Restricted t-wise $\mathcal{L}$-intersecting families on set systems

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Abstract

Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ be a set of s nonnegative integers with $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ and let $t \geq 2$. A family $\mathcal{F}$ of subsets of an n-element set is called t-wise $\mathcal{L}$-intersecting if the cardinality of the intersection of any t distinct members in $\mathcal{F}$ belongs to $\mathcal{L}$. We give the following improvement to Füredi-Sudakov Theorem: Let $t \geq 3$ and $\mathcal{F}$ be a t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. Then for $|\bigcap_{F\in\mathcal{F}} F| < \lambda_1$,

$$|\mathcal{F}| = o(n^s);$$

for $|\bigcap_{F\in\mathcal{F}} F| \geq \lambda_1$ and n sufficiently large

$$|\mathcal{F}| \leq \frac{k+s-1}{s+1}\binom{n-\lambda_1}{s} + \sum_{i\leq s-1}\binom{n-\lambda_1}{i}.$$

We also give a sharp upper bound for the size of a k-uniform t-wise $\mathcal{L}$-intersecting family in the case $s = 1$.

Keywords : intersecting family, Erdös-Ko-Rado Theorem, non-trivial intersecting family, $S_r(n, k, 1)$-design.

1 Introduction

Let $\mathcal{F}$ be a family of subsets of an n-element set $[n] = \{1, 2, \ldots, n\}$ and let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ be a set of s non-negative integers. The family $\mathcal{F}$ is called uniform if all its members have the same size. $\mathcal{F}$ is t-wise $\mathcal{L}$-intersecting if the cardinality of the intersection of any t distinct members in $\mathcal{F}$ belongs to $\mathcal{L}$. Suppose that $\lambda_1 < \lambda_2 < \ldots < \lambda_s$. We call a family $\mathcal{F}$ *non-trivial* if $|\bigcap_{F \in \mathcal{F}} F| < \lambda_1$.

In 1949, Bose [4] obtained the following intersection theorem.

Theorem 1.1 *If $\mathcal{F}$ is a family of subsets of X satisfying $|E \cap F| = \lambda$ for every pair of distinct subsets $E, F \in \mathcal{F}$, then $|\mathcal{F}| \leq n$.*

About 30 years ago, Deza, Erdös, Frankl [3] proved the following two theorems.

Theorem 1.2 *Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ be a set of s non-negative integers. If $\mathcal{F}$ is a k-uniform t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$, then*

$$|\mathcal{F}| \leq (t-1) \prod_{i=1}^{s} \frac{(n - \lambda_i)}{(k - \lambda_i)}$$

for $n > 2^k k^3$.

Theorem 1.3 *Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ be a set of s non-negative integers. If $\mathcal{F}$ is a k-uniform t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$ and $|\mathcal{F}| > cn^{s-1}$ (c=c(k) is a constant depending on k), then*

$$(\lambda_2 - \lambda_1)|(\lambda_3 - \lambda_2)| \cdots |(\lambda_r - \lambda_{s-1})|(k - \lambda_s).$$

In 1975, Ray-Chaudhuri and Wilson [11] derived the next result.

Theorem 1.4 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{F}$ is a k-uniform $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n}{s}.$$

As to nonuniform $\mathcal{L}$-intersecting families, Frankl and Wilson [5] obtained in 1981 the following celebrated result.

Theorem 1.5 *Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers. If $\mathcal{F}$ is an $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

In 2004, Füredi and Sudakov [6] gave the following theorem:

Theorem 1.6 *Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ be a set of s non-negative integers. If $\mathcal{F}$ is a t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$, then for n sufficiently large*

$$|\mathcal{F}| \leq \frac{k+s-1}{s+1}\binom{n}{s} + \sum_{i \leq s-1} \binom{n}{i}.$$

The main objective of this paper is to give the following theorem which provides an improvement to Theorem 1.6. We will also give a non-uniform version of the Deze-Erdös-Frankl type theorem.

Theorem 1.7 *Let $t \geq 3$ and $\mathcal{F}$ be a t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. Then for $|\bigcap_{F \in \mathcal{F}} F| < \lambda_1$,*

$$|\mathcal{F}| = o(n^s);$$

for $|\bigcap_{F \in \mathcal{F}} F| \geq \lambda_1$ and n sufficiently large

$$|\mathcal{F}| \leq \frac{k+s-1}{s+1}\binom{n-\lambda_1}{s} + \sum_{i \leq s-1} \binom{n-\lambda_1}{i}.$$

2 λ-intersecting family

In this section, we consider the case for $\mathcal{L} = \{\lambda\}$. The following result lowers the threshold number in Theorem and shows the upper bound is sharp, where a $S_r(n, k, 1)$-design is a collection of k-blocks (i.e., k-subsets of $[n]$) such that every element in $[n]$ appears in exactly r blocks.

Theorem 2.1. Let λ be a non-negative integer and let $3 \leq k \leq n$, $t \geq 3$ and $\lambda + 1 \leq k$. If $\mathcal{L}$ is a family of k-subsets of an n-elements set such that $|A_1 \cap \cdots \cap A_t| = \lambda$ for any collection of t-distinct members of $\mathcal{F}$, then for $n > \lambda - 1 + \frac{k(k-2)}{t-1}$,

$$|\mathcal{F}| \leq \frac{(n-\lambda)(t-1)}{k-\lambda}.$$

Moreover, the equality holds if and only if there exists a $S_{t-1}(n-\lambda, k-\lambda, 1)$-design.

Proof. First let us consider the case that $k \geq \lambda + 2$. If $\lambda = 0$, let us consider $\mathcal{F}$ as a hypergraph. Since $|A_1 \cap \cdots \cap A_t| = 0$ for any $A_1, \ldots, A_t \in \mathcal{F}$, the degree of each vertex of $\mathcal{F}$ is at most $t - 1$. Since every edge of $\mathcal{F}$ has k vertices, it follows that

$$|\mathcal{F}|k \leq n(t-1).$$

Hence $|\mathcal{F}| \leq \frac{n(t-1)}{k}$.

Next suppose that $\lambda > 0$ but there exist $A_1, \ldots, A_{t-1} \in \mathcal{F}$ such that $|A_1 \cap \cdots \cap A_{t-1}| = \lambda$. Then, for any other $F \in \mathcal{F}$, we have $|F \cap A_1 \cap \cdots \cap A_{t-1}| = \lambda$. Therefore all other members of $\mathcal{F}$ should contain the set $A = A_1 \cap \cdots \cap A_{t-1}$. Define a new set system $\mathcal{F}' = \{F \backslash A | F \in \mathcal{F}\}$. Then $|\mathcal{F}'| = |\mathcal{F}|$ and any t distinct members of $\mathcal{F}'$ have empty intersection. Also note that members of $\mathcal{F}'$ are subsets of an $(n-\lambda)$-set. Therefore it follows from the above discussion that

$$|\mathcal{F}| = |\mathcal{F}'| \leq \frac{(n-\lambda)(t-1)}{k-\lambda}.$$

Now we assume that the intersection of any $t-1$ members of $\mathcal{F}$ has a size different from λ. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$. Then $|A_{i_1} \cap \cdots A_{i_{t-1}}| > \lambda$ for $1 \leq i_1 \leq \cdots \leq i_{t-1}\}$. Let $A_1 \cap \cdots \cap A_{t-2} = A$. Then $|A_i \cap A| > \lambda$ for every $A_i \in \mathcal{F} - \{A_1, \ldots, A_{t-2}\}$. We claim that $A_i \cap A \neq A_j \cap A$ for any $A_i, A_j \in \mathcal{F} - \{A_1, \ldots, A_{t-2}\}$ with $i \neq j$. For otherwise if $A_i \cap A = A_j \cap A$ for some $t - 1 \leq i \neq j \leq m$, then $|A_i \cap A_j \cap A| = |A_j \cap A| > \lambda$, contradicting the assumption. Let $\mathcal{B} = \{A_i \cap A : t-2 < i \leq m\}$. Then any two sets in $\mathcal{B}$ have exactly λ elements in common. By Theorem 1.1, $|\mathcal{B}| \leq |A|$. Since $|A| \leq k-1$, we obtain $m \leq |\mathcal{B}| + t - 2 \leq k + t - 3$. When $n \geq \lambda - 1 + \frac{k(k-2)}{t-1}$, we have

$$k + t - 3 \leq \frac{(t-1)(n-\lambda)}{k-\lambda}.$$

Hence

$$|\mathcal{F}| \leq \frac{(t-1)(n-\lambda)}{k-\lambda}$$

when $n \geq \lambda - 1 + \frac{k(k-2)}{t-1}$.

Clearly the equality holds if and only if there are λ vertices contained by all elements of $\mathcal{F}$ and the degrees of other vertices of $[n]$ are all $t-1$, the existence of such a family corresponds to a $S_{t-1}(n-\lambda, k-\lambda, 1)$-design. □

To show the sharpness of the bound in Theorem 2.1, we next show the existence of a $S_{t-1}(n-\lambda, k-\lambda, 1)$-design.

Lemma 2.2. If there exists an $S_r(n, k, 1)$-design, then

$$k | nr.$$

Proof. If there exists an $S_r(n, k, 1)$-design, then the number of block is $\frac{nr}{k}$, which must be an integer. Hence we have $k|nr$. □

For the convenience, we label the n elements by the elements in $Z_n = \{0, 1, 2, \cdots, n-1\}$. Under the addition of the addition group Z_n, we divide all $\binom{n}{k}$ k-sunsets (blocks) of Z_n into equivalence classes as follows: two blocks $\{v_1, \ldots, v_k\}$ and $\{u_1, \ldots, u_k\}$ are equivalent if and only if

$$v_1 - u_1 = v_2 - u_2 = \cdots = v_k - u_k \pmod{n},$$

i.e., $\{u_1, \ldots, u_k\} = \{v_1 + h, \ldots, v_k + h\}$ for some integer h, where the sum is taking modulo n.

Suppose that all $\binom{n}{k}$ k-blocks are divided into equivalence classes $C_1, C_2, \ldots, C_q$, where $C_1 = \{\{x, x+1, \ldots, x+k-1\} : x \in Z_n\}$.

Lemma 2.3. For each $1 \leq i \leq q$, there exists a constant c_i such that every element in $[n]$ appears in exactly c_i blocks from C_i and $k|c_i n$, i.e., each C_i is a $S_{c_i}(n, k, 1)$-design.

Proof . Let i be any integer such that $1 \leq i \leq q$. Let $\{v_1, v_2, \cdots, v_k\} \in C_i$ and $B_i = \{\{v_1 + j, v_2 + j, \cdots, v_k + j\} | 0 \leq j \leq n-1\}$ (allowing repeats if there are any). Then every element in $[n]$ appears in exactly k blocks from B_i (counting repeats) and B_i is made up by

$d_i \geq 1$ copies of C_i. It follows that element in $[n]$ appears in exactly $c_i = \frac{k}{d_i}$ blocks from C_i. Moreover, since each C_i is a $S_{c_i}(n, k, 1)$-design, it follows from Lemma 2.2 that $k|c_i n$. $\square$

Theorem 2.4. For $k \geq 1$ and $t \geq 1$, a $S_r(n, k, 1)$-design exists if and only if $k|nr$.

Proof. By Lemma 2.2, we need only to prove the sufficiency. Let $k|nr$. Let $C_1, C_2, \ldots, C_q$ be the equivalence classes that partition the set of all k-subsets (k-blocks) of Z_n, where $C_1 = \{\{x, x+1, \ldots, x+k-1\} : x \in Z_n\}$. First we consider the case $r \leq k$. Let r_1 be the smallest factor of r such that $k|nr_1$ (if $k|n$, then $r_1 = 1$) and let $r = r_1 r_2$. Take

$$C_1' = \cup_{j=0}^{r_2-1} \{\{0 + ik + j, 1 + ik + j, \cdots, k - 1 + ik + j\} | 0 \leq i \leq \frac{nr_1}{k} - 1\}.$$

Then it is easy to see that $C_1' \subseteq C_1$ and every element in $[n]$ appears in exactly r blocks from C_1'. Thus C_1' is a $S_r(n, k, 1)$-design.

Now, we assume $r > k$. Then there exists an integer h such that

$$\sum_{i=2}^{h} c_i \leq r \leq \sum_{i=2}^{h} c_i + c_1.$$

Let $r' = r - \sum_{i=2}^{h} c_i$. Then $0 \leq r' \leq k$. Since $k|nr$ and $k|c_i n$ for each i by Lemma 2.3, $k|nr'$. From the previous case, we see that C_1 has a subset C_1' such that every element in $[n]$ appears in exactly r' blocks from C_1'. Now take $C = C_1' \cup (\cup_{j=2}^{h} C_j)$. Then every element in $[n]$ appears in exactly r blocks from C and so C is a $S_r(n, k, 1)$-design. $\square$

3 Non-trivial $\mathcal{L}$-intersecting families

We begin this section with the following theorem.

Theorem 3.1. Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_s\}$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s$ and let $t \geq 3$. Suppose $\mathcal{F}$ is a t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. If for any $A_1, \ldots, A_{t-1} \in \mathcal{F}$, $|A_1 \cap \cdots \cap A_{t-1}| > \lambda_1$, then

$$|\mathcal{F}| = o(n^s).$$

Proof . Let $\mathcal{F}$ be a t-wise $\mathcal{L}$-intersecting family. We denote by $D = F_1 \cap \cdots \cap F_{t-2}$ a smallest intersection of among all possible choices of $t-2$ members of $\mathcal{F}$ and Y be a smallest element of $\mathcal{F}$ (i.e., $|Y| \leq |F|$ for any other $F \in \mathcal{F}$). Denote $|D| = m_1$ and $|Y| = m_2$. Clearly, $m_1 \leq m_2$. For each $A_i \in \mathcal{F}\backslash\{F_1, \ldots, F_{t-2}\}$, let $B_i = D \cap A_i$. Then these B_i's form an $\mathcal{F}$-intersecting family $\mathcal{B}$ on $[m_1]$ in which we allow multisets (repeats of same sets).

Now we sperate the multisets from $\mathcal{B}$. Let $\mathcal{B}' = \{B_i | \exists j \neq i, B_i = B_j\}$ and $\mathcal{B}^* = \mathcal{B}\backslash\mathcal{B}'$. Then $\mathcal{B}^*$ is an $\mathcal{L}$-intersecting family of $[m_1]$. By Theorem 1.5, we have

$$|\mathcal{B}^*| \leq \sum_{i=0}^{s} \binom{m_1}{s}.$$

Next, we estimate the size of $\mathcal{B}'$. First we claim that for each $B_i \in \mathcal{B}'$, $|B_i| = \lambda_j$ for some $j \geq 2$. Suppose $B_i \in \mathcal{B}'$. Since there exists a B_j such that $B_i = B_j$, it follows that $|B_i| = |B_i \cap B_j| = |F_1 \cap \cdots \cap F_{t-2} \cap A_i \cap A_j| \in \mathcal{L}$. By the condition, for any $A_1, \ldots, A_{t-1} \in \mathcal{F}$, $|A_1 \cap \cdots \cap A_{t-1}| > \lambda_1$, thus $|B_i| > \lambda_1$. Hence the claim holds. Let $\mathcal{B}''$ be the set which consists of distinct members of $\mathcal{B}'$. Let $\mathcal{B}_j = \{B \in \mathcal{B}'' || B| = \lambda_j\}$, where $j = 2, \ldots, s$. Then each $\mathcal{B}_j$ is a λ_j-uniform $\{\lambda_1, \ldots, \lambda_{j-1}\}$-intersecting family of subsets of $[m_1]$. It follows from Theorem 1.4 that

$$|\mathcal{B}_j| \leq \binom{m_1}{j-1}.$$

For each $B \in \mathcal{B}_j$, Let

$$\mathcal{F}(B) = \{F - B | F \cap A_1 \cap \cdots \cap A_{t-2} = B \quad \text{and} \quad F \in \mathcal{F}\}.$$

Then $\mathcal{F}(B)$ is a t-wise $\{0, \lambda_{j+1} - \lambda_j, \ldots, \lambda_s - \lambda_j\}$-intersecting family. By Theorem 1.6, we have when n is sufficiently large,

$$\mathcal{F}(B) \leq \frac{t+s-j}{s-j+2} \binom{n}{s-j+1} + (t-1) \sum_{i \leq s-j} \binom{n}{i}.$$

Thus

$$|\mathcal{B}'| \leq \sum_{j=2}^{s} \binom{m_1}{j-1} \left(\frac{t+s-j}{s-j+2} \binom{n}{s-j+1} + (t-1) \sum_{i \leq s-j} \binom{n}{i} \right).$$

It follows that

$$|\mathcal{F}| = |\mathcal{B}'| + |\mathcal{B}^*| + (t-2) \tag{3.1}$$

$$\leq \sum_{i=0}^{s} \binom{m_1}{i} + \sum_{j=2}^{s} \binom{m_1}{j-1} \left(\frac{t+s-j}{s-j+2} \binom{n}{s-j+1} + (t-1) \sum_{i \leq s-j} \binom{n}{i} \right) + t - 2.$$

It is not difficult to see that if $m_1 \leq n^{(\lambda_s+t-2)/(\lambda_s+t-1)}$, then $|\mathcal{F}| \leq o(n^s)$ and the theorem holds. So we assume $m_1 > n^{(\lambda_s+t-2)/(\lambda_s+t-1)}$. Since $m_2 \geq m_1$, we may assume $m_2 \geq \lambda_s$ for n large enough, that is, the size of every element of $\mathcal{F}$ is no less than λ_s. Let $f(x) = \binom{x}{\lambda_s}$ if $x \geq \lambda_s - 1$ and $f(x) = 0$ otherwise, one can see that the function is monotone and convex so we can apply Jensen's inequality. For $A \in \binom{[n]}{\lambda_s}$, let $d_A = |\{F \in \mathcal{F} | A \subset F\}|$ which is the number of subsets in $\mathcal{F}$ containing A. Then

$$\binom{|\mathcal{F}|}{t} = \sum_{A \in \binom{[n]}{\lambda_s}} \binom{d_A}{t}.$$

It follows from Jensen's inequality that

$$\frac{\binom{|\mathcal{F}|}{t}}{\binom{n}{\lambda_s}} \geq \frac{\sum_{A \in \binom{[n]}{\lambda_s}} \binom{d_A}{t}}{\binom{n}{\lambda_s}} \geq \binom{\frac{\sum_A d_A}{\binom{n}{\lambda_s}}}{t}.$$

For a fixed A, there are d_A subsets in $\mathcal{F}$ which contain A and for a fixed $F \in \mathcal{F}$ there are $\binom{|F|}{\lambda_s}$ λ_s-subsets A's in it, so we have

$$\binom{\frac{\sum_A d_A}{\binom{n}{\lambda_s}}}{t} = \binom{\frac{\sum_{f \in \mathcal{F}} \binom{|F|}{\lambda_s}}{\binom{n}{\lambda_s}}}{t} \geq \binom{|\mathcal{F}| \frac{\binom{m_2}{\lambda_s}}{\binom{n}{\lambda_s}}}{t}.$$

We may assume $|\mathcal{F}| > n^{2(\lambda_s+1)/(t+\lambda_s-1)}$, for otherwise we would have $|\mathcal{F}| \leq o(n^s)$. Since $m_2 \geq m_1 \geq n^{(\lambda_s+t-2)/(\lambda_s+t-1)}$, the quantity $|\mathcal{F}| \frac{\binom{m_2}{\lambda_s}}{\binom{n}{\lambda_s}}$ tends to infinity as $n \to \infty$. Hence we have

$$\frac{1}{\binom{n}{\lambda_s} \lambda_s!} |\mathcal{F}|^t \geq \binom{|\mathcal{F}| \frac{\binom{m_2}{\lambda_s}}{\binom{n}{\lambda_s}}}{t} \geq \frac{1}{t!} \left(|\mathcal{F}| \frac{\binom{m_2}{\lambda_s}}{\binom{n}{\lambda_s}} - t + 1 \right)^t \geq \frac{1-\epsilon}{t!} \left(|\mathcal{F}| \frac{\binom{m_2}{\lambda_s}}{\binom{n}{\lambda_s}} \right)^t$$

for any $\epsilon > 0$ if n is large enough. Thus

$$(1+o(1)) \binom{n}{\lambda_s}^{t-1} \geq \binom{m_2}{\lambda_s}^t$$

which implies that $m_1 \leq m_2 \leq (1+o(1)) n^{(t-1)/t}$. It follows from (3.1) that

$$|\mathcal{F}| = o(n^s).$$

□

Proof of Theorem 1.7 . Let $\mathcal{L} = \{\lambda_1, \dots, \lambda_s\}$ be a set of s non-negative integers with $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ and let $\mathcal{F}$ be a t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. If $|\bigcap_{F\in\mathcal{F}} F| \geq \lambda_1$, then take A to be a subset of $\bigcap_{F\in\mathcal{F}} F$ such that $|A| = \lambda_1$. Define a new set system $\mathcal{F}' = \{F \backslash A | F \in \mathcal{F}\}$. Then $|\mathcal{F}'| = |\mathcal{F}|$ and the result follows by applying Theorem 1.6 to $\mathcal{F}'$.

Now assume that $|\bigcap_{F\in\mathcal{F}} F| < \lambda_1$. Then there do not exist $A_1, \dots, A_{t-1} \in \mathcal{F}$ such that $|A_1 \cap \cdots \cap A_{t-1}| = \lambda_1$. For otherwise $A_1 \cap \cdots \cap A_{t-1}$ is contained in every set in $\mathcal{F}$ which implies that $|\bigcap_{F\in\mathcal{F}} F| \geq \lambda_1$. Thus by Theorem 3.1, the result follows. □

In fact, if we restrict $\mathcal{F}$ to be a k-uniform family, we can obtain the following.

Theorem 3.2. Let $\mathcal{L} = \{\lambda_1, \dots, \lambda_s\}$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s$ and let $t \geq 3$. Suppose $\mathcal{F}$ is a k-uniform t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. If for any $A_1, \dots, A_{t-1} \in \mathcal{F}$, $|A_1 \cap \cdots \cap A_{t-1}| > \lambda_1$, then

$$|\mathcal{F}| \leq (t-1)\left[\binom{n}{s} - \binom{n-k}{s}\right] + t - 2.$$

Proof . We can prove this result by modifying the proof for Theorem 3.1. Since $\mathcal{F}$ is k-uniform, $|Y|$ in the proof above is k. Thus we have $m_1 \leq k$. It follows from a result in [8] that if $\mathcal{F}$ is k-uniform t-wise $\mathcal{L}$-intersecting family, then $|\mathcal{F}| \leq (t-1)\binom{n}{s}$. By (3.1) in the previous proof, we have

$$|\mathcal{F}| \leq (t-1)\sum_{i=1}^{s}\binom{k}{i}\binom{n-k}{s-i} + t - 2 = (t-1)\left[\binom{n}{s} - \binom{n-k}{s}\right] + t - 2.$$

The proof is completed. □

As an immediate consequence, we have the following corollary.

Corollary 3.3. Let $\mathcal{L} = \{\lambda_1, \dots, \lambda_s\}$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s$ and let $t \geq 3$. Suppose $\mathcal{F}$ is a non-trivial k-uniform t-wise $\mathcal{L}$-intersecting family of subsets of $[n]$. Then

$$|\mathcal{F}| \leq (t-1)\left[\binom{n}{s} - \binom{n-k}{s}\right] + t - 2.$$

4 An asymptotical bound

In this section,, we give a Deza-Erdös-Frankl type theorem for nonuniform families. First, we give the following lemma.

Lemma 4.1. Let $\mathcal{L} = \{0, \lambda_2, \ldots, \lambda_s\}$ with $\lambda_2 \geq 2$ and $t \geq 3$. Let $\mathcal{F}$ be a t-wise and $\mathcal{L}$-intersecting family of subsets of $[n]$. If λ_2 dose not divide every $\lambda_3, \ldots, \lambda_s$, then

$$|\mathcal{F}| = o(n^s)$$

Proof . If for any fixed ε, there exists n_0 such that when $n > n_0$ there exists a element x of $[n]$ satisfying $\deg_{\mathcal{F}}(x) < \varepsilon\binom{n-1}{s-1}$, then denote $\mathcal{F}[x] = \{F - x : x \in F \quad \text{and} \quad F \in \mathcal{F}\}$ and $\mathcal{F}' = \mathcal{F} - \mathcal{F}[x]$. We can get $|\mathcal{F}[x]| \leq \varepsilon\binom{n-1}{s-1} - 1$ and $\mathcal{F}'$ is a nonuniform t-wise and $\{\lambda_2, \ldots, \lambda_s\}$-intersecting family of $[n]-\{x\}$. When $n = n_0$, $|\mathcal{F}| \leq 2^{n_0}$. Thus by the induction process, we obtain

$$|\mathcal{F}| < \varepsilon\binom{n}{s}$$

for $n > n_0 + 2^{n_0}$ sufficiently large. Since ε is arbitrarily, we obtain $|\mathcal{F}| = o(n^s)$.

Now we will prove that for any fixed ε, there exists n_0 such that when $n > n_0$ there exists a element x of $[n]$ satisfying $\deg_{\mathcal{F}}(x) < \varepsilon\binom{n-1}{s-1}$. Suppose that for any $x \in [n]$, $|\mathcal{F}[x]| \geq \varepsilon\binom{n-1}{s-1}$. Since $\varepsilon\binom{n-1}{s-1} > o(n^{s-1})$ for n is large enough. Theorem 3.1 implies that there exist $A_1, \ldots A_{t-1}$ such that the size of their intersection is λ_2. Denote $A(x) = A_1 \cap \cdots \cap A_{t-1}$. Since $|F \cap A(x)| \in \mathcal{L}$ for any $F \in \mathcal{F}$ and $|F \cap A(x)| \leq \lambda_2$, $|F \cap A(x)| = 0$ or λ_2. Hence each set of $\mathcal{F}$ is either disjoint from $A(x)$ or contains it. The same argument holds for every vertex of $[n]$. It follows that if $x \neq y$, then $A(x)$ and $A(y)$ are either disjoint or coincide. Thus $[n]$ can be partitioned into m/λ_2 blocks from $\mathcal{A} = \{A(x)\}$. It implies that λ_2 divides n. For $F \in \mathcal{F}$, denote $H(F) = \{A \in \mathcal{A} : A \subset F\}$ and $\mathcal{H} = \{H(F) : F \in \mathcal{F}\}$. Then $|\mathcal{H}| = |\mathcal{F}|$ and $\mathcal{H}$ is a nonuniform, t-wise and $\mathcal{L}' = \{\lambda_i/\lambda_2 : \lambda_i \in \mathcal{L} \quad \text{and} \quad \lambda_i/\lambda_2 \quad \text{is an integer}\}$-intersecting family on n/λ_2 vertices. Since λ_2 does not divide each of $\lambda_3, \ldots, \lambda_s$, we have $|\mathcal{L}'| < |\mathcal{L}| = s$. Consider $\mathcal{H}[x] = \{H(F) : x \in F\}$. Note that $\mathcal{H}[x]$ is $\mathcal{L}' - \{0\}$-intersecting family. Theorem 1 in [8] implies that

$$|\mathcal{F}[x]| \leq (t-1)\sum_{i=0}^{|\mathcal{L}'|-1}\binom{n}{i} < \varepsilon\binom{n-1}{s-1}.$$

for n large enough. It contradicts the assumption. Hence for any fixed ε, there exists n_0 such that there is a element x of $[n]$ satisfying $\deg_{\mathcal{F}}(x) < \varepsilon\binom{n-1}{s-1}$. The proof is completed.□

Theorem 4.2. Let $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ with $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_s$ and $t \geq 3$. Suppose that $\mathcal{F}$ is a t-wise and $\mathcal{L}$-intersecting family of subsets of $[n]$. If there exists i such that $\lambda_{i+1} - \lambda_i$ dose not divide $\lambda_{i+2} - \lambda_{i+1}$, then we have

$$|\mathcal{F}| = o(n^s),$$

for n sufficiently large.

Proof . First let us consider the case $\lambda_1 = 0$. We use induction on s. When $s = 3$, $\mathcal{L} = \{0, \lambda_2, \lambda_3\}$, if λ_2 does not divide $\lambda_3 - \lambda_2$, then λ_2 does not divide λ_3. By Lemma 4.1, we have $\mathcal{F} = o(n^s)$ for n large enough. Hence the result holds for $s = 3$. Suppose that the result is true for $s-1$ and suppose $\mathcal{F}$ is a nonuniform t-wise $\mathcal{L} = \{0, \lambda_2, \ldots, \lambda_s\}$-intersecting family. If λ_2 does not divide every λ_i for $i \geq 3$, then Lemma 4.1 implies that $\mathcal{F} \leq o(n^s)$. Thus we need only to consider $\lambda_2 | \lambda_i$ for $i \geq 3$. Similar to the argument in Lemma 4.1, we have $|\mathcal{F}| = |\mathcal{H}|$, where $\mathcal{H}$ is a nonuniform t-wise $\mathcal{L}' = \{0, 1, \lambda_3/\lambda_2, \ldots, \lambda_s/\lambda_2\}$-intersecting family on n/λ_2 elements. For any $x \in [n]$, $\mathcal{H}[x]$ is a nonuniform t-wise $\{0, \frac{\lambda_3-\lambda_2}{\lambda_2}, \ldots, \frac{\lambda_s-\lambda_2}{\lambda_2}\}$-intersecting family on $(n/\lambda_2 - 1)$ elements. Since if $\lambda_{i+1} - \lambda_i$ does not divide $\lambda_{i+2} - \lambda_{i+1}$, then $\frac{\lambda_{i+1}-\lambda_i}{\lambda_2}$ does not divide $\frac{\lambda_{i+2}-\lambda_{i+1}}{\lambda_2}$. It follows from the condition of theorem that there exists i such that $\frac{\lambda_{i+1}-\lambda_i}{\lambda_2}$ does not divide $\frac{\lambda_{i+2}-\lambda_{i+1}}{\lambda_2}$. Hence Lemma 4.1 implies $|\mathcal{H}[x]| = o(n^{s-1})$. It follows that

$$|\mathcal{H}| \leq \frac{n}{k} \cdot o(n^{s-1}) = o(n^s)$$

for n large enough. Up to now, we verified the result for $\lambda_1 = 0$. Next for $\lambda_1 \geq 1$, If $\mathcal{F}$ is non-trivial, then Theorem 3.1 gives us that $|\mathcal{F}| \leq o(n^s)$. If $\mathcal{F}$ is trivial, then the argument above yields that $|\mathcal{F}| \leq o(n^s)$ for n large enough. The proof is completed. □

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