Minimal Permutations and 2-Regular Skew Tableaux

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Abstract. Bouvel and Pergola introduced the notion of minimal permutations in the study of the whole genome duplication-random loss model for genome rearrangements. Let $\mathcal{F}_d(n)$ denote the set of minimal permutations of length n with d descents, and let $f_d(n) = |\mathcal{F}_d(n)|$. They showed that $f_{n-2}(n) = 2^n - (n-1)n - 2$ and $f_n(2n) = C_n$, where C_n is the n-th Catalan number. Mansour and Yan proved that $f_{n+1}(2n+1) = 2^{n-2}nC_{n+1}$. In this paper, we consider the problem of counting minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents, and we show that they are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. Using the determinantal formula for the number of skew Young tableaux of a given shape, we find an explicit formula for $f_{n-3}(n)$. Furthermore, by using the Knuth equivalence, we give a combinatorial interpretation of a formula for a refinement of the number $f_{n+1}(2n+1)$.

Keywords: minimal permutation, 2-regular skew tableau, Knuth equivalence, RSK algorithm.

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1 Introduction

The notion of minimal permutations was introduced by Bouvel and Pergola in the study of genome evolution, see [3]. Such permutations are a basis of permutations that can be obtained from the identity permutation via a given number of steps in the duplication-random loss model, see [1, 3, 4, 6]. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. A duplication of π means the duplication of a fragment of consecutive elements of π in such a way that the duplicated fragment is put immediately after the original fragment. Suppose that $\pi_i\pi_{i+1}\cdots\pi_j$ is the fragment for duplication, then the duplicated sequence is

$$\pi_1\cdots\pi_{i-1}\pi_i\cdots\pi_j\pi_i\cdots\pi_j\pi_{j+1}\cdots\pi_n.$$

A random loss means to randomly delete one occurrence of each repeated element π_k for $i \le k \le j$, so that we get a permutation again. In the following example, the fragment

234 is duplicated, and the underlined elements are the occurrences of repeated elements that are supposed to be deleted,

$$1\,\overbrace{234}\,56 \rightsquigarrow 1\,\overbrace{234}\,\overbrace{234}\,56 \rightsquigarrow 1\underline{2}3\underline{4}2\underline{3}456 \rightsquigarrow 132456.$$

To describe the notation of minimal permutations, we give an overview of the descent set of a permutation and the patterns of subsequences of a permutation. Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$. In a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, a descent is a position i such that $i \leq n-1$ and $\pi_i > \pi_{i+1}$, whereas an ascent is a position i with $i \leq n-1$ and $\pi_i < \pi_{i+1}$. For example, the permutation $3145726 \in S_7$ has two descents 1 and 5 and has four ascents 2, 3, 4 and 6.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of distinct integers listed in increasing order, namely, $v_1 < v_2 < \cdots < v_n$. The standardization of a permutation π on V is the permutation $\mathrm{st}(\pi)$ on $[n]$ obtained from π by replacing v_i with i. For example, $\mathrm{st}(9425) = 4213$. A subsequence $\omega = \pi_{i(1)}\pi_{i(2)}\cdots\pi_{i(k)}$ of π is said to be of type σ or π contains a pattern σ if $\mathrm{st}(\omega) = \sigma$. We say that a permutation $\pi \in S_n$ contains a pattern $\tau \in S_k$ if there is a subsequence of π that is of type τ. For example, let $\pi = 263751498$. The subsequence 3549 is of type 1324, and so π contains the pattern 1324. We use the notation $\tau \prec \pi$ to denote that a permutation π contains the pattern τ.

A permutation π is called a minimal permutation with d descents if it is minimal in the sense that there exists no permutation σ with exactly d descents such that $\sigma \prec \pi$. Denote by $\mathcal{B}_d$ the set of minimal permutations with d descents. Bouvel and Pergola [3] have shown that the length, namely, the number of elements, of any minimal permutation in the set $\mathcal{B}_d$ is at least $d+1$ and at most $2d$. They also proved that in the whole genome duplication-random loss model, the permutations that can be obtained from the identity permutation in at most p steps can be characterized as permutations with $d = 2^p$ descents that avoid certain patterns.

Theorem 1.1 (Bouvel and Pergola [3]) *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation on $[n]$. Then π is a minimal permutation with d descents if and only if π is a permutation with d descents satisfying the following conditions:*

(i) It starts and ends with a descent;

(ii) If i is an ascent, that is, $\pi_i < \pi_{i+1}$, then $i \in \{2, 3, \ldots, n-2\}$ and $\pi_{i-1}\pi_i\pi_{i+1}\pi_{i+2}$ is of type 2143 *or* 3142.

Denote by $\mathcal{F}_d(n)$ the set of minimal permutations of length n with d descents and $f_d(n) = |\mathcal{F}_d(n)|$. Clearly, $f_d(n) = 0$ for $d \leq 0$ or $d \geq n$, and $f_d(d+1) = 1$ for all $d \geq 1$. Bouvel and Pergola proved that $f_n(2n)$ equals the n-th Catalan number, that is,

$$f_n(2n) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

and $f_{n-2}(n)$ is given by the formula

$$f_{n-2}(n) = 2^n - (n-1)n - 2.$$

Mansour and Yan [7] have shown that

$$f_{n+1}(2n+1) = 2^{n-2} n C_{n+1}. \tag{1.1}$$

As remarked by Bouvel and Pergola, it is an open problem to compute $f_d(n)$ for other cases of d. In this paper, we consider the enumeration of minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents. We show that such minimal permutations are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. Thus we may employ the determinantal formula for the number of skew Young tableaux of a given shape to compute the number $f_d(n)$. In this way, we derive an explicit formula for $f_{n-3}(n)$. Moreover, we obtain a formula for the refinement of the number $f_{n+1}(2n+1)$ and find a combinatorial proof by using the Knuth equivalence of permutations.

2 2-Regular skew tableaux

In this section, we establish a connection between minimal permutations and skew Young tableaux of certain shape. To describe our correspondence, let us give an overview of necessary terminology on Young tableaux as used in Stanley [9].

A partition of a positive integer n is defined to be a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. If λ is a partition of n, we write $\lambda \vdash n$, or $|\lambda| = n$. The Ferrers diagram of a partition λ is a diagram with left-justified rows in which the i-th row consists of λ_i dots. The conjugate partition λ' of λ is obtained by transposing the Ferrers diagram of λ. The positive terms λ_i are called the parts of λ, and the number of parts is denoted by $l(\lambda)$.

A standard Young tableau (SYT) of shape λ is an array $P = (P_{ij})$ of positive integers of shape λ that is strictly increasing in every row and in every column, where P_{ij} is the integer in the position (i, j) of P. The size of an SYT is its number of entries. If λ and μ are partitions with $\mu \subseteq \lambda$, namely, $\mu_i \leq \lambda_i$ for all i, we can define a Young tableau of skew shape λ/μ as a tableau on $[n]$ that is increasing in every row and every column. The number of boxes of the Young diagram of shape λ/μ is denoted by $|\lambda/\mu|$. For example, below are an SYT of shape (4, 3, 3, 1) and a skew Young tableau of shape (6, 5, 2, 2)/(3, 1):

$$\begin{array}{llll} 1 & 3 & 5 & 6 \\ 2 & 4 & 8 & \\ 7 & 9 & 11 & \\ 10 & & & \end{array} \qquad\qquad \begin{array}{cccccc} & & & 7 & 8 & 11 \\ & 1 & 5 & 9 & 10 & \\ 2 & 4 & & & & \\ 3 & 6 & & & & \end{array}.$$

If $|\lambda/\mu| = n$ and $l(\lambda) = r$, then the number of skew Young tableaux of shape λ/μ is given by

$$f^{\lambda/\mu} = n! \, \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1}^{r}, \tag{2.1}$$

see, for example, [9, Corollary 7.16.3] and [10]. Since $f^{\lambda/\mu} = f^{\lambda'/\mu'}$ for any skew shape λ/μ, (2.1) also takes the following form

$$f^{\lambda/\mu} = n! \, \det\left(\frac{1}{(\lambda'_i - \mu'_j - i + j)!}\right)_{i,j=1}^{t}, \tag{2.2}$$

where $|\lambda'/\mu'| = n$ and $l(\lambda') = t$.

Let $(a_1, \ldots, a_k)$ be a sequence of positive integers such that $a_i \geq 2$ for all i and $a_1 + a_2 + \cdots + a_k = n$. Let P be a skew Young tableau of size n with column lengths $a_1, a_2, \ldots, a_k$. We say that P is 2-regular if any two consecutive columns overlap exactly by two rows, namely, for any two consecutive columns there are exactly two rows containing elements in both columns. Denote by $\mathcal{P}_{a_1, a_2, \ldots, a_k}(n)$ the set of 2-regular skew tableaux with column lengths $a_1, a_2, \ldots, a_k$.

For example, the following skew Young tableau is 2-regular and it belongs to $\mathcal{P}_{4,2,5,3,2}(16)$:

$$\begin{array}{ccccc}
 & & & 6 & 8 \\
 & & 2 & 10 & 15 \\
 & & 5 & 11 & \\
 & & 9 & & \\
1 & 3 & 12 & & \\
4 & 7 & 14 & & \\
13 & & & & \\
16 & & & &
\end{array} . \tag{2.3}$$

For a permutation π of length n, a substring of π is a sequence of consecutive elements of π. A maximal decreasing substring of π is defined to be a decreasing substring that is not a substring of another decreasing substring. For example, the permutation $5\,2\,7\,3\,1\,4\,8\,9\,6$ contains five maximal decreasing substrings, namely, $5\,2$, 7, $3\,1$, 4, 8 and $9\,6$.

It is clear that any permutation π with $k-1$ ascents can be decomposed into k maximal decreasing substrings. To describe the ascent set, we find it convenient to use a sequence $(a_1, a_2, \ldots, a_k)$ to denote the lengths of the maximal decreasing substrings, and this sequence is called the ascent sequence of π. Then the ascent set of π is expressed as $\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}\}$.

Lemma 2.1 *Given a minimal permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$. Suppose $(a_1, a_2, \ldots, a_k)$ is its ascent sequence, then $a_i \geq 2$ for all i.*

Proof. By condition (i) of Theorem 1.1, π starts and ends with a descent, this implies that $a_1 \geq 2$ and $a_k \geq 2$. For each ascent $j = a_1 + \cdots + a_i$ of π, where $1 \leq i \leq k-1$, condition (ii) of Theorem 1.1 says that $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142, which means that both $j-1$ and $j+1$ are descents. Therefore, π contains no consecutive ascents. It follows that the length of every maximal decreasing substring is at least two. Thus we have $a_i \geq 2$. This completes the proof. ∎

Denote by $\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)$ the set of minimal permutations of length n with the ascent sequence $(a_1, a_2, \ldots, a_k)$, and let $F_{a_1, a_2, \ldots, a_k}(n) = |\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)|$. We show that there is a bijection between the set $\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)$ of minimal permutations and the set $\mathcal{P}_{a_1, a_2, \ldots, a_k}(n)$ of 2-regular skew tableaux.

Theorem 2.2 *Let $(a_1, a_2, \ldots, a_k)$ be a sequence of positive integers such that $a_1 + a_2 + \cdots + a_k = n$ and $a_i \geq 2$ for all i. Then the minimal permutations in the set $\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)$ are in one-to-one correspondence with the 2-regular skew tableaux in the set $\mathcal{P}_{a_1, a_2, \ldots, a_k}(n)$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a minimal permutation in $\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)$. We wish to transform π into a tableau P of the following form:

$$P = \begin{array}{cccc}
 & & \vdots & \\
 & & \vdots & \cdots \quad \cdots \\
 & \pi_{a_1+a_2} & \pi_{a_1+a_2+2} & \\
 & \pi_{a_1+a_2-1} & \pi_{a_1+a_2+1} & \\
 & \vdots & & \\
\pi_{a_1} & \pi_{a_1+2} & & \\
\pi_{a_1-1} & \pi_{a_1+1} & & \\
\vdots & & & \\
\pi_2 & & & \\
\pi_1 & & &
\end{array} . \tag{2.4}$$

More precisely, for $1 \leq i \leq k$, the i-th column of P is the i-th maximal decreasing substring of π, where the elements in the maximal decreasing substring are placed in increasing order along the column. Moreover, any two consecutive columns of P overlap exactly by two rows. We shall prove that P is a 2-regular skew tableau.

From the construction of P, we just need to show that both the rows and columns are increasing. It is easy to see that every column of P is strictly increasing. It remains to show that any row of P is also strictly increasing. To this end, it suffices to prove that the following array consisting of the four overlapping elements of two adjacent rows is increasing along the rows:

$$\begin{array}{cc} \pi_j & \pi_{j+2} \\ \pi_{j-1} & \pi_{j+1} \end{array}, \tag{2.5}$$

where $j = a_1 + a_2 + \cdots a_i\,(1 \le i \le k-1)$ is an ascent of π. From condition (ii) of Theorem 1.1, $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142. It follows that $\pi_{j-1} < \pi_{j+1}$ and $\pi_j < \pi_{j+2}$. We conclude that every row of P is increasing. So P is a 2-regular skew tableau in $\mathcal{P}_{a_1, a_2, \ldots, a_k}(n)$.

Conversely, let P be a 2-regular skew tableau in $\mathcal{P}_{a_1, a_2, \ldots, a_k}(n)$. We can easily construct a permutation π by reading off the elements of P from bottom to top and from left to right. It is routine to verify that π is a minimal permutation in $\mathscr{F}_{a_1, a_2, \ldots, a_k}(n)$. This completes the proof. ∎

For example, $\pi = 16\,13\,4\,1\,7\,3\,14\,12\,9\,5\,2\,11\,10\,6\,15\,8 \in \mathcal{F}_{11}(16)$ contains 5 maximal decreasing substrings and the 2-regular skew tableau corresponding to π is given by (2.3).

Some known results can be derived from the above bijection. For example, it can be seen that the minimal permutations in $\mathcal{F}_n(2n)$ are alternating permutations, in the sense that for any minimal permutation π in $\mathcal{F}_n(2n)$, we have

$$\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots < \pi_{2n-1} > \pi_{2n},$$

see [3]. Hence the 2-regular skew tableaux corresponding to the minimal permutations in $\mathcal{F}_n(2n)$ are of shape (n, n):

$$\begin{array}{cccccccc} \pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \cdots & \pi_{2n} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1} & \pi_{2i+1} & \cdots & \pi_{2n-1}. \end{array} \tag{2.6}$$

Thus, by (2.1), we have

$$\begin{aligned} f_n(2n) = f^{(n,n)} &= (2n)! \begin{vmatrix} \dfrac{1}{n!} & \dfrac{1}{(n+1)!} \\ \dfrac{1}{(n-1)!} & \dfrac{1}{n!} \end{vmatrix} \\ &= \frac{1}{n+1}\binom{2n}{n} \\ &= C_n. \end{aligned}$$

In view of the bijection given by Theorem 2.2, the enumeration of minimal permutations is equivalent to the enumeration of 2-regular skew tableaux.

Theorem 2.3 *Given an ascent sequence $\alpha = (a_1, a_2, \ldots, a_k)$, where $\sum_{i=1}^k a_i = n$ and*

$a_i \geq 2$, $1 \leq i \leq k$, *we have*

$$F_{a_1, a_2, \ldots, a_k} = n! \det(A) = n! \begin{vmatrix} \frac{1}{a_1!} & & & & A_{ij} & & & \\ 1 & \frac{1}{a_2!} & & & & & & \\ A_{3,1} & 1 & \frac{1}{a_3!} & & & & & \\ & A_{4,2} & 1 & \frac{1}{a_4!} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & A_{k-1,k-3} & 1 & \frac{1}{a_{k-1}!} & \\ & & & & A_{k,k-2} & 1 & \frac{1}{a_k!} \end{vmatrix}, \qquad (2.7)$$

where

$$A_{i,j} = 0, \qquad \textit{for} \quad j < i-2,$$

$$A_{i,i-2} = \begin{cases} 0, & \textit{for} \quad a_{i-1} > 2, \\ 1, & \textit{for} \quad a_{i-1} = 2, \end{cases}$$

$$A_{i,j} = \frac{1}{\left(\sum_{m=i}^{j} a_m - (j-i)\right)!}, \qquad \textit{for} \quad j > i.$$

Proof. First we need to determine the shape of the 2-regular skew tableau P defined in (2.4). Suppose that the shape of P is λ/μ. Since the i-th column of P is the i-th maximal decreasing substring, we have

$$\lambda'_i - \mu'_i = a_i, \quad \text{for} \quad 1 \leq i \leq k. \qquad (2.8)$$

Moreover, since any two consecutive columns overlap exactly by two rows in P, it follows that

$$\lambda'_i - \lambda'_{i+1} = a_i - 2, \quad \text{for} \quad 1 \leq i \leq k-1. \qquad (2.9)$$

Obviously $\lambda'_k = a_k$ and $\mu'_k = 0$. For $1 \leq i \leq k-1$, we have

$$\begin{aligned} \lambda'_i &= \lambda'_{i+1} + (a_i - 2) \\ &= \lambda'_{i+2} + (a_{i+1} - 2) + (a_i - 2) \\ &\vdots \\ &= a_k + (a_{k-1} - 2) + \cdots + (a_i - 2) \\ &= a_i + a_{i+1} + \cdots + a_k - 2(k-i). \end{aligned}$$

This implies $\mu'_i = \lambda'_i - a_i = a_{i+1} + \cdots + a_k - 2(k-i)$. Consequently,

$$F_{a_1,\, a_2, \ldots,\, a_k}(n) = f^{\lambda'/\mu'}, \tag{2.10}$$

where

$$\lambda' = (\lambda'_1,\, \lambda'_2, \ldots,\, \lambda'_k), \quad \lambda'_i = \sum_{j=i}^{k} a_j - 2(k-i), \quad \text{for} \quad 1 \le i \le k,$$

and

$$\mu' = (\mu'_1,\, \mu'_2, \ldots,\, \mu'_k), \quad \mu'_i = \lambda'_i - a_i = \sum_{j=i+1}^{k} a_j - 2(k-i), \quad \text{for} \quad 1 \le i \le k.$$

We continue to compute the number of 2-regular skew tableaux using formula (2.2). In view of (2.8) and (2.9), we have the following five cases.

(1) $j < i-2$, $\lambda'_i - \mu'_j = -(a_{j+1} + \cdots + a_{i-1}) + 2(i-j)$. So $\lambda'_i - \mu'_j - i + j = -(a_{j+1} + \cdots + a_{i-1}) + (i-j)$. Since each $a_m \ge 2 (1 \le m \le k)$, we have that $\lambda'_i - \mu'_j - i + j < 0$, which means $A_{i,j} = 0$.

(2) $j = i-2$, $\lambda'_i - \mu'_{i-2} = -a_{i-1} + 4$. Thus, if $a_{i-1} = 2$, then $\lambda'_i - \mu'_{i-2} - i + i - 2 = 0$ and $A_{i,\, i-2} = 1/0! = 1$. Otherwise, we have $a_{i-1} > 2$, so $\lambda'_i - \mu'_{i-2} - i + i - 2 < 0$, and $A_{i,\, i-2} = 0$.

(3) $j = i-1$, $\lambda'_i - \mu'_{i-1} = 2$. Then $\lambda'_i - \mu'_{i-1} - i + (i-1) = 1$, and $A_{i,\, i-1} = 1$.

(4) $j = i$. By (2.8), we have $\lambda'_i - \mu'_i = a_i$. Thus, $A_{i,\, i} = \dfrac{1}{a_i!}$.

(5) $j \ge i+1$. In this case, we have

$$A_{i,\, j} = \frac{1}{(\lambda'_i - \mu'_j - i + j)!} = \frac{1}{(a_i + \cdots + a_j - (j-i))!},$$

as desired. ∎

Now, we can compute the number of minimal permutations in $\mathcal{F}_d(n)$. Note that all the minimal permutations in $\mathcal{F}_d(n)$ have $n-d$ maximal decreasing substrings.

Corollary 2.4 *For $d+1 \le n \le 2d$, we have*

$$f_d(n) = \sum_{\substack{a_i \ge 2\, for\, 1 \le i \le n-d \\ a_1 + a_2 + \cdots + a_{n-d} = n}} F_{a_1,\, a_2, \ldots,\, a_{n-d}}. \tag{2.11}$$

Taking $d = n-2$ as an example, we immediately get the formula for $f_{n-2}(n)$ which is due to Bouvel and Pergola [3]. It is obvious that the minimal permutations in $\mathcal{F}_{n-2}(n)$ have only one ascent, which means that they have two maximal decreasing substrings. Suppose that the unique ascent is k, and the ascent sequence is $(k,\, n-k)$ for some $2 \le k \le n-2$. By Theorem 2.3, we have

$$F_{k,\,n-k} = n! \begin{vmatrix} \dfrac{1}{k!} & \dfrac{1}{(n-1)!} \\ 1 & \dfrac{1}{(n-k)!} \end{vmatrix} = \binom{n}{k} - n.$$

Therefore, by Corollary 2.4,

$$f_{n-2}(n) = \sum_{k=2}^{n-2} \left(\binom{n}{k} - n \right) = 2^n - 2 - n(n-1).$$

We finish the section with the computation of $f_{n-3}(n)$.

Theorem 2.5 *The number of minimal permutations of length n with $n-3$ descents is*

$$f_{n-3}(n) = 3^n - (n^2 - 2n + 4)2^{n-1} + \frac{1}{2}\left(n^4 - 7n^3 + 19n^2 - 21n + 2\right).$$

Proof. The minimal permutations in $\mathcal{F}_{n-3}(n)$ have three maximal decreasing substrings. Suppose that the ascent sequence is $(a,\, b,\, c)$, where $a+b+c = n$ and $a,\, b,\, c \ge 2$. According to Theorem 2.3, let

$$\begin{aligned} A_1 = n! &\begin{vmatrix} \dfrac{1}{a!} & \dfrac{1}{(a+1)!} & \dfrac{1}{(n-2)!} \\ 1 & \dfrac{1}{2!} & \dfrac{1}{(c+1)!} \\ 1 & 1 & \dfrac{1}{c!} \end{vmatrix} \\ &= \frac{n!}{a!2!c!} + \frac{n!}{(a+1)!(c+1)!} + \frac{n!}{(n-2)!} \\ &\quad - \frac{n!}{(n-2)!2!} - \frac{n!}{(a+1)!c!} - \frac{n!}{a!(c+1)!}, \end{aligned}$$

and let

$$A_2 = n! \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+b-1)!} & \frac{1}{(n-2)!} \\ 1 & \frac{1}{b!} & \frac{1}{(b+c-1)!} \\ 0 & 1 & \frac{1}{c!} \end{vmatrix}$$

$$= \frac{n!}{a!b!c!} + \frac{n!}{(n-2)!} - \frac{n!}{(a+b-1)!c!} - \frac{n!}{a!(b+c-1)!}.$$

By Corollary 2.4, we have

$$f_{n-3}(n) = \sum_{\substack{a,\,c\geq 2,\,b=2\\ a+b+c=n}} A_1 + \sum_{\substack{a,\,c\geq 2,\,b\geq 3\\ a+b+c=n}} A_2.$$

In order to simplify the computation, we reformulate the above equation into the following form:

$$\begin{aligned} f_{n-3}(n) &= \sum_{\substack{a,\,c\geq 2,\,b=2\\ a+b+c=n}} A_1 + \sum_{\substack{a,\,b,\,c\geq 2\\ a+b+c=n}} A_2 - \sum_{\substack{a,\,c\geq 2,\,b=2\\ a+b+c=n}} A_2 \\ &= \sum_{\substack{a,\,c\geq 2,\,b=2\\ a+b+c=n}} (A_1 - A_2) + \sum_{\substack{a,\,b,\,c\geq 2\\ a+b+c=n}} A_2. \end{aligned} \tag{2.12}$$

The first sum in (2.12) equals

$$\begin{aligned} \sum_{\substack{a,\,c\geq 2,\,b=2\\ a+b+c=n}} (A_1 - A_2) &= \sum_{\substack{a,\,c\geq 2\\ a+c=n-2}} \left(\frac{n!}{(a+1)!(c+1)!} - \binom{n}{2} \right) \\ &= \sum_{a=2}^{n-4} \left(\binom{n}{a+1} - \binom{n}{2} \right) \\ &= 2^n - 2n - 2 - (n-3)\binom{n}{2}. \end{aligned} \tag{2.13}$$

The second sum of (2.12) can be expressed as follows:

$$\sum_{\substack{a,\,b,\,c\geq 2\\ a+b+c=n}} A_2 = \sum_{\substack{a,\,b,\,c\geq 2\\ a+b+c=n}} \left(\binom{n}{a,\ b,\ c} + n(n-1) - n\binom{n-1}{c} - n\binom{n-1}{a} \right).$$

By the principle of inclusion-exclusion, we have

$$\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} \binom{n}{a,\ b,\ c} = 3^n - 3\sum_{b=0}^{n}\binom{n}{0,\ b,\ n-b} - 3\sum_{b=0}^{n-1}\binom{n}{1,\ b,\ n-1-b}$$

$$+ 3\binom{n}{0,\ 0,\ n} + 3\binom{n}{1,\ 1,\ n-2} + 6\binom{n}{0,\ 1,\ n-1}$$

$$= 3^n - 3\cdot 2^n - 3n\cdot 2^{n-1} + 3n^2 + 3n + 3. \tag{2.14}$$

Let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$, then

$$\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} n(n-1) = n(n-1)\cdot [x^n](x^2+x^3+\cdots)^3$$

$$= n(n-1)\binom{n-4}{2}. \tag{2.15}$$

Furthermore,

$$n\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} \left(\binom{n-1}{c} + \binom{n-1}{a}\right) = 2n\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} \binom{n-1}{a}.$$

It is easily seen that

$$\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} \binom{n-1}{a} = \sum_{a=2}^{n-4}\sum_{\substack{b,c\geq 2\\ b+c=n-a}} \binom{n-1}{a} = \sum_{a=2}^{n-4}\binom{n-1}{a}(n-3-a).$$

Since

$$\sum_{a=1}^{n-1} a\binom{n-1}{a} = \sum_{a=1}^{n-1} a\binom{n-1}{a} x^{a-1}\bigg|_{x=1} = (n-1)(1+x)^{n-2}\bigg|_{x=1} = (n-1)2^{n-2},$$

we find

$$2n\sum_{\substack{a,b,c\geq 2\\ a+b+c=n}} \binom{n-1}{a} = 2n(n-3)\sum_{a=2}^{n-4}\binom{n-1}{a} - 2n\sum_{a=2}^{n-4} a\binom{n-1}{a}$$

$$= 2n(n-3)\left(2^{n-1} - 2(n-1) - 2\right)$$

$$- 2n\left((n-1)2^{n-2} - 2(n-1) - (n-1)(n-2)\right)$$

$$= n(n-3)2^n - n(n-1)2^{n-1} - 2n^3 + 10n^2. \tag{2.16}$$

Combining (2.13), (2.14), (2.15) and (2.16), we have

$$f_{n-3}(n) = 3^n - (n^2 - 2n + 4)2^{n-1} + \frac{1}{2}\left(n^4 - 7n^3 + 19n^2 - 21n + 2\right),$$

as claimed. ∎

3 A refinement of the number $f_{n+1}(2n+1)$

In this section, we give a formula for a refinement of the number $f_{n+1}(2n+1)$. It can be derived from Theorem 2.2 and formula (2.1). We shall also provide a combinatorial proof by using the RSK algorithm and the Knuth equivalence of permutations.

Let $\mathcal{F}_{n+1}(2n+1)$ be the set of minimal permutations of length $2n+1$ with $n+1$ descents and $\pi = \pi_1\pi_2\cdots\pi_{2n+1} \in \mathcal{F}_{n+1}(2n+1)$. It can be seen that there is only one occurrence of consecutive descents in π, see [7]. This leads to a classification of the minimal permutations in $\mathcal{F}_{n+1}(2n+1)$ according to the positions of the consecutive descents.

Theorem 3.1 *For $1 \le i \le n$, let $\mathcal{M}_{2n+1,\,2i}$ denote the set of minimal permutations π in $\mathcal{F}_{n+1}(2n+1)$ such that both $2i-1$ and $2i$ are descents of π. Then we have*

$$|\mathcal{M}_{2n+1,\,2i}| = \binom{2n+1}{n-1}\binom{n-1}{i-1}. \tag{3.1}$$

It is easy to see that these 2-regular skew tableaux corresponding to the minimal permutations in $\mathcal{M}_{2n+1,\,2i}$ are the following skew Young tableaux of shape $(n,\,n,\,i)/(i-1)$:

$$\begin{array}{cccccccccc} & & & \pi_{2i+1} & \pi_{2i+3} & \pi_{2i+5} & \cdots & \pi_{2n-1} & \pi_{2n+1} \\ \pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \pi_{2i+4} & \cdots & \pi_{2n-2} & \pi_{2n} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1} & & & & & \end{array}. \tag{3.2}$$

Applying formula (2.1), we obtain

$$|\mathcal{M}_{2n+1,\,2i}| = f^{(n,\,n,\,i)/(i-1)} = (2n+1)!\begin{vmatrix} \frac{1}{(n-i+1)!} & \frac{1}{(n+1)!} & \frac{1}{(n+2)!} \\ \frac{1}{(n-i)!} & \frac{1}{n!} & \frac{1}{(n+1)!} \\ 0 & \frac{1}{(i-1)!} & \frac{1}{i!} \end{vmatrix}$$

$$= \binom{2n+1}{n}\binom{n+1}{i} + \binom{2n+1}{n-1}\binom{n-1}{i-1}$$

$$- \binom{2n+1}{n}\binom{n}{i} - \binom{2n+1}{n}\binom{n}{i-1}$$

$$= \binom{2n+1}{n-1}\binom{n-1}{i-1}.$$

This implies the following formula of Mansour and Yan [7]:

$$f_{n+1}(2n+1) = \sum_{i=1}^{n} f^{(n,n,i)/(i-1)} = 2^{n-1}\binom{2n+1}{n-1}.$$

We now proceed to give a combinatorial proof of Theorem 3.1 by using the Robinson-Schensted-Knuth (RSK) algorithm. Given a tableau T on a subset S of $[n]$ and an integer k in $[n]$ but not in S, the basic operation of the RSK algorithm is the row insertion. We shall use $T \leftarrow k$ to denote the tableau obtained from T after inserting k. The insertion path of this operation, denoted by $I(T \leftarrow k)$, is the set of positions where the elements are bumped to the next row during the insertion operation along with the position where last bumped element is placed.

For example, here are a tableau T and the tableau $T \longleftarrow 5$,

$$T = \begin{array}{ccccc} 1 & 2 & 4 & 7 & 20 \\ 3 & 6 & 10 & 15 & \\ 9 & 13 & 14 & 17 & \\ 11 & & & & \end{array}, \qquad T \longleftarrow 5 = \begin{array}{ccccc} 1 & 2 & 4 & \mathbf{5} & 20 \\ 3 & 6 & \mathbf{7} & 15 & \\ 9 & \mathbf{10} & 14 & 17 & \\ 11 & \mathbf{13} & & & \end{array}.$$

The insertion path $I(T \leftarrow 5)$ is $\{(1,4),\ (2,3),\ (3,2),\ (4,2)\}$, where the elements are shown in boldface.

The following properties of insertion paths will be needed, see [9].

(1) When we insert k into T, the insertion path $I(T \leftarrow k)$ moves to the left. More precisely, if $(r,\ s),\ (r+1,\ t) \in I(T \leftarrow k)$ then $t \leq s$.

(2) Suppose that $k > j$. Then $I((T \leftarrow j) \leftarrow k)$ lies strictly to the right of $I(T \leftarrow j)$. More precisely, if $(r,\ t) \in I((T \leftarrow j) \leftarrow k)$, and $(r,\ s) \in I(T \leftarrow j)$, then $t > s$. Moreover, $I((T \leftarrow j) \leftarrow k)$ can never go below the last position of $I(T \leftarrow j)$. In other words, if $T \leftarrow j$ ends in the u-th row and $(T \leftarrow j) \leftarrow k$ ends in the v-th row, then $v \leq u$.

The RSK algorithm establishes a bijection between permutations π in S_n and pairs (P, Q) of SYTs of the same shape on $[n]$. We write $\pi \xrightarrow{\text{RSK}} (P, Q)$, where P is the insertion tableau and Q is the recording tableau.

We shall also use the Knuth equivalence of permutations. Recall that a Knuth transformation switches two adjacent elements a and c of a permutation π if any element b satisfying $a < b < c$ is located next to a or c. According to this definition, for $a < b < c$, there are two kinds of Knuth transformations:

$$\cdots\, b\, a\, c \cdots \longleftrightarrow \cdots\, b\, c\, a \cdots, \tag{3.3}$$

$$\cdots\, a\, c\, b \cdots \longleftrightarrow \cdots\, c\, a\, b \cdots. \tag{3.4}$$

Two permutations π and σ in S_n are called Knuth equivalent (denoted by $\pi \overset{K}{\sim} \sigma$) if one can be obtained from the other by a sequence of Knuth transformations. The following characterization is due to Knuth [5].

Theorem 3.2 *Two permutations in S_n are Knuth equivalent if and only if they have the same insertion tableau.*

We now proceed to present a combinatorial proof of Theorem 3.1. The main idea goes as follows. For a minimal permutation π in $\mathcal{M}_{2n+1,\,2i}$, in order to determine the insertion tableau of π, we first construct a permutation π' in S_{2n+1} such that $\pi' \overset{K}{\sim} \pi$ so that π and π' have the same insertion tableau. As will be seen, the insertion tableau P' corresponding to the first $n+i$ elements of π' can be easily obtained. Moreover, we show that the insertion tableau of π' is of shape $(n,\, n+1-k,\, k)$, where $1 \leq k \leq \min\{n+1-i, i\}$. So we establish a bijection between the set $\mathcal{M}_{2n+1,\,2i}$ and a set of tableaux of certain shapes.

Given a permutation π in $\mathcal{M}_{2n+1,2i}$, we can construct a permutation π' in S_{2n+1} which is Knuth equivalent to π.

Lemma 3.3 *Let $\pi = \pi_1\pi_2\cdots\pi_{2n+1}$ in $\mathcal{M}_{2n+1,\,2i}$. The permutation*

$$\pi' = \pi_1\pi_2\cdots\pi_{2i-1}\pi_{2i}\pi_{2i+2}\cdots\pi_{2n}\pi_{2i+1}\pi_{2i+3}\cdots\pi_{2n+1} \in S_{2n+1} \tag{3.5}$$

and π are Knuth equivalent.

Proof. For notational clarity, we write

$$\pi = \pi_1\pi_2\cdots\pi_{2i-1}\pi_{2i}\boxed{\pi_{2i+1}}\pi_{2i+2}\boxed{\pi_{2i+3}}\cdots\pi_{2n}\boxed{\pi_{2n+1}},$$

where the elements $\pi_{2i+1}, \ldots, \pi_{2n+1}$ are put into boxes. Let us recall condition (ii) of Theorem 1.1. That is, for $2 \leq j \leq 2n$, if j is an ascent of π, then $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142. It follows that $\pi_{j-1}\pi_j\pi_{j+1}$ is of the form bac and $\pi_j\pi_{j+1}\pi_{j+2}$ is of the form acb, where $a < b < c$.

We now consider the substring $\pi_{2i}\pi_{2i+1}\pi_{2i+2}\cdots\pi_{2n}\pi_{2n+1}$ of π. Since π is a minimal permutation in $\mathcal{M}_{2n+1,2i}$, it is easy to see that for $i \leq j \leq n-1$, $2j+1$ is an ascent of π.

This implies that $\pi_{2i}\pi_{2i+1}\pi_{2i+2}$ is of the form bac and for $i \le j \le n-2$, $\pi_{2j+3}\pi_{2j+4}\pi_{2j+5}$ is of the form acb. This allows us to apply a series of Knuth transformations to switch adjacent elements that are after π_{2i} and before π_{2n+1}. In this way, we may move π_{2i+2} to the right of π_{2i} and move π_{2n-1} to the left of π_{2n+1}, as demonstrated below:

$$\begin{aligned}
\pi &= \pi_1 \cdots \underbrace{\pi_{2i}\boxed{\pi_{2i+1}}\pi_{2i+2}}_{bac}\boxed{\pi_{2i+3}}\pi_{2i+4}\boxed{\pi_{2i+5}}\pi_{2i+6}\cdots\boxed{\pi_{2n-1}}\pi_{2n}\boxed{\pi_{2n+1}}\\
&\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\pi_{2i+2}\boxed{\pi_{2i+1}}\underbrace{\boxed{\pi_{2i+3}}\pi_{2i+4}\boxed{\pi_{2i+5}}}_{acb}\pi_{2i+6}\boxed{\pi_{2i+7}}\cdots\boxed{\pi_{2n-1}}\pi_{2n}\boxed{\pi_{2n+1}}\\
&\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\pi_{2i+2}\boxed{\pi_{2i+1}}\pi_{2i+4}\boxed{\pi_{2i+3}}\underbrace{\boxed{\pi_{2i+5}}\pi_{2i+6}\boxed{\pi_{2i+7}}}_{acb}\pi_{2i+8}\boxed{\pi_{2i+9}}\pi_{2i+10}\boxed{\pi_{2i+11}}\cdots\\
&\vdots\\
&\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\underbrace{\pi_{2i+2}\boxed{\pi_{2i+1}}\pi_{2i+4}}_{bac}\boxed{\pi_{2i+3}}\pi_{2i+6}\boxed{\pi_{2i+5}}\pi_{2i+8}\boxed{\pi_{2i+7}}\cdots\pi_{2n}\boxed{\pi_{2n-1}}\boxed{\pi_{2n+1}}.
\end{aligned}$$

For the substring $\pi_{2i+2}\pi_{2i+1}\pi_{2i+4}\cdots\pi_{2n-1}$ of the above permutation, by the second condition of Theorem 1.1, we see that $\pi_{2i+1} < \pi_{2i+2} < \pi_{2i+4}$. Hence $\pi_{2i+2}\pi_{2i+1}\pi_{2i+4}$ is of the form bac. Furthermore, for $i \le j \le n-4$, we have $\pi_{2j+3} < \pi_{2j+5} < \pi_{2j+6}$. It follows that $\pi_{2j+3}\pi_{2j+6}\pi_{2j+5}$ is of the form acb. Now we may apply a series of Knuth transformations to move π_{2i+4} to the right of π_{2i+2} and to move π_{2n-3} to the left of π_{2n-1}, see the illustration as follows:

$$\begin{aligned}
\pi &\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\pi_{2i+2}\pi_{2i+4}\boxed{\pi_{2i+1}}\underbrace{\boxed{\pi_{2i+3}}\pi_{2i+6}\boxed{\pi_{2i+5}}}_{acb}\pi_{2i+8}\boxed{\pi_{2i+7}}\cdots\pi_{2n}\boxed{\pi_{2n-1}}\boxed{\pi_{2n+1}}\\
&\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\pi_{2i+2}\pi_{2i+4}\boxed{\pi_{2i+1}}\pi_{2i+6}\boxed{\pi_{2i+3}}\underbrace{\boxed{\pi_{2i+5}}\pi_{2i+8}\boxed{\pi_{2i+7}}}_{acb}\pi_{2i+10}\boxed{\pi_{2i+9}}\cdots\\
&\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i}\pi_{2i+2}\underbrace{\pi_{2i+4}\boxed{\pi_{2i+1}}\pi_{2i+6}}_{bac}\boxed{\pi_{2i+3}}\pi_{2i+8}\boxed{\pi_{2i+5}}\pi_{2i+10}\cdots\pi_{2n}\boxed{\pi_{2n-3}}\boxed{\pi_{2n-1}}\boxed{\pi_{2n+1}}.
\end{aligned}$$

Continuing the above process, we eventually obtain π',

$$\begin{aligned}
\pi &\overset{\mathrm{K}}{\sim} \pi_1 \cdots \pi_{2i-1}\pi_{2i}\pi_{2i+2}\pi_{2i+4}\cdots\pi_{2n-2}\pi_{2n}\boxed{\pi_{2i+1}}\boxed{\pi_{2i+3}}\boxed{\pi_{2i+5}}\cdots\boxed{\pi_{2n-3}}\boxed{\pi_{2n-1}}\boxed{\pi_{2n+1}}\\
&= \pi'.
\end{aligned}$$

Thus π and π' are Knuth equivalent. This completes the proof. ∎

For example, for $n = 4$ and $i = 2$, consider the minimal permutation $\pi = 3\,2\,6\,5\,1\,8\,4\,9\,7$ in $\mathcal{M}_{9,4}$. The permutation π' is Knuth equivalent to π via the following transformations:

$$\pi = 3\,2\,6\,5\,\boxed{1}\,8\,\boxed{4}\,9\,\boxed{7}$$

$$\stackrel{\mathrm{K}}{\sim} 3\,2\,6\,5\,8 \boxed{1}\boxed{4}\,9\,\boxed{7}$$
$$\stackrel{\mathrm{K}}{\sim} 3\,2\,6\,5\,8 \boxed{1}\,9\,\boxed{4}\boxed{7}$$
$$\stackrel{\mathrm{K}}{\sim} 3\,2\,6\,5\,8\,9 \boxed{1}\boxed{4}\boxed{7}$$
$$= \pi'.$$

Corollary 3.4 *Let $\pi = \pi_1\pi_2\cdots\pi_{2n+1}$ be a minimal permutation in $\mathcal{M}_{2n+1,2i}$, and let π' be given as in* (3.5)*. Then the insertion tableau of π' is given by*

$$\Big(\big(\cdots((P' \leftarrow \pi_{2i+1}) \leftarrow \pi_{2i+3})\cdots\big) \leftarrow \pi_{2n-1}\Big) \leftarrow \pi_{2n+1}, \tag{3.6}$$

where

$$P' = \begin{array}{ccccccc} \pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \cdots & \pi_{2n} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1} & & & \end{array}. \tag{3.7}$$

Proof. Let $\tilde{\pi} = \pi_1\pi_2\cdots\pi_{2i-1}\pi_{2i}$. Since π is a minimal permutation in $\mathcal{M}_{2n+1,2i}$, we find that $\tilde{\pi}$ is a minimal permutation in $\mathcal{F}_i(2i)$. Thus $\tilde{\pi}$ is an alternating permutation, or a down-up permutation, in the sense that

$$\pi_1 > \pi_2 < \pi_3 > \pi_4 \cdots < \pi_{2i-1} > \pi_{2i}.$$

By condition (ii) of Theorem 1.1, for $2 \le j \le 2i-2$, if j is an ascent of $\tilde{\pi}$, then $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142. It follows that the insertion tableau of $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ takes the following form

$$\begin{array}{cc} \pi_j & \pi_{j+2} \\ \pi_{j-1} & \pi_{j+1} \end{array}.$$

Hence we deduce that the insertion tableau of $\tilde{\pi}$ is

$$\begin{array}{cccc} \pi_2 & \pi_4 & \cdots & \pi_{2i} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1} \end{array}. \tag{3.8}$$

It is easily checked that

$$\pi_{2i} < \pi_{2i+2} < \cdots < \pi_{2n-2} < \pi_{2n}.$$

Thus, these elements are placed to the right of π_{2i} after they are inserted into P'. This implies that the insertion tableau of the first $n+i$ elements of π' is given as follows

$$P' = \begin{array}{ccccccc} \pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \cdots & \pi_{2n} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1} & & & \end{array}. \tag{3.9}$$

It follows that the insertion tableau of π' can be obtained by

$$\Big(\big(\cdots((P' \leftarrow \pi_{2i+1}) \leftarrow \pi_{2i+3})\cdots\big) \leftarrow \pi_{2n-1}\Big) \leftarrow \pi_{2n+1}. \tag{3.10}$$

This completes the proof. ∎

Since $\pi \overset{K}{\sim} \pi'$, we see that (3.10) is also the insertion tableau of π. The following theorem gives the shape of (3.10).

Theorem 3.5 *For any $1 \leq i \leq n$, let $j = \min\{n-i+1, i\}$. Then P_{n-i} is of shape $(n, n-k+1, k)$ for some $1 \leq k \leq j$.*

Proof. Let
$$P_m = (\cdots(P' \leftarrow \pi_{2i+1}) \leftarrow \cdots) \leftarrow \pi_{2i+2m+1},$$
where $0 \leq m \leq n-i$. First we show that for $0 \leq m \leq n-i$, each P_m has three rows and the length of the first row of P_m is n.

We proceed by induction on m. We begin with the insertion of π_{2i+1} into P'. Since $2i-1$ and $2i$ are consecutive descents of π, we have
$$\pi_{2i+1} < \pi_{2i} < \pi_{2i-1}.$$
It follows that π_{2i+1} bumps some element α in the first row of P' that is weakly to the left of π_{2i} into the second row, and α bumps some element β in the second row of P' that is weakly to the left of π_{2i-1} into the third row. Hence P_0 has three rows and is of shape $(n,\ i,\ 1)$.

Assume that P_{m-1} has three rows with the first row containing n elements. Consider the insertion path of $P_{m-1} \leftarrow \pi_{2i+2m+1}$. Since $\pi_{2i+2m+1} > \pi_{2i+2m-1}$, the insertion path $I(P_{m-1} \leftarrow \pi_{2i+2m+1})$ does not go below the last position of $I(P_{m-2} \leftarrow \pi_{2i+2m-1})$. Thus P_m also has three rows. Furthermore, since $2i+2m$ is a descent of π, we have $\pi_{2i+2m+1} < \pi_{2i+2m}$. It follows that $\pi_{2i+2m+1}$ bumps some element in the first row of P_{m-1} that is weakly to the left of π_{2i+2m} into the second row. Thus the length of the first row of P_m remains to be n. So we conclude that P_{n-i} has three rows with the first row containing n elements.

Suppose that the shape of P_{n-i} is $(n, n-k+1, k)$ for some k. It remains to prove that $k \leq j$, where $j = \min\{n-i+1, i\}$. Since there are $n-i+1$ elements that are inserted into P', it follows that the third row of P_{n-i} contains at most $n-i+1$ elements. Hence $k \leq n-i+1$.

We now prove $k \leq i$. We need the following property: Assume that there exists some m, where $i-1 \leq m \leq n-i$, such that P_{m-1} has $i-1$ elements in the third row, and after $\pi_{2i+2m+1}$ is inserted, the last bumped element is placed at the position $(3, i)$. Then the second row of P_m contains i elements.

Assume to the contrary that there are more than i elements in the second row of P_m. Since P' has i elements in the second row, we infer that there exists some $u < m$ such that P_{u-1} contains less than i elements in the third row and the last position of $I(P_{u-1} \leftarrow \pi_{2i+2u+1})$ is at the end of the second row of P_u. So all the insertions of

$\pi_{2i+2u+3}, \ldots, \pi_{2i+2m+1}$ will stop at the second row. In other words, insertion path of P_m does not contain the position $(3, i)$, which is a contradiction. Thus, we deduce that P_m also has i elements in the second row.

Since the insertion path of an element moves left, it follows that $I(P_{m-1} \leftarrow \pi_{2i+2m+1})$ contains the position $(2, i)$. Moreover, we see that $(2, i)$ is at the end of the second row of P_m. Hence the insertions of $\pi_{2i+2m+3}, \ldots, \pi_{2n+1}$ will stop at the end of the second row. In other words, the number of elements on the third row remain unchanged. This implies that the number of elements in the third row is at most i. Thus we have $k \leq i$, and so the proof is complete. ∎

For example, for $n = 4$ and $i = 2$, we have $j = 2$. Let $\pi = 3\,2\,6\,5\,1\,8\,4\,9\,7$ be a minimal permutation in $\mathcal{M}_{9,4}$. The insertion tableau P_2 can be obtained by inserting 1, 4, 7 into the tableau

$$P' = \begin{array}{cccc} 2 & 5 & 8 & 9 \\ 3 & 6 & & \end{array} .$$

Notice that $I(P_0 \leftarrow 4)$ contains $(2, 2)$ and $(3, 2)$. The insertion of 7 stops at the second row, as shown below:

$$P_0 = P' \leftarrow 1 = \begin{array}{cccc} \mathbf{1} & 5 & 8 & 9 \\ \mathbf{2} & 6 & & \\ \mathbf{3} & & & \end{array} , \qquad P_1 = P_0 \leftarrow 4 = \begin{array}{cccc} 1 & \mathbf{4} & 8 & 9 \\ 2 & \mathbf{5} & & \\ 3 & \mathbf{6} & & \end{array} ,$$

and

$$P_2 = P_1 \leftarrow 7 = \begin{array}{cccc} 1 & 4 & \mathbf{7} & 9 \\ 2 & 5 & \mathbf{8} & \\ 3 & 6 & & \end{array} .$$

Conversely, we shall show that the SYTs as described in the above theorem are in one-to-one correspondence with minimal permutations in $\mathcal{M}_{2n+1,2i}$. Let $\mathcal{P}_{2n+1,\,k}$ be the set of SYTs of shape $(n,\, n+1-k,\, k)$, where $1 \leq k \leq n+1-k$. Let $j = \min\{n+1-i,\, i\}$, and let

$$\mathcal{T}_{2n+1,\,j} = \bigcup_{k=1}^{j} \mathcal{P}_{2n+1,\,k}.$$

Then we have the following correspondence.

Theorem 3.6 *There is a bijection between the set $\mathcal{T}_{2n+1,\,j}$ of SYTs and the set $\mathcal{M}_{2n+1,2i}$ of minimal permutations.*

Proof. By Theorem 2.2, we know that there is a bijection between 2-regular tableaux of shape $(n,\, n,\, i)/(i-1)$ and minimal permutations in $\mathcal{M}_{2n+1,2i}$. As we have proved in Theorem 3.5 a minimal permutation in $\mathcal{M}_{2n+1,2i}$ can be transformed to an SYT in $\mathcal{T}_{2n+1,j}$, we only need to show that for any $1 \leq k \leq j$, we can transform an SYT of shape

$(n,\, n+1-k,\, k)$ to a 2-regular skew tableau of shape $(n,n,i)/(i-1)$. Then we obtain a bijection between the set $\mathcal{T}_{2n+1,\,j}$ of SYTs and the set $\mathcal{M}_{2n+1,2i}$ of minimal permutations.

Let P be an SYT of shape $(n,\, n+1-k,\, k)$, where $1 \le k \le \min\{n-i+1, i\}$. We shall apply the inverse bumping process to the $n-i+1$ elements in the third row and at the end of the second row of P, so that we obtain a sequence of $n-i+1$ elements and a tableau of shape (n,i). Then we get a 2-regular skew tableau of shape $(n,n,i)/(i-1)$.

The detailed procedure to construct a 2-regular skew tableau from P is as follows. Let

$$S = \{P_{i_1 j_1},\, P_{i_2 j_2}, \ldots, P_{i_{n-i+1} j_{n-i+1}}\}$$

be the set consisting of the $n-i-k+1$ elements at the end of the second row and all the k elements in the third row of P, where $j_1 \ge j_2 \cdots \ge j_{n-i+1}$, and if $j_t = j_{t+1}$, then $i_t < i_{t+1}$. Intuitively, the elements in S are ordered from northeast to southwest.

For $1 \le s \le n-i+1$, we shall apply the inverse bumping procedure to $P_{i_s j_s}$ and recursively construct a sequence of skew tableaux Q^s. As will be shown, the last tableau Q^{n-i+1} is the 2-regular skew tableau that corresponds to P.

As the first step, we apply the inverse bumping procedure to $P_{i_1 j_1}$. Denote by P^1 the resulting tableau and suppose that $\pi_{u_1 v_1}$ is the element that is bumped out of P. Let Q^1 be the tableau obtained by putting $\pi_{u_1 v_1}$ on top of P^1_{1n}. It is easy to see that $\pi_{u_1 v_1} < P^1_{1n}$. From the construction of Q^1, we find that $Q^1_{2n} = P^1_{1n}$. It follows that $\pi_{u_1 v_1} < Q^1_{2n}$. Thus both the rows and columns of Q^1 are increasing.

We now assume that we have constructed the tableau Q^s. Apply the inverse bumping procedure to the element $P_{i_{s+1} j_{s+1}}$ in the tableau Q^s. Let P^{s+1} denote the resulting tableau. By the properties of the RSK algorithm, the entire inverse insertion path of $P_{i_{s+1} j_{s+1}}$ lies strictly to the left of that of $P_{i_s j_s}$. Hence the inverse insertion path of $P_{i_{s+1} j_{s+1}}$ does not extend to the right of the $(n-s+1)$-st column of Q^s. Furthermore, the element $\pi_{u_{s+1} v_{s+1}}$ that is bumped out of Q^s is smaller than the element $\pi_{u_s v_s}$ that is bumped out of Q^{s-1}. Thus we deduce that $\pi_{u_{s+1} v_{s+1}} < \pi_{u_s v_s}$.

Let Q^{s+1} be the tableau obtained by putting $\pi_{u_{s+1} v_{s+1}}$ on top of $P^{s+1}_{2(n-s)}$. Since $\pi_{u_{s+1} v_{s+1}}$ lies weakly to the left of $P^{s+1}_{2(n-s)}$ and $Q^{s+1}_{2(n-s)} = P^{s+1}_{2(n-s)}$, we see that $\pi_{u_{s+1} v_{s+1}} < Q^{s+1}_{2(n-s)}$. So we reach the conclusion that Q^{s+1} is strictly increasing along the rows and down the columns.

When the above procedure terminates, we get a tableau Q^{n-i+1}. We have shown that both the rows and columns of Q^{n-i+1} are increasing. On the other hand, it is easy to see that Q^{n-i+1} is of shape $(n,n,i)/(i-1)$. Thus Q^{n-i+1} is a 2-regular skew tableau of shape $(n,n,i)/(i-1)$ and the proof is complete. ∎

For example, for $n = 4$, $i = 2$, we have $j = 2$. Let

$$P = \begin{array}{cccc} 1 & 4 & 7 & 9 \\ 2 & 5 & \mathbf{8} & \\ \mathbf{3} & \mathbf{6} & & \end{array}$$

be an SYT in $\mathcal{T}_{9,2}$. Applying the inverse bumping procedure to the elements $8, 6$ and 3, we see that 7, 4, 1 are bumped out of P, so we obtain a 2-regular skew tableau of shape $(4,4,2)/(1)$, as given below:

$$Q^1 = \begin{array}{cccc} & & & 7 \\ 1 & 4 & 8 & 9 \\ 2 & 5 & & \\ \mathbf{3} & \mathbf{6} & & \end{array}, \qquad Q^2 = \begin{array}{cccc} & & 4 & 7 \\ 1 & 5 & 8 & 9 \\ 2 & 6 & & \\ \mathbf{3} & & & \end{array},$$

and

$$Q^3 = \begin{array}{cccc} & 1 & 4 & 7 \\ 2 & 5 & 8 & 9 \\ 3 & 6 & & \end{array}.$$

Thus 326518497 is the corresponding minimal permutation in $\mathcal{M}_{9,4}$.

We are now in a position to compute the number of SYTs of shape $(n,\, n+1-k,\, k)$. Let us recall the hook length formula. For a partition λ of n, the number of SYTs of shape λ is given by

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where u ranges over squares of the shape λ and $h(u)$ is the hook length of u, that is $h(u) = \lambda_i + \lambda'_j - i - j + 1$.

By the hook length formula, it is easy to show that $\binom{2n+1}{n-1}$ equals the number of SYTs of shape $(n,\, n,\, 1)$, see [8]. In fact, we have the following more general formula.

Theorem 3.7 *For $2 \leq k \leq [\frac{n+1}{2}]$, the number of SYTs of shape $(n,\, n+1-k,\, k)$ equals*

$$|\mathcal{P}_{2n+1,\,k}| = \frac{n-2k+2}{k-1}\binom{n-1}{k-2}\binom{2n+1}{n-1}. \tag{3.11}$$

Proof. We proceed by induction on k. When $k = 2$, by the hook length formula, it is easy to show that the number of SYTs of shape $(n,\, n-1,\, 2)$ is

$$(n-2)\binom{2n+1}{n-1}.$$

For $k > 2$, assume that (3.11) is true for $k-1$. Considering the hook lengths of shape $(n,\, n-k+1,\, k)$ and the hook lengths of shape $(n,\, n-k+2,\, k-1)$, we deduce that

$$\frac{|\mathcal{P}_{2n+1,\,k-1}|}{|\mathcal{P}_{2n+1,\,k}|} = \frac{(n-2k+2)(n-k+2)}{(n-2k+4)(k-1)}.$$

Thus the number of SYTs of shape $(n,\, n-k+1,\, k)$ equals

$$|\mathcal{P}_{2n+1,\,k}| = \frac{(n-2k+2)(n-k+2)}{(n-2k+4)(k-1)}\frac{(n-2k+4)}{k-2}\binom{n-1}{k-3}\binom{2n+1}{n-1}$$

$$= \frac{n-2k+2}{k-1}\binom{n-1}{k-2}\binom{2n+1}{n-1}.$$

This completes the proof. ∎

We are now ready to finish the combinatorial proof of Theorem 3.1. We use induction on i. Clearly, the theorem holds for $i = 1$. Assume that for $2 \leq i \leq n-1$,

$$\left|\mathcal{M}_{2n+1,\,2(i-1)}\right| = \binom{n-1}{i-2}\binom{2n+1}{n-1}.$$

By Corollary 2.4 and Theorem 3.7, we obtain

$$\begin{aligned}
|\mathcal{M}_{2n+1,\,2i}| &= \left|\mathcal{M}_{2n+1,\,2(i-1)}\right| + |\mathcal{P}_{2n+1,\,i}| \\
&= \left(\binom{n-1}{i-2} + \frac{n-2i+2}{i-1}\binom{n-1}{i-2}\right)\binom{2n+1}{n-1} \\
&= \binom{n-1}{i-1}\binom{2n+1}{n-1},
\end{aligned}$$

as desired. Thus the proof is complete by induction.

Note added in proof. The results in Section 2 have been independently obtained by Bouvel and Ferrari [2].

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