear that μ is a partition and $\mu_i - \mathrm{c}(\pi(i))$ is divisible by 3. We only need to show that (μ, π) is standard. We have the following cases.

Case 1: $\lambda_i > \lambda_{i+1}$. In this case, we have $\lambda_i+3a_i+\mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1}+3a_{i+1}+\mathrm{c}(\pi(i+1))$, since $\lambda_i - \lambda_{i+1} \geq 3, a_i \geq a_{i+1}$ and $|\mathrm{c}(\pi(i)) - \mathrm{c}(\pi(i+1))| < 3$.

Case 2: $\lambda_i = \lambda_{i+1}$. We further consider the following two subcases:

(i) If $\pi(i) > \pi(i+1)$, then it is easy to verify that

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

(ii) If $\pi(i) < \pi(i+1)$ and $\pi(i), \pi(i+1)$ have the same subscript, then we have

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i = \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

Otherwise, if $\pi(i)$ and $\pi(i+1)$ have different subscripts, then we see that the subscript of $\pi(i)$ is greater than that of $\pi(i+1)$. This implies that

$$\lambda_i + 3a_i + \mathrm{c}(\pi(i)) = \mu_i > \mu_{i+1} = \lambda_{i+1} + 3a_{i+1} + \mathrm{c}(\pi(i+1)).$$

Now we see that the labeled partition (μ, π) is standard. Conversely, given a labeled partition $(\mu, \pi) \in Q_\pi$, we can uniquely recover the partition $\lambda \in P_n^3$. ∎

Consequently, we obtain the following formula.

Theorem 2.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in S_n^3} q^{\mathrm{fmaj}_3(\pi)} = [3]_q [6]_q \cdots [3n]_q.$$

Proof. We consider the following equivalent form of (1.3):

$$\frac{1}{(q^3;q^3)_n} \sum_{\pi \in S_n^3} q^{\mathrm{fmaj}_3(\pi)} = \frac{1}{(1-q)^n},$$

where

$$(q^3;q^3)_n = (1-q^3)(1-q^6)\cdots(1-q^{3n}).$$

Let W_n be the set of sequences of n nonnegative integers. It is clear that $\frac{1}{(q^3;q^3)_n}$ and $\frac{1}{(1-q)^n}$ are the generating functions for numbers of partitions in P_n^3 and W_n, respectively. Therefore, it suffices to construct a bijection $\phi\colon (\lambda, \pi) \to s$ from (P_n^3, S_n^3) to W_n such that $|\lambda|+\mathrm{fmaj}_3(\pi) = |s|$, where $|s|$ denotes the sum of entries of s. The bijection ϕ can be described as follows:

Step 1. Use the bijection in Lemma 2.2 to derive a standard labeled partition (μ, π) from (λ, π).

Step 2. Based on the two row representation of the labeled partition (μ, π), we permute the columns to make the second row become the identity permutation by ignoring the subscripts of the elements in π. Let s denote the first row of the array.

It is not difficult to see that the above procedure is reversible. The inverse of ϕ consists of four steps.

Step 1. For a sequence $s = (s(1), s(2), \ldots, s(n)) \in W_n$, we construct a two row array

$$\begin{pmatrix} s(1) & s(2) & \cdots & s(n) \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Step 2. For each element $i \in [n]$, we may construct a colored permutation $1_{c_1} 2_{c_2} \cdots n_{c_n}$, where $c_i = s(i) \pmod 3$. Clearly, we have $s^*(i) = s(i) - c_i$ is divisible by 3. So we are led to the following array

$$\begin{pmatrix} s^*(1) & s^*(2) & \cdots & s^*(n) \\ 1_{c_1} & 2_{c_2} & \cdots & n_{c_n} \end{pmatrix}.$$

Step 3. Permute the columns of the above array to make the first row $s^*(j_1)s^*(j_2)\cdots s^*(j_n)$ in decreasing order. Moreover, we order the elements in the second row in increasing order if they correspond to the same elements in the first row. We denote the resulted labeled partition by

$$\begin{pmatrix} s^*(j_1) & s^*(j_2) & \cdots & s^*(j_n) \\ \delta(1)_{e_1} & \delta(2)_{e_2} & \cdots & \delta(n)_{e_n} \end{pmatrix}.$$

Step 4. Recover the initial labeled partition (λ, π) from the array produced in Step 3 by the following rule:

$$(\lambda^*, \pi) = \begin{pmatrix} s^*(j_1) - 3a_1 & s^*(j_2) - 3a_2 & \cdots & s^*(j_n) - 3a_n \\ \delta(1)_{e_1} & \delta(2)_{e_2} & \cdots & \delta(n)_{e_n} \end{pmatrix},$$

where a_k is the number of descents in $\delta(k)_{e_k} \cdots \delta(n)_{e_n}$.

It is easy to see that the above procedure is feasible. Moreover, one can verify that $\phi \cdot \phi^{-1} =$ id and $\phi^{-1} \cdot \phi =$ id, where id denotes the identity map. This completes the proof. ∎

Let us give an example. Let $n = 7$, $\lambda = (18, 18, 18, 9, 9, 6, 3)$ and $\pi = 3_2\, 4_2\, 6_0\, 5_1\, 7_2\, 2_1\, 1_2$. Then we obtain $s = (5, 10, 29, 29, 16, 27, 14)$ via the following steps:

$$\begin{pmatrix} 18 & 18 & 18 & 9 & 9 & 6 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix} \xrightarrow{\text{Step 1}} \begin{pmatrix} 29 & 29 & 27 & 16 & 14 & 10 & 5 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix}$$

$$\xrightarrow{\text{Step 2}} (5, 10, 29, 29, 16, 27, 14).$$

The reverse process from s to (λ, π) are demonstrated as follows:

$$(5, 10, 29, 29, 16, 27, 14)$$

$$\xrightarrow{\text{Step 1}} \begin{pmatrix} 5 & 10 & 29 & 29 & 16 & 27 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow{\text{Step 2}} \begin{pmatrix} 3 & 9 & 27 & 27 & 15 & 27 & 12 \\ 1_2 & 2_1 & 3_2 & 4_2 & 5_1 & 6_0 & 7_2 \end{pmatrix}$$

$$\xrightarrow{\text{Step 3}} \begin{pmatrix} 27 & 27 & 27 & 15 & 12 & 9 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix} \xrightarrow{\text{Step 4}} \begin{pmatrix} 18 & 18 & 18 & 9 & 9 & 6 & 3 \\ 3_2 & 4_2 & 6_0 & 5_1 & 7_2 & 2_1 & 1_2 \end{pmatrix}.$$

3 Labeled Partitions and q-Derangements Numbers

In this section, we give a combinatorial treatment of a relation on the q-derangement numbers for S_n^k. This relation leads to the formula of Faliharimalala and Zeng for $d_n^k(q)$. We will give the proof for the case $k = 3$. It is easy to see that the argument applies to the general case.

Following Wachs [14] and Chow [8], we define the reduction of a colored permutation σ on a set of positive integers $A = \{a_1 < a_2 < \cdots < a_k\}$ by substituting the element a_i with i while keeping the color. Keep in mind that a positin i is called a fixed point of a colored permutation $\pi(1)\pi(2)\cdots\pi(n)$ if $\pi(i)=i_0$. Then the *derangement part* of a colored permutation $\sigma\in S_n^3$, denoted by $dp(\sigma)$, is the reduction of the sequence obtained from σ by removing the fixed elements. For example, $dp(8_0\,1_2\,5_1\,4_0\,3_1\,6_0\,7_1\,2_2)=6_0\,1_2\,4_1\,3_1\,5_1\,2_2$.

Then we have the following extension of the relation due to Wachs [14]:

Theorem 3.1. *Given $\alpha\in\mathscr{D}_k^3$, for $0\le k\le n$ we have*

$$\sum_{dp(\sigma)=\alpha,\sigma\in S_n^3} q^{\mathrm{fmaj}_3(\sigma)} = q^{\mathrm{fmaj}_3(\alpha)}\begin{bmatrix} n\\ k\end{bmatrix}_{q^3}. \tag{3.9}$$

It should be noted that the above theorem can be proved by the method of Wachs [14] which has been extended by Chow [8] to signed permutations. We will give a combinatorial proof based on labeled partitions with colored permutations.

For any $\pi=\pi(1)\pi(2)\cdots\pi(k)\in S_k^3$, we can insert a fixed point j with $1\le j\le k+1$ into π to obtain a permutation $\bar\pi$ in S_{k+1}^3 given by

$$\bar\pi=\pi'(1)\pi'(2)\cdots\pi'(j-1)j_0\pi'(j)\cdots\pi'(k),$$

where

$$\pi'(i)=\begin{cases}(c(\pi(i)))d(\pi(i)), & \text{if } d(\pi(i))<j,\\ (c(\pi(i)))(d(\pi(i))+1), & \text{otherwise.}\end{cases}$$

In other words, $\bar\pi$ is the unique permutation with i being a fixed point such that the reduction of the sequence obtained from $\bar\pi$ by deleting the element at position i equals π. For example, let $\pi=4_2\,1_0\,2_0\,6_1\,5_1\,3_2$. Then we get $5_2\,1_0\,3_0\,2_0\,7_1\,6_1\,4_2$ when we insert 3 into π.

Proof of Theorem 3.1. First, we reformulate the relation (3.9) in the equivalent form

$$\frac{1}{(q^3;q^3)_n}\sum_{dp(\sigma)=\alpha,\sigma\in S_n^3} q^{\mathrm{fmaj}_3(\sigma)} = \frac{1}{(q^3;q^3)_k(q^3;q^3)_{n-k}}q^{\mathrm{fmaj}_3(\alpha)}, \tag{3.10}$$

and use labeled partitions to give combinatorial proof of the above relation. Let R_α be the set of colored permutations $\sigma\in S_n^3$ such that $dp(\sigma)=\alpha$. We proceed to establish a bijection $\theta\colon(\lambda,\sigma)\to(\beta,\gamma)$ from (P_n^3,R_α) to (P_k^3,P_{n-k}^3) such that

$$|\lambda|+\mathrm{fmaj}_3(\sigma)=|\beta|+|\gamma|+\mathrm{fmaj}_3(\alpha). \tag{3.11}$$

The bijection consists of the following steps.

Step 1. Apply the bijection g_σ given in Lemma 2.2 to get a standard labeled partition (λ^*,σ) from λ.

Step 2. Let the fixed points and non-fixed points of σ be $\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_{n-k})$ and $\sigma(j_1),\sigma(j_2),\ldots,\sigma(j_k)$. We decompose λ^* into two parts, namely, $\lambda^*(i_1),\lambda^*(i_2),\ldots,\lambda^*(i_{n-k})$ and $\lambda^*(j_1),\lambda^*(j_2),\ldots,\lambda^*(j_k)$.

Let $\gamma=(\lambda^*(i_1),\lambda^*(i_2),\ldots,\lambda^*(i_{n-k}))$ and $\beta^*=(\lambda^*(j_1),\lambda^*(j_2),\ldots,\lambda^*(j_k))$.

Step 3. Apply g_α^{-1} to (β^*, α) and denote the resulted partition by β.

To show that the above procedure is feasible, we need to show that β^* generated in Step 2 satisfies the condition that (β^*, α) belongs to Q_α so that one can apply g_α^{-1}.

Observe that for any $1 \leq q \leq k$, $\sigma(j_q)$ and $\alpha(q)$ have the same subscript since $\alpha(q)$ is obtained by the reduction operation. It follows that

$$\beta^*(q) - \mathrm{c}(\alpha(q)) = \lambda^*(j_q) - \mathrm{c}(\alpha(q))$$

is divisible by 3 for any $1 \leq q \leq k$. To prove that (β^*, α) is standard, it suffices to show if $\sigma(p) > \sigma(q)$ with $\sigma(p+1), \ldots, \sigma(q-1)$ being at the positions of fixed points, then $\lambda_p^* > \lambda_q^*$. When $q = p+1$, we conclude that $\lambda_p^* > \lambda_q^*$ from the fact that (λ^*, σ) is standard. When $q > p+1$, it is easy to see that we have either $\sigma(p) > \sigma(p+1)$ or $\sigma(q-1) > \sigma(q)$. Therefore, we have either $\lambda_p^* > \lambda_{p+1}^*$ or $\lambda_{q-1}^* > \lambda_q^*$. Since λ^* is a partition, we find that $\lambda_p^* > \lambda_q^*$. Hence the bijection is well defined.

It remains to show that the above procedure is reversible. We proceed to construct the inverse map η from (P_k^3, P_{n-k}^3) to (P_n^3, R_α), which consists of three steps.

Step 1. Apply g_α to β and denote the resulted partition by $(\tilde{\beta}, \alpha)$.

Step 2. Let $(\tilde{\lambda}^0, \sigma^0) = (\tilde{\beta}, \alpha)$. We insert γ_i into $(\tilde{\lambda}^{i-1}, \sigma^{i-1})$ to get $(\tilde{\lambda}^i, \sigma^i)$. Find the first position r in $\tilde{\lambda}^{i-1}$ such that the insertion of γ_i to this position will produce a partition. We denote this partition by $\tilde{\lambda}^i$. Obviously, we have $\tilde{\lambda}_{r-1}^i > \tilde{\lambda}_r^i = \gamma_i$. Suppose that $\tilde{\lambda}_r^i = \cdots = \tilde{\lambda}_t^i > \tilde{\lambda}_{t+1}^i$ for some $t \geq r$. If $r = t$ then we set $s = r$. Otherwise, from left to right, we look for a position s satisfying $\sigma^{i-1}(s-1) < s_0 \leq \sigma^{i-1}(s)$ (here we treat $\sigma^{i-1}(r-1)$ as $-\infty$ and $\sigma^{i-1}(t+1)$ as ∞). In this way, we obtain σ^i from σ^{i-1} by inserting s_0 as a fixed point. In fact, this procedure guarantees that the subsequence $\sigma^i(r), \sigma^i(r+1), \ldots, \sigma^i(t)$ is increasing. That is, $(\tilde{\lambda}^i, \sigma^i)$ is a standard labeled partition. On the other hand, since $\gamma \in P_{n-k}^3$ and each fixed point has subscript 0, we have γ_i is divisible by 3 for each $1 \leq i \leq n-k$ and thus $(\tilde{\lambda}^i, \sigma^i) \in Q_{\sigma^i}$.

Step 3. Apply $g_{\sigma^{n-k}}^{-1}$ to $(\tilde{\lambda}^{n-k}, \sigma^{n-k})$ and denote the resulted partition by λ^{n-k}.

We claim that λ^{n-k} and σ^{n-k} equal λ and σ respectively. Then we see that η is the inverse of θ. From Lemma 2.2, it is easily seen that $\beta^* = \tilde{\beta}$. Since $\tilde{\lambda}^{n-k}$ is the partition obtained from $\tilde{\beta}$ by inserting $\gamma_1, \ldots, \gamma_{n-k}$, we have $\lambda^* = \tilde{\lambda}^{n-k}$.

It is now necessary to show that $\sigma^{n-k} = \sigma$. It suffices to verify σ^{n-k} and σ have the same fixed points. By removing the common fixed points, we may assume that the first fixed point f (f_0) of σ is different from that of f' (f_0') of σ^{n-k}. We have

$$\sigma(f-1) < f_0 \leq \sigma(f+1) - 1.$$

Since f' is the first position we aim to find, we have $f' < f$. On the other hand, it is clear that $\lambda^*(f) = \lambda^*(f')$. Since (λ^*, σ) and $(\lambda^*, \sigma^{n-k})$ are both standard labeled partitions, we find

$$\sigma(f') < \sigma(f'+1) < \cdots < \sigma(f),$$

and

$$\sigma^{n-k}(f') < \sigma^{n-k}(f'+1) < \cdots < \sigma^{n-k}(f).$$

Based on the fact that $\lambda^*(f) = \lambda^*(f')$, $\sigma^{n-k}(f)$ and $\sigma^{n-k}(f')$ have the same subscript, we conclude that $\sigma^{n-k}(f)$ has the subscript 0 as $\sigma^{n-k}(f')$.

Now we see that $\sigma(f) = f$ and $\sigma^{n-k}(f') = f'$. Since

$$\sigma(f') < \sigma(f'+1) < \cdots < \sigma(f)$$

and $\sigma(f) = f$, we can deduce that $\sigma(f') \leq f'$. Since f is the first fixed point of σ, we obtain $\alpha(f') = \sigma(f') < f'$. From the construction of σ^{n-k}, if follows that $\sigma^{n-k}(f') \leq \alpha(f') < f'$ which contradicts the assumption that $\sigma^{n-k}(f') = f'$ (f' is a fixed point of σ^{n-k}).

Therefore, we have $\sigma = \sigma^{n-k}$. Again by Lemma 2.2, we conclude that $\lambda = \lambda^{n-k}$. Hence η is the inverse map of θ. This completes the proof. ∎

For example, let $n = 8, \lambda = (18, 12, 12, 12, 9, 9, 6, 3)$ and $\sigma = 5_2\, 1_0\, 2_0\, 4_0\, 8_1\, 6_0\, 7_1\, 3_2$. Then we have

$$g_\sigma(\lambda) = \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 7_1 & 6_1 & 3_2 \end{pmatrix}.$$

The fixed points of σ are 4_0 and 6_0, and $\alpha = dp(\sigma) = 4_2\, 1_0\, 2_0\, 6_1\, 5_1\, 3_2$. Decomposing $(29, 21, 21, 21, 16, 15, 10, 5)$, we get $((29, 21, 21, 16, 10, 5), (21, 15))$. Applying g_α^{-1} to $\beta^* = (29, 21, 21, 16, 10, 5)$ gives $\beta = (18, 12, 12, 9, 6, 3)$ and $\gamma = (21, 15)$.

Conversely, given $\alpha = 4_2\, 1_0\, 2_0\, 6_1\, 5_1\, 3_2$ and $(\beta, \gamma) = ((18, 12, 12, 9, 6, 3), (21, 15))$, we have $\tilde{\beta} = (29, 21, 21, 16, 10, 5)$. The insertion process is illustrated as follows:

$$\begin{pmatrix} 29 & 21 & 21 & 16 & 10 & 5 \\ 4_2 & 1_0 & 2_0 & 6_1 & 5_1 & 3_2 \end{pmatrix} \xrightarrow{\gamma_1=21} \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 7_1 & 6_1 & 3_2 \end{pmatrix}$$

$$\xrightarrow{\gamma_2=15} \begin{pmatrix} 29 & 21 & 21 & 21 & 16 & 15 & 10 & 5 \\ 5_2 & 1_0 & 2_0 & 4_0 & 8_1 & 6_0 & 7_1 & 3_2 \end{pmatrix}.$$

So we get $\tilde{\lambda}^{n-k} = (29, 21, 21, 21, 16, 15, 10, 5)$, $\sigma^{n-k} = 5_2\, 1_0\, 2_0\, 4_0\, 8_1\, 6_0\, 7_1\, 3_2$. Finally, we find $\lambda^{n-k} = g_{\sigma^{n-k}}^{-1} = (18, 12,\ 12, 12, 9, 9, 6, 3)$.

4 Involutions on Labeled Partitions

In this section, we give a combinatorial interpretation of the formula of Gessel and Simon in terms of an involution on labeled partitions. This involution can be easily extended to type B. Hence we also give a combinatorial proof of a formula of Adin, Gessel and Roichman on the signed q-counting of fmaj indices of signed permutations.

Recall that the sign of a signed permutations is defined in terms of generators of B_n as a Coxeter group. Consider the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$ of B_n, where

$$s_0 := [-1, 2, 3, \ldots, n], \quad \text{and} \quad s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n]$$

for $1 \leq i \leq n-1$. Then the sign of a signed permutation π is defined by

$$\operatorname{sign}(\pi) := (-1)^{l(\pi)},$$

where $l(\pi)$ is the standard length of π with respect to the generators of B_n.

The following theorem is due to Gessel and Simon [15].

Theorem 4.1.
$$\sum_{\pi\in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)} = [1]_q[2]_{-q}[3]_q[4]_{-q}\cdots[n]_{(-1)^{n-1}q}. \tag{4.12}$$

A combinatorial proof of the above formula has been given by Wachs [15]. Here we will give an involution on labeled partitions and we will show that this involution can be easily extended to the following type B formula due to Adin, Gessel and Roichman [3].

Theorem 4.2.
$$\sum_{\pi\in B_n} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)} = [2]_{-q}[4]_q\cdots[2n]_{(-1)^n q}. \tag{4.13}$$

To describe our involution on labeled partitions as a proof of the formula (4.12), we may reformulate in an equivalent form:
$$\frac{\sum_{\pi\in S_n} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)}}{(q;q)_n} = \frac{1}{(1-q)(1+q)(1-q)(1+q)\cdots(1-(-1)^{n-1}q)}. \tag{4.14}$$

Proof of Theorem 4.1. We consider two cases according to the parity of n.

Case 1. n is even, i.e., $n=2k$. Then (4.14) takes the form
$$\frac{\sum_{\pi\in S_{2k}} \mathrm{sign}(\pi) q^{\mathrm{maj}(\pi)}}{(q;q)_{2k}} = \frac{1}{(1-q^2)^k}. \tag{4.15}$$
Clearly, the right hand side of (4.15) is the generating function of sequences $(a_1, a_2, \ldots, a_{2k-1}, a_{2k})$ satisfying $a_{2i-1} = a_{2i}$ for $i = 1, 2, \ldots, k$. It is also easy to see that the left hand side of (4.15) is the generating function of labeled partitions on S_n with at most $2k$ parts under the assumption that a labeled partition (λ, π) carries the sign of the permutation π. To be more specific, such labeled partitions are called signed labeled partitions. We proceed to construct an involution on the set H of signed labeled partitions (λ, π) such that the generating function of the fixed points of this involution equals the right hand side of (4.15). This involution consists of three steps.

Step 1. Let (λ, π) be a labeled partition such that $\pi \in S_{2k}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2k} \geq 0$. If $|\pi^{-1}(1) - \pi^{-1}(2)| \neq 1$, then we define
$$\phi^1(\pi)(i) = \begin{cases} \pi(i), & i \neq \pi^{-1}(1) \text{ and } \pi^{-1}(2), \\ 2, & i = \pi^{-1}(1), \\ 1, & i = \pi^{-1}(2). \end{cases}$$
Obviously, (λ, π) and $(\lambda, \phi^1(\pi))$ have opposite signs and $\mathrm{maj}(\pi) = \mathrm{maj}(\phi^1(\pi))$. Therefore, we have $\mathrm{maj}(\pi) + |\lambda| = \mathrm{maj}(\phi^1(\pi)) + |\lambda|$, and so these two elements cancel each other. If $|\pi^{-1}(1) - \pi^{-1}(2)| = 1$, then we see that $\mathrm{maj}(\pi) \neq \mathrm{maj}(\phi^1(\pi))$.

We now use H^1 to denote the set of signed labeled partitions (λ, π) such that $|\pi^{-1}(1) - \pi^{-1}(2)| = 1$. Repeating the above procedure, we continue to cancel out some elements in H^1. At this time, we consider the positions of the elements 3 and 4. Similarly, if $|\pi^{-1}(3) - \pi^{-1}(4)| \neq 1$, then we define
$$\phi^2(\pi)(i) = \begin{cases} \pi(i), & i \neq \pi^{-1}(3) \text{ and } \pi^{-1}(4), \\ 4, & i = \pi^{-1}(3), \\ 3, & i = \pi^{-1}(4). \end{cases}$$

It follows that (λ,π) and $(\lambda,\phi^2(\pi))$ have the opposite signs and that

$$\mathrm{maj}(\pi)+|\lambda| = \mathrm{maj}(\phi^2(\pi))+|\lambda|.$$

In other words, the two elements cancel out in the set H^1.

Now, we use H^2 to denote the subset of H^1 such that $|\pi^{-1}(3)-\pi^{-1}(4)|=1$. Iterating this process, we may consider the elements $\{5,6\},\{7,8\},\ldots,\{2k-1,2k\}$ and denote the set obtained at the last step by H^k. Finally, we obtain $H^k\subseteq H^{k-1}\subseteq\cdots\subseteq H^1$. In the intermediate steps, we can defined the functions ϕ^i for $i=1,2,\ldots,k$. It is not difficult to see that the labeled partition (λ,π) in H^k has the property that

$$|\pi^{-1}(1)-\pi^{-1}(2)|=1, |\pi^{-1}(3)-\pi^{-1}(4)|=1,\ldots,|\pi^{-1}(2k-1)-\pi^{-1}(2k)|=1.$$

Namely, any odd number $2i-1$ is next to $2i$ in π for all $i=1,\ldots,k$.

Step 2. For any labeled partition

$$(\lambda,\pi)=\begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(2)} & \lambda_{\pi^{-1}(1)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 2 & 1 & \cdots & \pi(2k) \end{pmatrix},$$

we define $(f^1(\lambda),g^1(\pi))$ to be the labeled partition

$$(f^1(\lambda),g^1(\pi))=\begin{pmatrix} \lambda_1+1 & \cdots & \lambda_{\pi^{-1}(2)}+1 & \lambda_{\pi^{-1}(1)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 1 & 2 & \cdots & \pi(2k) \end{pmatrix},$$

where $f^1(\lambda)$ is the partition obtained from λ by adding 1 to the first $\pi^{-1}(2)$ parts of λ and $g^1(\pi)$ is the permutation obtained from π by exchanging the positions of 1 and 2.

Clearly, (λ,π) and $(f^1(\lambda),g^1(\pi))$ have opposite signs. Also, we have

$$\mathrm{maj}(\pi)+|\lambda| = \mathrm{maj}(g^1(\pi))+|f^1(\lambda)|.$$

Therefore (λ,π) and $(f^1(\lambda),g^1(\pi))$ cancel out in H^k. Notice that the resulted labeled partition $(f^1(\lambda),g^1(\pi))$ has the additional property that $f^1(\lambda)_{\pi^{-1}(1)}$ is greater than $f^1(\lambda)_{\pi^{-1}(2)}$. By inspection, we see that after cancellation, the remaining elements in H^k are of the following form

$$(\lambda,\pi)=\begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(1)} & \lambda_{\pi^{-1}(2)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 1 & 2 & \cdots & \pi(2k) \end{pmatrix}$$

where $\lambda_{\pi^{-1}(1)}=\lambda_{\pi^{-1}(2)}$. Let H_1^k denote the set of remaining elements in H^k that of the above form.

We continue the above process for the H_1^k with respect the relative positions of 3 and 4. It is easy to check that for any labeled partition (λ,π) in H_1^k, 1 appears before 2 in π and $\lambda_{\pi^{-1}(1)}=\lambda_{\pi^{-1}(2)}$. Now, for any element $(\lambda,\pi)\in H_1^k$, if

$$(\lambda,\pi)=\begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(4)} & \lambda_{\pi^{-1}(3)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 4 & 3 & \cdots & \pi(2k) \end{pmatrix},$$

then we can find another labeled partition $(f^2(\lambda), g^2(\pi)) \in H_1^k$

$$(f^2(\lambda), g^2(\pi)) = \begin{pmatrix} \lambda_1+1 & \cdots & \lambda_{\pi^{-1}(4)}+1 & \lambda_{\pi^{-1}(3)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 3 & 4 & \cdots & \pi(2k) \end{pmatrix}.$$

Again, (λ, π) and $(f^2(\lambda), g^2(\pi))$ cancel each other in H_1^k. Notice that $f^2(\lambda)_{\pi^{-1}(3)}$ is greater than $f^2(\lambda)_{\pi^{-1}(4)}$. So the remaining labeled partitions after the above cancelation are of the form

$$(\lambda, \pi) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_{\pi^{-1}(3)} & \lambda_{\pi^{-1}(4)} & \cdots & \lambda_{2k} \\ \pi(1) & \cdots & 3 & 4 & \cdots & \pi(2k) \end{pmatrix},$$

where $\lambda_{\pi^{-1}(3)} = \lambda_{\pi^{-1}(4)}$. Then we can denote the set of the remaining labeled partitions by H_2^k and continue the above process. Eventually, we get $H_k^k \subseteq H_{k-1}^k \subseteq \cdots \subseteq H_1^k$. Moreover, in the process we have defined the functions f^i and g^i for $i = 1, 2, \ldots, k$.

It is easy to see that for any labeled partition (λ, π) in H_k^k and for any $i \in \{1, \ldots, k\}$, $2i-1$ appears immediately before $2i$ and $\lambda_{\pi^{-1}(2i-1)} = \lambda_{\pi^{-1}(2i)}$. Clearly, all the labeled partitions in H_k^k have positive signs.

Step 3. Permute the columns of the labeled partitions (λ, π) in H_k^k so that the elements in π are rearranged in increasing order. Taking the first row of the resulted two row array, we will get a sequence $(a_1, a_2, \ldots, a_{2k-1}, a_{2k})$ such that $a_{2i-1} = a_{2i}$ $(i = 1, \ldots, k)$ whose generating function is the right hand side of (4.15).

It is easy to see that the relation (4.15) can be justified by the above algorithm. Hence Theorem 4.1 holds when n is even.

Case 2. n is odd, i.e., $n = 2k+1$. We need to show that

$$\frac{\sum_{\pi \in S_{2k+1}} \operatorname{sign}(\pi) q^{\operatorname{maj}(\pi)}}{(q;q)_{2k+1}} = \frac{1}{(1-q^2)^k (1-q)}. \tag{4.16}$$

This case is analogous to the case when n is even. We may employ the same operations in Step 1 and Step 2 by ignoring the element $2k+1$ while making the pairs $\{1, 2, \}, \{3, 4\}, \ldots, \{2k-1, 2k\}$. The only difference lies in Step 3 when we take the first row of the resulted two row array, we get a sequence $(a_1, a_2, \ldots, a_{2k-1}, a_{2k}, a_{2k+1})$ such that $a_{2i-1} = a_{2i}$ $(i = 1, \ldots, k)$. Moreover, a_{2k+1} can be any positive integer. This completes the proof of the relation (4.16).

In fact, we have constructed a sign reversing involution

$$(\theta, \chi)\colon (\lambda, \pi) \to (\theta(\lambda), \chi(\pi)).$$

Specifically, the map (θ, χ) is defined by

$$(\theta(\lambda), \chi(\pi)) = \begin{cases} (\lambda, \phi^1(\pi)), & \text{if}(\lambda, \pi) \in H \setminus H^1, \\ (\lambda, \phi^2(\pi)), & \text{if}(\lambda, \pi) \in H^1 \setminus H^2, \\ \cdots & \\ (\lambda, \phi^k(\pi)), & \text{if}(\lambda, \pi) \in H^{k-1} \setminus H^k, \\ (f^1(\lambda), g^1(\pi)), & \text{if}(\lambda, \pi) \in H^k \setminus H^k_1, \\ (f^2(\lambda), g^2(\pi)), & \text{if}(\lambda, \pi) \in H^k_1 \setminus H^k_2, \\ \cdots & \\ (f^k(\lambda), g^k(\pi)), & \text{if}(\lambda, \pi) \in H^k_{k-1} \setminus H^k_k, \\ (\lambda, \pi), & \text{if}(\lambda, \pi) \in H^k_k, \end{cases}$$

where $\phi^i(\pi)$, $f^i(\lambda)$ and $g^i(\pi)$ are defined in the above algorithm. It is easy to verify that the map induces sign reversing, that is, if (λ, π) is not a fixed point of the map (θ, χ), then we have $\text{sign}(\theta(\lambda), \chi(\pi)) = -\text{sign}(\lambda, \pi)$ and $|\theta(\lambda)| + \text{maj}(\chi(\pi)) = |\lambda| + \text{maj}(\pi)$. The fixed points of the map (θ, χ) correspond to the right hand side of (4.12). This completes the proof. ∎

We now turn to the Theorem 4.2, and we need a characterization of the length function of signed permutations [6, Propostion 3.1 and Corollary 3.2].

Lemma 4.3. *Let $\sigma \in B_n$, we have*

$$l(\sigma) = \text{inv}\,(\sigma) + \sum_{\{1 \le i \le n | \sigma(i) < 0\}} |\sigma(i)|,$$

where inv(σ) is defined with respect to the order

$$\bar{n} < \cdots < \bar{1} < 1 < \cdots < n.$$

Note that in the definition of the fmaj on B_n we have imposed the order

$$\bar{1} < \cdots < \bar{n} < 1 < \cdots < n$$

or in the notation of colored permutations,

$$1_1 < \cdots < n_1 < 1_0 < \cdots < n_0.$$

The above lemma is useful for the construction of a sign reversing involution for the formula (4.13) for B_n. Given a signed permutation $\sigma \in B_n$, we may construct a signed permutation σ' as follows. If 1 and 2 have different signs or 1 and 2 have the same sign but are not adjacent in σ, then we exchange 1 and 2 without changing the signs. By Lemma 4.3, we see that the σ' and σ have opposite signs and $\text{fmaj}(\sigma) = \text{fmaj}(\sigma')$.

For example, let $\sigma = 4_0\, 2_1\, 5_1\, 1_0\, 3_1$. Then we have $\sigma' = 4_0\, 1_1\, 5_1\, 2_0\, 3_1$. Clearly, σ and σ' have opposite signs.

Using the above sign change rule, we can extend the involution for Theorem 4.1 to Theorem 4.2. The detailed proof is omitted. ∎

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