Set Systems with Cross $\mathcal{L}$-Intersection and k-Wise $\mathcal{L}$-Intersecting Families

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Abstract

We prove some results involving cross $\mathcal{L}$-intersections of two families of subsets of $[n] = \{1, 2, \ldots, n\}$. As a consequence, we derive the following results: (1) Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s positive integers. If $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $X = [n]$ satisfying $|F_i - F_j| \in \mathcal{L}$ for $i \neq j$, then

$$m \leq \sum_{i=0}^{s} \binom{n-1}{i}.$$

(2) Let p be a prime, $k \geq 2$, and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Suppose $\mathcal{F}$ is a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_1 \cap \cdots \cap F_k| \pmod p \in \mathcal{L}$ for any collection of k distinct sets from $\mathcal{F}$. If $n > (r+1)(s-2r+2)$, then

$$|\mathcal{F}| \leq (k-1) \sum_{i=s-2r+1}^{s} \binom{n-1}{i}.$$

The first result improves a result of Frankl about families with given difference sizes between subsets and the second result gives an improvement to a theorem by Grolmusz-Sudakov and a theorem by W. Cao, K.W. Hwang, and D.B. West.

1 Introduction

Throughout our paper, we use the set $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{F}$ is *t-uniform* if it is a set of t-subsets of $[n]$. We call a family $\mathcal{F}$ of subsets of $[n]$ an *intersecting* family if every pair of distinct subsets $F_i, F_j \in \mathcal{F}$ have a nonempty intersection. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of nonnegative or positive integers. A family $\mathcal{F}$ of subsets of $X = [n]$ is called *k-wise $\mathcal{L}$-intersecting* if $|F_1 \cap F_2 \cap \cdots \cap F_k| \in \mathcal{L}$ for every collection of k distinct members from $\mathcal{F}$. When $k = 2$, a 2-wise $\mathcal{L}$-intersecting family is simply called *$\mathcal{L}$-intersecting*.

In 1961, Erdös-Ko-Rado [7] proved the classical result as follows:

Theorem 1.1 *Suppose $\mathcal{F}$ be a k-uniform intersecting family of subsets of $[n]$ with $n \geq 2k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. And for $n > 2k$, equality holds only if $\mathcal{F}$ consists of all k-subsets containing a common element.*

Since then, many researchers have worked on various kinds of intersecting families, see [1-3,6,4,5,8-13,15,16-18]. In 1981, Frankl and Wilson [10] obtained the following celebrated result.

Theorem 1.2 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

This result is best possible as shown by the set of all subsets of size at most s of an n-set. In 1984, Frankl [8] proved the following similar result for set systems with given difference sizes between subsets, where a Sperner family $\mathcal{F}$ is a family of subsets of $X = [n]$ such that $E \not\subseteq F$ for any two distinct subsets $E, F \in \mathcal{F}$.

Theorem 1.3 *Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{F}$ is a Sperner family of subsets of $[n]$ satisfying that $|F - F'| \pmod{p} \in L$ for all distinct pair $F, F' \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

Here, we will give the following improvement to Theorem 1.3.

Theorem 1.4 *Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{F}$ is a family of subsets of $[n]$ satisfying that $|F - F'| \pmod p \in L$ for all distinct pair $F, F' \in \mathcal{F}$. Then*

$$m \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

Note that for any two sets A and B, $A - B = A \cap \overline{B}$, where $\overline{B}$ is the complement of B. Theorem 1.4 follows directly from the following result about cross-intersecting two families by taking $\mathcal{A} = \mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ and $\mathcal{B} = \{\overline{F_1}, \overline{F_2}, \ldots, \overline{F_m}\}$. The next theorem can also be viewed as a variation to Bollobás's Theorem on cross intersecting families in [2].

Theorem 1.5 *Let p be a prime and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{1, 2, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of $[n]$ such that $|A_i \cap B_j| \pmod p \in \mathcal{L}$ whenever $i \neq j$. If $|A_i \cap B_i| \pmod p \notin \mathcal{L}$ and $n \notin A_i \cap B_i$ for each $i \leq m$, then*

$$m \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}.$$

In section 3, we will prove the next result about cross-intersecting two families which can be used to derive results about k-wise $\mathcal{L}$-intersecting families.

Theorem 1.6 *Let p be a prime and let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, 2, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two families of subsets of X satisfying that*

(1) $n \notin A_i \cap B_i$ for $1 \leq i \leq b$ and $n \in A_i$ for $b < i \leq m$;

(2) $|A_i \cap B_j| \pmod p \in \mathcal{L}$ for $1 \leq j < i \leq m$;

(3) $|A_i \cap B_i| \pmod p \notin \mathcal{L}$ for every $1 \leq i \leq m$.

(4) $|A_i| \pmod p \in K$ for every $1 \leq i \leq m$.

If $n > (r+1)(s-2r+2)$, then

$$m \leq \sum_{i=s-2r+1}^{s} \binom{n-1}{i}.$$

As a consequence, we can prove the following result about k-wise $\mathcal{L}$-intersecting families which improves both Theorem 2 in Grolmusz and Sudakov [13] and the main theorem in Cao, Hwang and West [4]. The following result also gives a better bound than those in [11,12] when $|\mathcal{L}| > |\{l \pmod p) | l \in \mathcal{L}\}|$.

Theorem 1.7 *Let p be a prime, $k \geq 2$, and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, 2, \ldots, p-1\}$. Suppose $\mathcal{F}$ is a family of subsets of X such that $|F_1 \cap F_2 \cap \cdots \cap F_k| \pmod p \in \mathcal{L}$ for every collection of k distinct members from $\mathcal{F}$ and $|F| \pmod p \in K$ for every $F \in \mathcal{F}$. If $n > (r+1)(s-2r+2)$, then*

$$|\mathcal{F}| \leq (k-1) \sum_{i=s-2r+1}^{s} \binom{n-1}{i}.$$

When K is a set of r consecutive integers, we can prove the following slightly better bound for k-wise $\mathcal{L}$-intersecting families than that in Theorem 1.7.

Theorem 1.8 *Let p be a prime, $k \geq 2$, and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k, k+1, \ldots, k+r-1\}$ be two distinct subsets of $\{0, 1, \ldots, p-1\}$. Suppose $\mathcal{F}$ is a family of subsets of X such that $|F_1 \cap F_2 \cap \cdots \cap F_k| \pmod p \in \mathcal{L}$ for every collection of k distinct members from $\mathcal{F}$ and $|F| \pmod p \in K$ for every $F \in \mathcal{F}$. If $n > (r+1)(s-2r+2)$, then*

$$|\mathcal{F}| \leq (k-1) \sum_{i=s-r}^{s} \binom{n-1}{i}.$$

2 Proof of Theorem 1.5

We will use $x = (x_1, x_2, \ldots, x_n)$ to denote a vector of n variables with each variable x_i taking values 0 and 1. A polynomial $f(x)$ in variables x_i, $1 \leq i \leq n$, is called *multilinear* if the power of each variable x_i in each term is at most one. Clearly, if each variable x_i takes only the value 0 or 1, then any polynomial in variables x_i, $1 \leq i \leq n$, is multilinear since any positive power of a variable x_i may be replaced by one. For any subset F of $[n]$, we define *the characteristic vector* of F to be the vector $v_F = (v_{F_1}, v_{F_2}, \ldots, v_{F_n}) \in R^n$ with $v_{F_i} = 1$ if $i \in F$ and $v_{F_i} = 0$ otherwise. For $x, y \in R^n$, let $x \cdot y = \sum_{i=1}^n x_i y_i$ denote their standard inner product.

Proof of Theorem 1.5 For each $B_i \in \mathcal{B}$, we define the multilinear polynomial of degree at most s by

$$f_{B_i} = \prod_{l \in \mathcal{L}} (v_{B_i} \cdot x - l),$$

where v_{B_i} is the characteristic vector of B_i. Then $f_{B_i}(v_{A_j}) = \prod_{l \in \mathcal{L}} (|A_j \cap B_i| - l) = 0 \pmod p$ for $i \neq j$ and $f_{B_i}(v_{A_i}) \neq 0 \pmod p$ as $|A_i \cap B_i| \pmod p \notin \mathcal{L}$ for each $i \leq m$.

Let W be the family of subsets of $[n]$ with size at most s which contain n. Now for each $I \in W$, define

$$g_I(x) = \prod_{j \in I} x_j,$$

which is a multilinear polynomial with a degree at most s.

We now proceed to show that these polynomials in

$$\{f_{B_i} | 1 \leq i \leq m\} \cup \{g_I | I \in W\}$$

are linearly independent over the field $\mathbf{F}_p$. Suppose that the following linear combination of these polynomials are equal to zero:

$$\sum_{i=1}^{m} \alpha_i f_{B_i}(x) + \sum_{I \in W} \beta_I g_I(x) = 0. \tag{2.1}$$

Claim 1. $\alpha_i = 0$ for each i with $n \notin A_i$.

To the contrary, suppose that i' is a subscript such that $n \notin A_{i'}$ and $\alpha_{i'} \neq 0$. Since $n \notin A_{i'}$, $g_I(v_{A_{i'}}) = 0$ for each $I \in W$. By evaluating Eq. (2.1) with $x = v_{A_{i'}}$, we have $\alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0 \pmod p$ which implies $\alpha_{i'} = 0$, a contradiction. So the claim holds.

Claim 2. $\beta_I = 0$ for every $I \in W$. By Claim 1, we obtain

$$\sum_{n \in A_i} \alpha_i f_{B_i}(x) + \sum_{I \in W} \beta_I g_I(x) = 0. \tag{2.2}$$

Since $n \in A_i$ and $n \notin A_i \cap B_i$, we have $n \notin B_i$. Therefore, x_n does not appear in the first sum of Eq. (2.2). Setting $x_n = 0$ in Eq. (2.2) gives us

$$\sum_{n \in A_i} \alpha_i f_{B_i}(x) = 0, \tag{2.3}$$

and so

$$\sum_{I \in W} \beta_I g_I(x) = 0. \tag{2.4}$$

Suppose that I' is the minimal subset such that $\beta_{I'} \neq 0$. Note that $g_{I'}(v_{I'}) = 1$ and $g_I(v_{I'}) = 0 \pmod p$ for any $I \in W$ with $I \neq I'$ and $|I| \geq |I'|$. Setting $x = v_{I'}$ in Eq. (2.4), we obtain $\beta_{I'} g_{I'}(v_{I'}) = 0 \pmod p$, which implies $\beta_{I'} = 0$, a contradiction. Thus the claim is true.

By Claims 1 and 2, we only need to show that $\alpha_i = 0$ for each i with $n \in A_i$. To the contrary, suppose i' is a subscript that $n \in A_{i'}$ and $\alpha_{i'} \neq 0$. Evaluating (2.3) with $x = v_{A_{i'}}$, we obtain $\alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0 \pmod p$ which implies $\alpha_{i'} = 0$, a contradiction.

In summary, we have shown that f_{B_i}'s and g_I's are linearly independent over $\mathbf{F}_p$. Since the set of all monomials in variables $x_i,\ 1 \le i \le n$, of degree at most s forms a basis for the vector space of multilinear polynomials of degree at most s, it follows that

$$m + \sum_{i=0}^{s-1} \binom{n-1}{i} \le \sum_{i=0}^{s} \binom{n}{i},$$

which implies that

$$m \le \sum_{i=0}^{s} \binom{n-1}{i}.$$

This completes the proof. $\square$

3 Proof of Theorems 1.6–1.8

To prove Theorem 1.6, we need the following lemma which is Lemma 3.6 in [1]. We say a set $H = \{h_1, h_2, \ldots, h_t\} \subseteq [n]$ has a gap of size $\ge d$ (where the h_i are arranged in increasing order) if either $h_1 \ge d-1$, or $n - h_t \ge d-1$, or $h_{i+1} - h_i \ge d$ for some i $(1 \le i \le t-1)$.

Lemma 3.1. Let p be a prime and $H \subseteq \{0, 1, \ldots, p-1\}$ be a set of integers such that the set $(H + p\mathbf{Z}) \cap \{0, 1, \ldots, n\}$ has a gap $\ge d+1$, where $d \ge 0$. Let f denote the following polynomial in n variables

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n} x_j - h \right).$$

Then the set of polynomials $\{f(x) \prod_{j \in I} x_j \| I| \le d-1\}$ is linearly independent over $\mathbf{F}_p$.

The following proof is alone the same line as the proof of Theorem 1.11 in [5] with some important differences.

Proof of Theorem 1.6. For each $B_i \in \mathcal{B}$, we define

$$f_{B_i}(x) = \prod_{j=1}^{s} (v_{B_i} \cdot x - l_j),$$

where $x = (x_1, x_2, \ldots, x_n)$ with each x_j taking values 0 or 1, v_{B_i} is the characteristic vector of B_i, and $v_{B_i} \cdot x$ is the standard inner product. Then each $f_{B_i}(x)$ is a multilinear polynomial of degree at most s. It is clear from condition (2) that $f_{B_j}(v_{A_i}) = 0 \pmod p$ for $i > j$ as $v_{B_j} \cdot v_{A_i} = |A_i \cap B_j| \pmod p \in \mathcal{L}$.

Let Q be the family of subsets of $X = [n]$ with size at most s which contain n. Then $|Q| = \sum_{i=0}^{s-1} \binom{n-1}{i}$. For each $L \in Q$, define

$$q_L(x) = (1 - x_n) \prod_{j \in L, j \neq n} x_j.$$

Then each $q_L(x)$ is a multilinear polynomial of degree at most s.

Let $H = \{k_i - 1 | k_i \in K\} \cup K$. Then $|H| \leq 2r$. Set

$$f(x) = \prod_{h \in H} \left(\sum_{j=1}^{n-1} x_j - h \right).$$

Let W be the family of subsets of $[n]$ with sizes at most $s - 2r$ which do not contain n, then $|W| = \sum_{i=0}^{s-2r} \binom{n-1}{i}$. For each $I \in W$, define

$$A_I(x) = f(x) \prod_{j \in I} x_j.$$

Then each $A_I(x)$ is a multilinear polynomial of degree at most s.

We now show that the polynomials in

$$\{f_{B_i}(x) | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent over $\mathbf{F}_p$. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_{B_i}(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{3.1}$$

Claim 1. $\alpha_i = 0$ for each $i > b$ (i.e., $n \in A_i$).

Suppose, to the contrary, that i' is the largest subscript such that $i' > b$ and $\alpha_{i'} \neq 0$. Since $n \in A_{i'}$, $q_L(v_{A_{i'}}) = 0$ for every $L \in Q$. Recall that $f_{B_j}(v_{A_{i'}}) = 0 \pmod p$ for $j < i'$ and $f(v_{A_{i'}}) = 0 \pmod p$. By evaluating Eq. (3.1) with $x = v_{A_{i'}}$, we obtain that $\alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0 \pmod p$. Since $f_{B_{i'}}(v_{A_{i'}}) \neq 0 \pmod p$, we have $\alpha_{i'} = 0$, a contradiction. Thus, Claim 1 holds.

Claim 2. $\alpha_i = 0$ for each $i \leq b$.

Applying Claim 1, we get

$$\sum_{i \leq b} \alpha_i f_{B_i}(x) + \sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{3.2}$$

Suppose, to the contrary, that i' is the largest subscript subscript such that $i' \leq b$ and $\alpha_{i'} \neq 0$. Let $v^*_{A_{i'}} = v_{A_{i'}} + (0, 0, \ldots, 0, 0, 1)$ (namely, making $x_n = 1$ in $v^*_{A_{i'}}$). Then $q_L(v^*_{A_{i'}}) = 0$ for every $L \in Q$. Note that $f_{B_j}(v^*_{A_{i'}}) = f_{B_j}(v_{A_{i'}})$ for each $j \leq b$ as $n \notin B_j$. For each $I \in W$, since $f(v^*_{A_{i'}}) = 0 \pmod p$, $A_I(v^*_{A_{i'}}) = 0 \pmod p$. By evaluating Eq. (3.2) with $x = v^*_{A_{i'}}$, we obtain $\alpha_{i'} f_{B_{i'}}(v^*_{A_{i'}}) = \alpha_{i'} f_{B_{i'}}(v_{A_{i'}}) = 0 \pmod p$ which implies $\alpha_{i'} = 0$, a contradiction. Thus, the claim is verified.

Claim 3. $\beta_L = 0$ for each $L \in Q$.

By Claims 1 and 2, we obtain

$$\sum_{L \in Q} \beta_L q_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0. \tag{3.3}$$

Rewrite Eq. (3.3) as

$$\left[\sum_{L \in Q} \beta_L q'_L(x) + \sum_{I \in W} \mu_I A_I(x)\right] - \left(\sum_{L \in Q} \beta_L q'_L(x)\right) x_n = 0, \tag{3.4}$$

where $q'_L = \prod_{j \in L, j \neq n} x_j$. Note that x_n does not appear in the first parenthesis of Eq. (3.4). Setting $x_n = 0$ in Eq. (3.4) gives us

$$\sum_{L \in Q} \beta_L q'_L(x) + \sum_{I \in W} \mu_I A_I(x) = 0$$

and

$$\left(\sum_{L \in Q} \beta_L q'_L(x)\right) x_n = 0.$$

By setting $x_n = 1$, we obtain

$$\sum_{L \in Q} \beta_L q'_L(x) = 0.$$

It is not difficult to see that the polynomials $q'_L(x)$, $L \in Q$, are linearly independent. Therefore, we conclude that $\beta_L = 0$ for each $L \in Q$.

By Claims 1-3, we now have

$$\sum_{I \in W} \mu_I A_I(x) = 0. \tag{3.5}$$

Recall that $H = \{k_i - 1 | k_i \in K\} \cup K$, $H \subseteq \{0, 1, \ldots, p-1\}$ with r pairs of consecutive integers $k_i - 1$ and k_i, $1 \leq i \leq r$. Since $n > (r+1)(s - 2r + 2)$, H has a gap at least $s - 2r + 2$. By applying Lemma 3.1 with $d = s - 2r + 1$, we conclude that the set of polynomials $\{A_I(x) = x_I f(x) | I \in W\}$ is linearly independent over $\mathbf{F}_p$, and so $\mu_I = 0$ for each $I \in W$ in Eq. (3.5).

In summary, we have shown that the polynomials in

$$\{f_{B_i}(x) | 1 \le i \le m\} \cup \{q_L(x) | L \in Q\} \cup \{A_I(x) | I \in W\}$$

are linearly independent. Since the set of all monomials in variables x_i, $1 \le i \le n$, of degree at most s forms a basis for the vector space of multilinear polynomials of degree at most s, it follows that

$$m + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-2r} \binom{n-1}{i} \le \sum_{i=0}^{s} \binom{n}{i}$$

which implies that

$$m \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This completes the proof. $\square$

Before we proceed further, we give the following remark.

Remark 3.1. Note that if the set $K = \{k_1, k_2, \ldots, k_r\}$ is a set of r consecutive integers, then $|H| = r + 1$ for $H = \{k_i - 1 | 1 \le i \le r\} \cup \{k_i | 1 \le i \le r\}$. Therefore, if we replace W in the proof of Theorem 1.6 by the family of all subsets of $[n]$ with sizes at most $s - r - 1$ which do not contain n, then we get

$$m \le \sum_{i=s-r}^{s} \binom{n-1}{i}.$$

Proof of Theorem 1.7. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s non-negative integers and $k \ge 2$. Suppose that $\mathcal{F}$ is a k-wise $\mathcal{L}$-intersecting family of subsets of X. We repeat the following procedure until $\mathcal{F}$ is empty to produce two families $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ satisfying the conditions in Theorem 1.6.

Suppose we have defined pairs $\{A_j, B_j\}$ for $j \le i-1$. At round i, if $\mathcal{F} \ne \emptyset$, then we define pair $\{A_i, B_i\}$ as follows: Whenever there exists $F \in \mathcal{F}$ such that $n \notin F$, choose $F_1 \in \mathcal{F}$ with $n \notin F_1$; otherwise choose any $F_1 \in \mathcal{F}$. Let $F_1, F_2, \ldots, F_d$ be a maximal collection of subsets from $\mathcal{F}$ such that $| \cap_{j=1}^{d'} F_j| \ (mod\ p) \notin \mathcal{L}$ for all $1 \le d' \le d$, but $| \cap_{j=1}^{d} F_j \cap F'| \ (mod\ p) \in \mathcal{L}$ for any additional set $F' \in \mathcal{F}$. Clearly, by the assumption, such collection always exists and $1 \le d \le k - 1$. Denote $A_i = F_1$ and $B_i = \cap_{j=1}^{d} F_j$ and remove $F_1, F_2, \ldots, F_d$ from $\mathcal{F}$. Note that as a result of this process, we obtain two families $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and

$\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ satisfying the conditions in Theorem 1.6 and $m \geq |\mathcal{F}|/(k-1)$. Thus it follows from Theorem 1.6 that

$$|\mathcal{F}| \leq (k-1)m \leq (k-1) \sum_{i=s-2r+1}^{s} \binom{n-1}{i}.$$

This completes the proof. $\square$

Theorem 1.8 can be proved in exactly the same way as Theorem 1.7 by applying Remark 3.1 instead of Theorem 1.6.

Concluding remark. We remark here that with almost identical proofs, one can obtain results similar to Theorems 1.6–1.8 by change the condition $n > (r+1)(s-2r+2)$ to the condition $\min k_i > \max l_j$.

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