

The Generating Function for the Dirichlet Series $L_m(s)$

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Abstract. The Dirichlet series $L_m(s)$ are of fundamental importance in number theory. Shanks defined the generalized Euler and class numbers in connection with the Dirichlet series, denoted by $\{s_{m,n}\}_{n \geq 0}$. We obtain a formula for the exponential generating function $s_m(x)$ of $s_{m,n}$, where m is an arbitrary positive integer. In particular, for $m > 1$, say, $m = bu^2$, where b is square-free and $u > 1$, we show that $s_m(x)$ can be expressed as a linear combination of the four functions $w(b, t) \sec(btx)(\pm \cos((b-p)tx) \pm \sin(ptx))$, where p is a nonnegative integer not exceeding b , $t|u^2$ and $w(b, t) = K_b t/u$ with K_b being a constant depending on b . Moreover, the Dirichlet series $L_m(s)$ can be easily computed from the generating function formula for $s_m(x)$. Finally, we show that the main ingredient in the formula for $s_{m,n}$ has a combinatorial interpretation in terms of the Λ -alternating augmented m -signed permutations defined by Ehrenborg and Readdy. More precisely, when m is square-free, this answers a question posed by Shanks concerning a combinatorial interpretation of the numbers $s_{m,n}$. When m is not square-free and $m = bu^2$, the numbers $K_b^{-1} s_{m,n}$ can be written as a linear combination of the number of Λ -alternating augmented bt -signed permutations with integer coefficients, where $t|u^2$.

Keywords: Dirichlet series, generalized Euler and class number, Λ -alternating augmented m -signed permutation, r -cubical lattice, Springer number

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1 Introduction

The Dirichlet series

$$L_m(s) = \sum_{\substack{l > 0 \\ \text{odd } l}} \left(\frac{-m}{l} \right) \frac{1}{l^s}, \quad (1.1)$$

where $(-m/l)$ is the Jacobi symbol, originate in the distribution of primes into arithmetic progressions, the class number of binary quadratic forms, as well as the distribution of the Legendre and Jacobi symbols. They play a crucial role in the computation of certain number-theoretic constants, see [3, 6, 12, 14]. Several approaches have been developed for the computation of $L_m(s)$, see, for example, Shanks [15, 16, 18].

The generalized Euler and class numbers were introduced by Shanks for the computation of the Dirichlet series $L_m(s)$ [15, 17]. These numbers are also related to derivative polynomials and Euler polynomials, see Hoffman [7] and Shanks [17].

In this paper, we obtain the generating functions for the generalized Euler and class numbers. Let us recall the definition of the generalized Euler and class numbers $s_{m,n}$ ($m \geq 1, n \geq 0$), introduced by Shanks,

$$s_{m,n} = \begin{cases} c_{m,\frac{n}{2}} & \text{if } n \text{ is even,} \\ d_{m,\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

where $c_{m,n}$ and $d_{m,n}$ are given by

$$c_{m,n} = (2n)!L_m(2n+1)(K_m\sqrt{m})^{-1} \left(\frac{\pi}{2m}\right)^{-2n-1}, \quad (1.2)$$

$$d_{m,n} = (2n-1)!L_{-m}(2n)(K_m\sqrt{m})^{-1} \left(\frac{\pi}{2m}\right)^{-2n}, \quad (1.3)$$

in which

$$K_m = \begin{cases} \frac{1}{2} & \text{if } m = 1, \\ 1 & \text{otherwise.} \end{cases} \quad (1.4)$$

Set

$$\begin{aligned} c_m(x) &= \sum_{n \geq 0} c_{m,n} \frac{x^{2n}}{(2n)!}, \\ d_m(x) &= \sum_{n \geq 1} d_{m,n} \frac{x^{2n-1}}{(2n-1)!}, \\ s_m(x) &= \sum_{n \geq 0} s_{m,n} \frac{x^n}{n!}. \end{aligned}$$

Clearly,

$$s_m(x) = c_m(x) + d_m(x).$$

By the definitions (1.2) and (1.3), we have

$$\begin{aligned} \sum_{n \geq 0} L_m(2n+1)x^{2n} &= \frac{K_m\pi\sqrt{m}}{2m} c_m\left(\frac{\pi}{2m}x\right), \\ \sum_{n \geq 1} L_{-m}(2n)x^{2n-1} &= \frac{K_m\pi\sqrt{m}}{2m} d_m\left(\frac{\pi}{2m}x\right). \end{aligned}$$

Therefore, if we set

$$\hat{L}_m(s) = \begin{cases} L_{-m}(s+1) & \text{if } s \text{ is odd;} \\ L_m(s+1) & \text{if } s \text{ is even,} \end{cases}$$

then

$$\sum_{s \geq 0} \hat{L}_m(s)x^s = \frac{K_m\pi\sqrt{m}}{2m} s_m\left(\frac{\pi}{2m}x\right). \quad (1.5)$$

It follows from (1.5) that $\hat{L}_m(s)$ is determined by $s_m(x)$. In other words, the generating function $s_m(x)$ leads to a quick way to compute $\hat{L}_m(s)$.

Consider $c_{m,n}$ and $d_{m,n}$ as entries of the infinite matrices C and D , respectively. Then the first column of C forms the sequence of class numbers in connection with primitive binary quadratic forms, and the first row of C forms the sequence of secant numbers, corresponding to up-down permutations, also called alternating permutations, of even length. Meanwhile, the first row of D forms the sequence of tangent numbers corresponding to up-down permutations of odd length. Recall that both secant numbers and tangent numbers are called Euler numbers. This is why the numbers $s_{m,n}$ are called generalized Euler and class numbers.

Shanks [17] found recurrence relations for $c_{m,n}$ and $d_{m,n}$ with respect to the index n , from which it follows that $c_{m,n}$ and $d_{m,n}$ are integers. For example, we have

$$\sum_{i=0}^n (-4)^i \binom{2n}{2i} c_{2,n-i} = (-1)^n$$

and

$$\sum_{i=0}^{n-1} (-4)^i \binom{2n-1}{2i} d_{2,n-i} = (-1)^{n-1}.$$

In fact, due to the well-known Euler product of the Dirichlet series $L_m(s)$ (see [8, 11]), it can be easily shown that $c_{m,n}$ and $d_{m,n}$ are positive.

Shanks [17] raised the following question: Whether all of the generalized Euler and class numbers may have some combinatorial interpretation? The combinatorial interpretations of $s_{m,n}$ for $m = 1, 2, 3, 4$ have been found. Let $(s_{m,n})_{n \geq 0}$ denote the conjoined sequence

$$(c_{m,0}, d_{m,1}, c_{m,1}, d_{m,2}, c_{m,2}, d_{m,3}, \dots).$$

For $m = 1$, the sequence $(1, 1, 1, 2, 5, 16, \dots)$ is listed as A000111 in Sloane [19], which is called the sequence of Euler numbers, enumerating the number of alternating permutations on $[n] = \{1, 2, \dots, n\}$.

For $m = 2$, the sequence $(1, 1, 3, 11, 57, 361, \dots)$ is numbered A001586 in [19], which is also called the sequence of Springer numbers which arise in the work of Springer on the theory of Weyl groups.

For $m = 3$, the sequence $(1, 2, 8, 46, 352, 3362, \dots)$ is listed as A007289 in [19], and we call it the sequence of Ehrenborg and Readdy numbers. The combinatorial interpretation of this sequence in terms of alternating 3-signed permutations has been given by Ehrenborg and Readdy [4].

For $m = 4$, a combinatorial interpretation of the sequence $(1, 4, 16, 128, 1280, 16384, \dots)$ has been given implicitly by Ehrenborg and Readdy [5] in terms of non-augmented André R -signed permutations on $[n]$ with $R = (4, 4, \dots, 4)$.

For $m \geq 5$, we shall give formulas for the generating function for $s_m(x)$. For the case when m is square-free, we shall give a combinatorial interpretation for $s_{m,n}$. In the general case, we see that the number $s_{m,n}$ or $2s_{m,n}$ is a linear combination with integer coefficients of some numbers with a combinatorial interpretation.

For $m = 1, 2, 3, 4$, it is known that

$$\begin{aligned} s_1(x) &= \sec x + \tan x, \\ s_2(x) &= \frac{\cos x + \sin x}{\cos 2x}, \\ s_3(x) &= \frac{\sin 2x + \cos x}{\cos 3x}, \\ s_4(x) &= \sec 4x + \tan 4x. \end{aligned}$$

From our formulas, we get the following expressions for $m = 5, 6, 7$,

$$\begin{aligned} s_5(x) &= \frac{\cos 4x + \sin x}{\cos 5x} + \frac{\cos 2x + \sin 3x}{\cos 5x}, \\ s_6(x) &= \frac{\cos 5x + \sin x}{\cos 6x} + \frac{\cos x + \sin 5x}{\cos 6x}, \\ s_7(x) &= \frac{\cos 3x + \sin 4x}{\cos 7x} + \frac{\cos x + \sin 6x}{\cos 7x} - \frac{\cos 5x + \sin 2x}{\cos 7x}. \end{aligned}$$

This paper is organized as follows. In Section 2, we compute the generating function $s_m(x)$ when m is square-free, while in Section 3 we consider the case when m is not square-free. Section 4 is devoted to the combinatorial interpretation of the numbers $s_{m,n}$ in terms of m -signed permutations as introduced by Ehrenborg and Readdy when m is square-free.

2 Computation for $s_m(x)$ when m is square-free

In this section, we compute the generating function $s_m(x)$ when $m > 1$ and m is square-free. For $0 \leq p \leq m$, we adopt the following notation introduced in [4],

$$\Lambda_{m,p}(x) := \frac{\cos((m-p)x) + \sin(px)}{\cos(mx)}. \quad (2.6)$$

When m is square-free, we shall not encounter the case that m is a multiple of 4. We shall have three formulas for $s_m(x)$ depending on the residue of m modulo 4.

Theorem 2.1 *Assume that m is square-free and $m = 4t + 3$. Then*

$$s_m(x) = \sum_{k=1}^t \binom{k}{m} \Lambda_{m,4k}(x) + \sum_{k=t+1}^{2t+1} \binom{k}{m} \Lambda_{m,2m-4k}(x). \quad (2.7)$$

Theorem 2.2 *Assume that m is square-free and $m = 4t + 1$. Then*

$$s_m(x) = \sum_{k=1}^t \binom{k}{m} \Lambda_{m,m-4k}(x) - \sum_{k=t+1}^{2t} \binom{k}{m} \Lambda_{m,4k-m}(x). \quad (2.8)$$

Theorem 2.3 *Assume that m is square-free and $m = 4t + 2$. Then*

$$s_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\frac{-m}{k} \right) \Lambda_{m,k}(x). \quad (2.9)$$

To prove the above theorems, let us first recall the following formula of $L_m(2n+1)$ obtained by Shanks [15, 17].

Lemma 2.4 *Suppose that $m > 1$ and m is square-free. Then $L_m(2n+1)$ can be expressed as a linear combination of the Fourier series $S_{2n+1}(x)$. More precisely,*

$$L_m(2n+1) = \frac{2}{\sqrt{m}} \sum_k \epsilon_k S_{2n+1}(y_k),$$

where the Jacobi symbols ϵ_k and rational numbers y_k are uniquely determined by m , and $S_{2n+1}(x)$ is defined by

$$S_{2n+1}(x) = \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^{2n+1}}.$$

Furthermore, we have

$$c_m(x) = \frac{1}{\cos(mx)} \sum_k \epsilon_k \cos(mx(1 - 4y_k)).$$

In fact, Shanks has given an explicit procedure to determine the constants ϵ_k and y_k . To compute ϵ_k and y_k , we use the definition (1.1) of the series $L_m(s)$ and express the Jacobi symbol $\left(\frac{-m}{l}\right)$ as a linear combination of sines according to the following expansion, see [9].

Proposition 2.5 *Assume that l is odd and m satisfies the following two conditions: $m \equiv 1 \pmod{4}$ or $m \equiv 8$ or $12 \pmod{16}$ and $p^2 \nmid m$ for any odd prime p . Then we have*

$$\left(\frac{m}{l}\right) = \frac{1}{\sqrt{m}} \sum_{r=1}^{|m|} \left(\frac{m}{r}\right) e^{2\pi i lr/|m|}. \quad (2.10)$$

In particular, when $m \equiv 3 \pmod{4}$, we have $-m \equiv 1 \pmod{4}$ and we can use the above expansion for $\left(\frac{-m}{l}\right)$. Similarly, when $m \equiv 1 \pmod{4}$, we see that $-4m \equiv 12 \pmod{16}$ so that we can compute $\left(\frac{-4m}{l}\right)$ by using the above formula. Finally, when $m \equiv 2 \pmod{4}$, we find $-4m \equiv 8 \pmod{16}$ so that we can compute $\left(\frac{-4m}{l}\right)$. Note that when l is odd, we have

$$\left(\frac{-4m}{l}\right) = \left(\frac{-m}{l}\right).$$

Thus, the Jacobi symbol $\left(\frac{-m}{l}\right)$ can be determined by the above procedure for $m > 1$.

On the other hand, Shanks [15, 17] provided the following formula for $L_{-m}(2n)$.

Lemma 2.6 *Suppose that $m > 1$ and m is square-free. Then $L_{-m}(2n)$ can be expressed as a linear combination of the Fourier series $C_{2n}(x)$. To be more specific,*

$$L_{-m}(2n) = \frac{2}{\sqrt{m}} \sum_k \epsilon'_k C_{2n}(y'_k),$$

where the Jacobi symbols ϵ'_k and rational numbers y'_k are uniquely determined by m , and $C_{2n}(x)$ is defined by

$$C_{2n}(x) = \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^{2n}}.$$

Moreover,

$$d_m(x) = \frac{1}{\cos(mx)} \sum_k \epsilon'_k \sin(mx(1 - 4y'_k)).$$

Similarly, Shanks has shown how to compute the constants ϵ'_k and y'_k . In order to compute ϵ'_k and y'_k , recall that

$$L_{-m}(s) = \sum_{\substack{l>0 \\ \text{odd } l}} \left(\frac{m}{l}\right) \frac{1}{l^s},$$

where the Jacobi symbol $\left(\frac{m}{l}\right)$ can be expressed as a linear combination of cosines, resorting to Proposition 2.5.

For the case $m \equiv 3 \pmod{4}$, we find $4m \equiv 12 \pmod{16}$ so that we can use the above expansion for $\left(\frac{4m}{l}\right)$. When $m \equiv 1 \pmod{4}$, we can also compute $\left(\frac{m}{l}\right)$ by using the above expansion. Finally, when $m \equiv 2 \pmod{4}$, we see that $4m \equiv 8 \pmod{16}$ and $\left(\frac{4m}{l}\right)$ can be determined in the same manner. Note that when l is odd, the following relation holds

$$\left(\frac{4m}{l}\right) = \left(\frac{m}{l}\right).$$

Thus, the Jacobi symbol $\left(\frac{m}{l}\right)$ can be determined for $m > 1$.

Keep in mind that m is assumed to be square-free. Set

$$\hat{c}_m(x) = \cos(mx)c_m(x), \quad \hat{d}_m(x) = \cos(mx)d_m(x), \quad \hat{s}_m(x) = \cos(mx)s_m(x).$$

Proof of Theorem 2.1. Since $m \equiv 3 \pmod{4}$, by using the expansion (2.10) for $\left(\frac{-m}{l}\right)$ and $\left(\frac{4m}{l}\right)$, we have

$$\epsilon_k = \left(\frac{k}{m}\right), \quad y_k = \frac{k}{m}, \quad \epsilon'_k = \left(\frac{m}{k}\right), \quad y'_k = \frac{k}{4m},$$

which imply that

$$\begin{aligned} L_m(2n+1) &= \frac{2}{\sqrt{m}} \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) S_{2n+1}\left(\frac{k}{m}\right), \\ L_{-m}(2n) &= \frac{2}{\sqrt{m}} \sum_{\text{odd } k < m} \left(\frac{m}{k}\right) C_{2n}\left(\frac{k}{4m}\right). \end{aligned}$$

Therefore, by Lemmas 2.4 and 2.6, we obtain

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{k=1}^{(m-1)/2} \binom{k}{m} \cos(m-4k)x + \sum_{\text{odd } k < m} \binom{m}{k} \sin(m-k)x.$$

Suppose that $m = 4t + 3$. It follows that

$$\begin{aligned} \hat{c}_m(x) &= \sum_{k=1}^t \binom{k}{m} \cos(m-4k)x + \sum_{k=t+1}^{2t+1} \binom{k}{m} \cos(m-4k)x \\ &= \sum_{k=1}^t \binom{k}{m} \cos(m-4k)x + \sum_{k=t+1}^{2t+1} \binom{k}{m} \cos(4k-m)x, \\ \hat{d}_m(x) &= \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \binom{m}{k} \sin(m-k)x. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \hat{s}_m(x) &= \sum_{k=1}^t \left(\binom{k}{m} \cos(m-4k)x + \binom{m}{m-4k} \sin(4k)x \right) \\ &\quad + \sum_{k=t+1}^{2t+1} \left(\binom{k}{m} \cos(4k-m)x + \binom{m}{4k-m} \sin(2m-4k)x \right). \end{aligned}$$

It remains to verify that

$$\binom{k}{m} = \binom{m}{m-4k} \quad (2.11)$$

for $1 \leq k \leq t$ and

$$\binom{k}{m} = \binom{m}{4k-m} \quad (2.12)$$

for $t+1 \leq k \leq 2t+1$.

As for (2.11), since both m and $m-4k$ are odd positive numbers, if they are coprime, then by the law of quadratic reciprocity, we find

$$\begin{aligned} \binom{m}{m-4k} &= - \binom{m-4k}{m} = - \binom{-4k}{m} = - \binom{-4}{m} \binom{k}{m} \\ &= - \binom{-1}{m} \binom{4}{m} \binom{k}{m} = \left(\frac{2}{m} \right)^2 \binom{k}{m} = \binom{k}{m}. \end{aligned}$$

Otherwise, suppose that the greatest common divisor of m and $m-4k$ is greater than 1, i.e., $(m, m-4k) > 1$. From the definition of the Jacobi symbol, it is clear that

$$\binom{m}{m-4k} = 0.$$

We wish to show that

$$\binom{k}{m} = 0.$$

Since $(m, m - 4k) | m$, $(m, m - 4k) | (m - 4k)$ and $m - (m - 4k) = 4k$, we deduce that

$$(m, m - 4k) | 4k. \quad (2.13)$$

But the fact that m is odd implies that $((m, m - 4k), 4) = 1$. Thus from (2.13) it follows that $(m, m - 4k) | k$. So we conclude that $(m, m - 4k) | (m, k)$ and $(m, k) > 1$, which yields

$$\left(\frac{k}{m}\right) = \left(\frac{m}{m - 4k}\right) = 0.$$

Similarly, (2.12) can be checked via the following steps if $4k - m$ and m are coprime,

$$\begin{aligned} \left(\frac{m}{4k - m}\right) &= \left(\frac{4k - m}{m}\right) = \left(\frac{-(m - 4k)}{m}\right) = \left(\frac{-1}{m}\right) \left(\frac{m - 4k}{m}\right) \\ &= -\left(\frac{m - 4k}{m}\right) = -\left(\frac{-4k}{m}\right) = \left(\frac{k}{m}\right). \end{aligned}$$

For the case $(m, 4k - m) > 1$, by the definition of the Jacobi symbol, it is evident that

$$\left(\frac{m}{4k - m}\right) = 0.$$

In this case, we claim that

$$\left(\frac{k}{m}\right) = 0.$$

Since $(m, 4k - m) | m$, $(m, 4k - m) | (4k - m)$ and $m + (4k - m) = 4k$, we deduce that

$$(m, 4k - m) | 4k. \quad (2.14)$$

Because m is odd, we see that $((m, 4k - m), 4) = 1$. Thus from (2.14) it follows that $(m, 4k - m) | k$. Therefore, we obtain that $(m, 4k - m) | (m, k)$ and $(m, k) > 1$, which implies

$$\left(\frac{k}{m}\right) = \left(\frac{m}{4k - m}\right) = 0.$$

So we find that

$$\begin{aligned} s_m(x) &= \sum_{k=1}^t \left(\frac{k}{m}\right) \frac{\cos(m - 4k)x + \sin(4k)x}{\cos mx} \\ &\quad + \sum_{k=t+1}^{2t+1} \left(\frac{k}{m}\right) \frac{\cos(4k - m)x + \sin(2m - 4k)x}{\cos mx}. \end{aligned}$$

This completes the proof. ■

Proof of Theorem 2.2. Under the condition $m \equiv 1 \pmod{4}$, by applying the expansion (2.10) to $\left(\frac{-4m}{l}\right)$ and $\left(\frac{m}{l}\right)$, we get

$$\epsilon_k = \left(\frac{-m}{k}\right), \quad y_k = \frac{k}{4m}, \quad \epsilon'_k = \left(\frac{k}{m}\right), \quad y'_k = \frac{k}{m}.$$

It follows that

$$L_m(2n+1) = \frac{2}{\sqrt{m}} \sum_{\text{odd } k < m} \left(\frac{-m}{k}\right) S_{2n+1}\left(\frac{k}{4m}\right),$$

$$L_{-m}(2n) = \frac{2}{\sqrt{m}} \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) C_{2n}\left(\frac{k}{m}\right).$$

In view of Lemmas 2.4 and 2.6, we find

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{\text{odd } k < m} \left(\frac{-m}{k}\right) \cos(m-k)x + \sum_{k=1}^{(m-1)/2} \left(\frac{k}{m}\right) \sin(m-4k)x.$$

Writing $m = 4t + 1$, we obtain

$$\begin{aligned} \hat{d}_m(x) &= \sum_{k=1}^t \left(\frac{k}{m}\right) \sin(m-4k)x + \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \sin(m-4k)x \\ &= \sum_{k=1}^t \left(\frac{k}{m}\right) \sin(m-4k)x - \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \sin(4k-m)x, \\ \hat{c}_m(x) &= \sum_{\substack{k=1 \\ \text{odd } k}}^{4t-1} \left(\frac{-m}{k}\right) \cos(m-k)x. \end{aligned}$$

Consequently,

$$\begin{aligned} \hat{s}_m(x) &= \sum_{k=1}^t \left(\left(\frac{k}{m}\right) \sin(m-4k)x + \left(-\frac{m}{m-4k}\right) \cos(4k)x \right) \\ &\quad + \sum_{k=t+1}^{2t} \left(-\left(\frac{k}{m}\right) \sin(4k-m)x + \left(-\frac{m}{4k-m}\right) \cos(2m-4k)x \right). \end{aligned}$$

Finally, we need to show that

$$\left(\frac{k}{m}\right) = \left(-\frac{m}{m-4k}\right) \tag{2.15}$$

for $1 \leq k \leq t$ and

$$-\left(\frac{k}{m}\right) = \left(-\frac{m}{4k-m}\right) \tag{2.16}$$

for $t+1 \leq k \leq 2t$.

To verify (2.15), we shall consider two cases. If $(m, m-4k) = 1$, since both m and $m-4k$ are odd, we may employ the law of quadratic reciprocity to deduce that

$$\begin{aligned} \left(-\frac{m}{m-4k}\right) &= \left(\frac{-1}{m-4k}\right) \left(\frac{m}{m-4k}\right) = \left(\frac{m}{m-4k}\right) = \left(\frac{m-4k}{m}\right) = \left(\frac{-4k}{m}\right) \\ &= \left(\frac{-4}{m}\right) \left(\frac{k}{m}\right) = \left(\frac{-1}{m}\right) \left(\frac{2}{m}\right)^2 \left(\frac{k}{m}\right) = \left(\frac{k}{m}\right). \end{aligned}$$

For the case $(m, m - 4k) > 1$, by definition,

$$\left(-\frac{m}{m-4k}\right) = 0.$$

We wish to show that

$$\left(\frac{k}{m}\right) = 0.$$

Since $(m, m - 4k) | m$, $(m, m - 4k) | (m - 4k)$ and $m - (m - 4k) = 4k$, we get

$$(m, m - 4k) | 4k. \quad (2.17)$$

However, the fact that m is odd implies that $((m, m - 4k), 4) = 1$. Thus, from (2.17) we see that $(m, m - 4k) | k$. It follows that $(m, m - 4k) | (m, k)$ and $(m, k) > 1$. Hence

$$\left(\frac{k}{m}\right) = \left(-\frac{m}{m-4k}\right) = 0.$$

To prove (2.16), we also consider two cases. If m and $4k - m$ are coprime, then we have

$$\begin{aligned} -\left(-\frac{m}{4k-m}\right) &= -\left(\frac{-1}{4k-m}\right) \left(\frac{m}{4k-m}\right) = \left(\frac{m}{4k-m}\right) = \left(\frac{4k-m}{m}\right) \\ &= \left(\frac{-(m-4k)}{m}\right) = \left(\frac{-1}{m}\right) \left(\frac{m-4k}{m}\right) = \left(\frac{-4k}{m}\right) = \left(\frac{k}{m}\right). \end{aligned}$$

If $(m, 4k - m) > 1$, then we get

$$-\left(-\frac{m}{4k-m}\right) = 0.$$

We shall show that

$$\left(\frac{k}{m}\right) = 0.$$

Since $(m, 4k - m) | m$, $(m, 4k - m) | (4k - m)$ and $m + (4k - m) = 4k$, we find

$$(m, 4k - m) | 4k. \quad (2.18)$$

But m is odd, we see that $((m, 4k - m), 4) = 1$. From (2.18) it follows that $(m, 4k - m) | k$. This implies that $(m, 4k - m) | (m, k)$ and $(m, k) > 1$. Consequently,

$$\left(\frac{k}{m}\right) = -\left(-\frac{m}{4k-m}\right) = 0.$$

Thus we obtain

$$\begin{aligned} s_m(x) &= \sum_{k=1}^t \left(\frac{k}{m}\right) \frac{\cos(4k)x + \sin(m-4k)x}{\cos mx} \\ &\quad - \sum_{k=t+1}^{2t} \left(\frac{k}{m}\right) \frac{\cos(2m-4k)x + \sin(4k-m)x}{\cos mx}. \end{aligned}$$

This completes the proof. ■

Proof of Theorem 2.3. Since $m \equiv 2 \pmod{4}$, by using the expansion (2.10) for $\left(\frac{-4m}{t}\right)$ and $\left(\frac{4m}{t}\right)$, we obtain

$$\epsilon_k = \left(\frac{-m}{k}\right), \quad y_k = \frac{k}{4m}, \quad \epsilon'_k = \left(\frac{m}{k}\right), \quad y'_k = \frac{k}{4m}.$$

Hence

$$\begin{aligned} L_m(2n+1) &= \frac{2}{\sqrt{m}} \sum_{\text{odd } k < m} \left(\frac{-m}{k}\right) S_{2n+1}\left(\frac{k}{4m}\right), \\ L_{-m}(2n) &= \frac{2}{\sqrt{m}} \sum_{\text{odd } k < m} \left(\frac{m}{k}\right) C_{2n}\left(\frac{k}{4m}\right). \end{aligned}$$

By Lemmas 2.4 and 2.6, we see that

$$\hat{s}_m(x) = \hat{c}_m(x) + \hat{d}_m(x) = \sum_{\text{odd } k < m} \left(\frac{-m}{k}\right) \cos(m-k)x + \sum_{\text{odd } k < m} \left(\frac{m}{k}\right) \sin(m-k)x.$$

Writing $m = 4t + 2$, since

$$\left(\frac{4t+2}{2t+1}\right) = 0,$$

we obtain

$$\hat{s}_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\left(\frac{-m}{k}\right) \cos(m-k)x + \left(\frac{m}{m-k}\right) \sin(k)x \right).$$

Finally, it follows from $-k \equiv m - k \pmod{m}$ that

$$\left(\frac{-m}{k}\right) = \left(\frac{m}{m-k}\right) \tag{2.19}$$

for $1 \leq k \leq 4t + 1$. Hence we conclude that

$$s_m(x) = \sum_{\substack{k=1 \\ \text{odd } k}}^{4t+1} \left(\frac{-m}{k}\right) \frac{\cos(m-k)x + \sin(k)x}{\cos mx}.$$

This completes the proof. ■

For $m = 5, 6, 7$, the generating functions $s_m(x)$ have been given in the Introduction.

3 Computation for $s_m(x)$ when m is not square-free

In this section, we obtain an expression for $s_m(x)$ for the case when m is not square-free. Assume that m can be divided by a square u^2 and $u > 1$. Recall that the constant K_b is given by (1.4).

Theorem 3.1 *Suppose that $m = bu^2$, where b is square-free and $u > 1$. Then we can express $s_m(x)$ as a linear combination of the four functions*

$$w(b, t) \sec(btx) (\pm \cos((b-p)tx) \pm \sin(ptx)),$$

where p is a nonnegative integer not exceeding b , $t|u^2$ and $w(b, t) = K_b t/u$.

The idea of the proof is to establish two recursive relations (3.21) and (3.22) between $s_{m,n}$ and $s_{b,n}$. Then we express $s_m(x)$ as a linear combination of the functions $c_b(tx)$ and $d_b(tx)$ by considering the two cases according to whether there exist odd prime factors u_i of u with residues 3 modulo 4. Since b is square-free, $c_b(tx)$ and $d_b(tx)$ can be evaluated by using the formulas in the previous section.

Proof. Let us start with the following relation given by Shanks [15]

$$L_m(s) = L_b(s) \prod_{u_i|u} \left(1 - \left(\frac{-b}{u_i} \right) \frac{1}{u_i^s} \right), \quad (3.20)$$

where the product ranges over odd primes u_i (if any) that divide u . To be precise, in case there are no odd prime factors, the empty product is defined to be 1. From the definitions (1.2) and (1.3) it follows that

$$c_{m,n} = K_b u (u^2)^{2n} \prod_i \left(u_i^{2n+1} - \left(\frac{-b}{u_i} \right) \right) \left(\prod_i \frac{1}{u_i} \right)^{2n+1} c_{b,n}, \quad (3.21)$$

$$d_{m,n} = K_b u (u^2)^{2n-1} \prod_i \left(u_i^{2n} - \left(\frac{b}{u_i} \right) \right) \left(\prod_i \frac{1}{u_i} \right)^{2n} d_{b,n}. \quad (3.22)$$

For the purpose of computing $s_m(x)$ for the case when m is not square-free, we need to consider the two cases according to whether there exist $u_i \equiv 3 \pmod{4}$ among the k odd factors u_1, u_2, \dots, u_k of u .

Case 1: $u_i \equiv 1 \pmod{4}$ for $1 \leq i \leq k$. In this case, we see that

$$\left(\frac{-b}{u_i} \right) = \left(\frac{b}{u_i} \right).$$

Suppose that among the factors u_1, u_2, \dots, u_k there are k_1 primes u_1, u_2, \dots, u_{k_1} satisfying $\left(\frac{b}{u_i} \right) = 1$ for $1 \leq i \leq k_1$, k_2 primes $u_{k_1+1}, u_{k_1+2}, \dots, u_{k_1+k_2}$ satisfying $\left(\frac{b}{u_{k_1+j}} \right) = -1$ for $1 \leq j \leq k_2$, and k_3 primes $u_{k_1+k_2+1}, u_{k_1+k_2+2}, \dots, u_{k_1+k_2+k_3}$ satisfying $\left(\frac{b}{u_{k_1+k_2+l}} \right) = 0$ for $1 \leq l \leq k_3$, where $k_1 + k_2 + k_3 = k$. From (3.21) it is clear that

$$c_{m,n} = K_b u (u^2)^{2n} \frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \times \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} c_{b,n}. \quad (3.23)$$

Let

$$f_c = \frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \times \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}}.$$

In this notation, (3.23) can be rewritten as

$$c_{m,n} = K_b u (u^2)^{2n} f_c c_{b,n},$$

which implies that

$$c_m(x) = K_b u \sum_{n \geq 0} f_c c_{b,n} \frac{(u^2 x)^{2n}}{(2n)!}. \quad (3.24)$$

Since

$$\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) = \prod_{i=1}^{k_1} u_i^{2n+1} - \sum_{i=1}^{k_1} \left(\prod_{j=1, j \neq i}^{k_1} u_j \cdots u_{i-1} u_{i+1} \cdots u_{k_1} \right)^{2n+1} + \cdots + (-1)^{k_1}$$

and

$$\prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1) = \prod_{j=1}^{k_2} u_{k_1+j}^{2n+1} + \sum_{j=1}^{k_2} \left(\prod_{i=1, i \neq j}^{k_2} u_{k_1+i} \cdots u_{k_1+i-1} u_{k_1+i+1} \cdots u_{k_1+k_2} \right)^{2n+1} + \cdots + 1,$$

we can expand f_c as follows

$$\begin{aligned} f_c = & 1 + \sum_{j=1}^{k_2} \frac{1}{u_{k_1+j}^{2n+1}} + \cdots + \frac{1}{\prod_{j=1}^{k_2} u_{k_1+j}^{2n+1}} - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n+1}} - \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \frac{1}{u_i^{2n+1} u_{k_1+j}^{2n+1}} + \cdots \\ & - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n+1} \prod_{j=1}^{k_2} u_{k_1+j}^{2n+1}} + \cdots + \frac{(-1)^{k_1}}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}}. \end{aligned} \quad (3.25)$$

Plugging (3.25) into (3.24), we find that $c_m(x)$ is a linear combination of the functions $c_b(tx)$, where $t|u^2$ and the coefficient of $c_b(tx)$ equals $K_b t/u$.

Similarly, we have

$$d_{m,n} = K_b u (u^2)^{2n-1} \frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \times \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} d_{b,n}. \quad (3.26)$$

Let

$$f_d = \frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \times \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}},$$

then (3.26) can be rewritten as

$$d_{m,n} = K_b u (u^2)^{2n-1} f_d d_{b,n},$$

which leads to the following relation

$$d_m(x) = K_b u \sum_{n \geq 1} f_d d_{b,n} \frac{(u^2 x)^{2n-1}}{(2n-1)!}. \quad (3.27)$$

Again we may expand f_d as follows

$$\begin{aligned}
f_d = & 1 + \sum_{j=1}^{k_2} \frac{1}{u_{k_1+j}^{2n}} + \cdots + \frac{1}{\prod_{j=1}^{k_2} u_{k_1+j}^{2n}} - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n}} - \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \frac{1}{u_i^{2n} u_{k_1+j}^{2n}} + \cdots \\
& - \sum_{i=1}^{k_1} \frac{1}{u_i^{2n} \prod_{j=1}^{k_2} u_{k_1+j}^{2n}} + \cdots + \frac{(-1)^{k_1}}{\prod_{i=1}^{k_1+k_2} u_i^{2n}}. \tag{3.28}
\end{aligned}$$

Substituting (3.28) into (3.27), we find that $d_m(x)$ is a linear combination of the functions $d_b(tx)$, where $t|u^2$ the coefficient of $d_b(tx)$ equals $K_b t/u$. Furthermore, we see that $c_m(x)$ and $d_m(x)$ have the same coefficients for the linear combinations. In other words, the relation for $c_m(x)$ and $c_b(tx)$ is still valid after changing $c_m(x)$ and $c_b(tx)$ to $d_m(x)$ and $d_b(tx)$, respectively. Therefore, in this case, $s_m(x)$ can be expressed as a sum of the functions $w(b, t)(c_b(tx) + d_b(tx))$, where $t|u^2$ and $w(b, t) = K_b t/u$.

Case 2: Among the k primes u_1, u_2, \dots, u_k , there exists q primes $u_{i_1}, u_{i_2}, \dots, u_{i_q}$ with residue 3 modulo 4. To compute $s_m(x)$, we first consider the case when $q = 1$, and then argue that the case $q > 1$ can be dealt with in the same way.

Since $q = 1$, we assume that the first $k - 1$ odd primes u_1, u_2, \dots, u_{k-1} satisfy $u_i \equiv 1 \pmod{4}$ for $1 \leq i \leq k - 1$, and suppose that the last prime u_k satisfies $u_k \equiv 3 \pmod{4}$, or

$$\left(\frac{-b}{u_k}\right) = -\left(\frac{b}{u_k}\right).$$

We now define the indices k_1, k_2 and k_3 as in Case 1 except that $k_1 + k_2 + k_3 = k - 1$ since there are $k - 1$ primes with residue 1 modulo 4. This leads us to consider two subcases according to whether $\left(\frac{-b}{u_k}\right)$ equals 0. Keep in mind that we have $q = 1$ in these two subcases.

On the one hand, in order to use the two recursive relations (3.21) and (3.22) between $s_{m,n}$ and $s_{b,n}$, we may assume that $\left(\frac{-b}{u_k}\right) = 0$. Therefore, the term u_k on the denominator and the same term on the numerator cancel each other in (3.21). This argument also applies to the relation (3.22). In other words, there is no need to consider the occurrence of the term u_k^{2n+1} in (3.21) and the terms u_k^{2n} in (3.22). In this sense, it remains to consider the other $k - 1$ primes u_1, u_2, \dots, u_{k-1} such that $u_i \equiv 1 \pmod{4}$ for $1 \leq i \leq k - 1$. By the argument in Case 1, we see again that $s_m(x)$ can be expressed as a sum of the terms $w(b, t)(c_b(tx) + d_b(tx))$, where $t|u^2$ and $w(b, t) = K_b t/u$.

On the other hand, we should consider the case when $\left(\frac{-b}{u_k}\right) = 1$ or $\left(\frac{-b}{u_k}\right) = -1$. Since the proofs for these two cases are similar, we only give the proof for the case $\left(\frac{-b}{u_k}\right) = -1$. From (3.21) it follows that

$$\begin{aligned}
c_{m,n} &= K_b u(u^2)^{2n} \frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \times \frac{u_k^{2n+1} + 1}{u_k^{2n+1}} c_{b,n} \\
&= K_b u(u^2)^{2n} \left(\frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} + \frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{u_k^{2n+1} \prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \right) c_{b,n}.
\end{aligned}$$

Let

$$g_b(x) = K_b u \sum_{n \geq 0} \left(\frac{\prod_{i=1}^{k_1} (u_i^{2n+1} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n+1} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n+1}} \right) c_{b,n} \frac{(u^2 x)^{2n}}{(2n)!},$$

then we have

$$c_m(x) = g_b(x) + \frac{1}{u_k} g_b(x/u_k). \quad (3.29)$$

In fact, by the argument in Case 1, we see that $g_b(x)$ is a linear combination of the functions $c_b(tx)$, where $t|u^2$ and the coefficient of $c_b(tx)$ equals $K_b t/u$.

Similarly, by (3.22) we find

$$\begin{aligned} d_{m,n} &= K_b u (u^2)^{2n-1} \frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} \times \frac{u_k^{2n} - 1}{u_k^{2n}} d_{b,n} \\ &= K_b u (u^2)^{2n-1} \left(\frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} - \frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{u_k^{2n} \prod_{i=1}^{k_1+k_2} u_i^{2n}} \right) d_{b,n}. \end{aligned}$$

Let

$$h_b(x) = K_b u \sum_{n \geq 1} \left(\frac{\prod_{i=1}^{k_1} (u_i^{2n} - 1) \prod_{j=1}^{k_2} (u_{k_1+j}^{2n} + 1)}{\prod_{i=1}^{k_1+k_2} u_i^{2n}} \right) d_{b,n} \frac{(u^2 x)^{2n-1}}{(2n-1)!}.$$

We get

$$d_m(x) = h_b(x) - \frac{1}{u_k} h_b(x/u_k). \quad (3.30)$$

Again, from the reasoning in Case 1 it follows that $h_b(x)$ is a linear combination of the functions $d_b(tx)$, where $t|u^2$ and the coefficient of $d_b(tx)$ equals $K_b t/u$. Combining the equations (3.29) and (3.30) yields the following relation

$$s_m(x) = g_b(x) + h_b(x) + \frac{1}{u_k} (g_b(x/u_k) - h_b(x/u_k)).$$

Thus we obtain that $s_m(x)$ can also be expressed as a linear combination of the functions $w(b, t)(\pm c_b(tx) \pm d_b(tx))$, where $t|u^2$ and $w(b, t) = K_b t/u$.

Finally, as mentioned before we shall show that the justification for the above two subcases can be applied to the case for $q > 1$. Let us give an example for $q = 2$. Suppose that u_{k-1} and u_k are the last two primes such that $u_{k-1} \equiv 3 \pmod{4}$, $u_k \equiv 3 \pmod{4}$, and $\left(\frac{-b}{u_{k-1}}\right) = -1$, $\left(\frac{-b}{u_k}\right) = -1$. Using the functions $g_b(x)$ and $h_b(x)$ as given before, we can deduce the following formula for $s_m(x)$,

$$\begin{aligned} s_m(x) &= g_b(x) + h_b(x) + \frac{1}{u_{k-1}} (g_b(x/u_{k-1}) - h_b(x/u_{k-1})) + \frac{1}{u_k} (g_b(x/u_k) - h_b(x/u_k)) \\ &\quad + \frac{1}{u_{k-1} u_k} (g_b(x/u_{k-1} u_k) + h_b(x/u_{k-1} u_k)). \end{aligned}$$

Clearly, $s_m(x)$ can also be written as a linear combination of the functions $w(b, t)(\pm c_b(tx) \pm d_b(tx))$, where $t|u^2$ and $w(b, t) = K_b t/u$. For the other conditions on u_{k-1} and u_k , the computation can be done in the same spirit.

In summary, $s_m(x)$ can be expressed as a linear combination of the functions $w(b, t)(\pm c_b(tx) \pm d_b(tx))$, where $t|u^2$ and $w(b, t) = K_b t/u$. Since b is square-free, $c_b(tx)$ and $d_b(tx)$ can be written as a linear combination of the functions $\sec(btx) \cos((b-p)tx)$ and $\sec(btx) \sin(ptx)$, respectively, by using the formulas in the previous section. This completes the proof. \blacksquare

Here we give three examples corresponding to the above three cases. For Case 1, suppose that $m = 3(5 \times 13)^2 = 3(65)^2 = 3 \times 4225 = 12675$. Then we have

$$s_{12675}(x) = 65s_3(4225x) - 5s_3(325x) + 13s_3(845x) - s_3(65x),$$

where

$$s_3(x) = \sec(3x)(\sin 2x + \cos x).$$

For the first subcase of Case 2, suppose that $m = 6(5 \times 3)^2 = 3(15)^2 = 6 \times 225 = 1350$. Then we get

$$s_{1350}(x) = 15s_6(225x) - 3s_6(45x),$$

where

$$s_6(x) = \sec(6x)(\cos 5x + \sin x) + \sec(6x)(\cos x + \sin 5x).$$

For the second subcase of Case 2, assume that $m = 225 = (5 \times 3)^2$. We find

$$\begin{aligned} 2s_{225}(x) = & 15(c_1(225x) + d_1(225x)) - 3(c_1(45x) + d_1(45x)) + 5(c_1(75x) - d_1(75x)) \\ & - (c_1(15x) - d_1(15x)), \end{aligned}$$

where

$$c_1(x) = \sec x, \quad d_1(x) = \tan x.$$

4 A combinatorial interpretation for $s_{m,n}$

In this section, we aim to give a combinatorial interpretation of $s_{m,n}$ when m is square-free based on its generating function formula $s_m(x)$. Let us recall the known combinatorial interpretations of $s_{m,n}$ when $m = 1, 2, 3, 4$.

For $m = 1$, $(s_{1,n})_{n \geq 0}$ is called the sequence of Euler numbers. Let E_n be the n -th Euler number, that is, the number of up-down permutations on $[n] = \{1, 2, \dots, n\}$, which are also called snakes of type A_{n-1} by Arnol'd [2]. The following generating function is due to André [1]:

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Note that Springer also gave an explanation of the Euler numbers in terms of the irreducible root system A_{n-1} and derived the generating function of André in this context.

For $m = 2$, the sequence $(s_{2,n})_{n \geq 0}$ turns out to be the sequence of Springer numbers of the irreducible root system B_n [20]. Purtill [13] has found an interpretation of this sequence. Let P_n be the n -th entry of this sequence, whereas Purtill used the notation E_n^\pm . He has shown that P_n equals the number of André signed permutations on $[n]$. On the other hand, it has been shown by Arnol'd [2] that S_n also counts the number of snakes of type B_n . Hoffman [7] has derived the generating function of the number of snakes of type B_n by giving a direct combinatorial proof.

For $m = 3$, the sequence $(s_{3,n})_{n \geq 0}$ has been studied by Ehrenborg and Readdy [4]. Let F_n denote the n -th Ehrenborg and Readdy number, which was denoted by $|ER_n|$, see Hoffman [7]. It has been shown that F_n equals the number Λ -alternating augmented 3-signed permutations on $[n]$. Meanwhile, Hoffman [7] presented another combinatorial interpretation of the sequence in the case $m = 3$ in terms of ER_n -snakes in the spirit of the snakes of type A_{n-1} and B_n .

For $m = 4$, the sequence $(s_{4,n})_{n \geq 0}$ has also been studied by Ehrenborg and Readdy [5, P. 719]. In fact, they introduced the concept of non-augmented André R -signed permutations on a finite set N with $|N| = n$, where $R = (r_i)_{i \in N}$ is a vector indexed by N . They have shown that the exponential generating function for the number of non-augmented André R -signed permutations with the n -tuple $R = (r, r, \dots, r)$ is $\sec(rx) + \tan(rx)$. Therefore, for the n -tuple $R = (4, 4, \dots, 4)$, it follows that the number of non-augmented André R -signed permutation is counted by $s_{4,n}$.

We proceed to give a combinatorial interpretation for $s_{m,n}$ when m is square-free based on the generating function $s_m(x)$. More precisely, in this case we shall show that $s_m(x)$ can be expressed by the generating function $\Lambda_{m,p}(x)$ for the numbers $\Lambda_{m,p,n}$ defined by Ehrenborg and Readdy [4]. Consequently, when m is square-free, the numbers $s_{m,n}$ can be written as a linear combination of the numbers $\Lambda_{m,p,n}$ with coefficients being 1 or -1 . With the aid of the combinatorial interpretation of the generating function due to Ehrenborg and Readdy, we see that $s_{m,n}$ can be explained in terms of maximal chains in the m -cubical lattice.

When m is not square-free, write $m = bu^2$. The numbers $s_{m,n}$ or $2s_{m,n}$ can be written as a linear combination of the numbers $\Lambda_{bt,pt,n}$ with integer coefficients, where $t|u^2$ and $0 \leq p \leq b$. However, these integer coefficients are obtained from the generating function of $s_m(x)$. It would be interesting to find a combinatorial explanation of these coefficients when m is not square-free.

We now need some definitions. Ehrenborg and Readdy [4] defined a poset called the Sheffer poset, which can be viewed as a generalization of the binomial poset introduced by Stanley [21]. As an important example, they studied the r -cubical lattice, which is a set of ordered r -tuples (A_1, A_2, \dots, A_r) of subsets from an infinite set I together with the reverse inclusion order and with a minimum element $\hat{0}$ adjoined. Note that the r -cubical lattice has been studied by Metropolis, Rota, Strehl and White [10]. Ehrenborg and Readdy further generalized the concept of R -labelings to linear edge-labelings. By considering the set of maximal chains in the interval $[\hat{0}, \hat{1}]$ on the Hasse diagram of the r -cubical lattice, they derived a formula for the number of Λ -alternating augmented r -signed permutations.

To describe the definition of Λ -alternating augmented r -signed permutations, Ehrenborg and Readdy constructed a linear edge-labeling on the Hasse diagram of the r -cubical lattice. To be more specific, for an edge corresponding to the cover relation $A < B$ with $A \neq \hat{0}$, let (i, j)

denote its label where i equals the unique index such that $A_i \neq B_i$ and j takes the singleton element in $A_i - B_i$. Let G be the label of the edge corresponding to $\hat{0} < A$, which is called the special element by Ehrenborg and Readdy. Then an augmented r -signed permutation is a list $(G, (i_1, j_1), (i_2, j_2), \dots, (i_n, j_n))$, where $i_1, i_2, \dots, i_n \in [r]$, and (j_1, j_2, \dots, j_n) forms a permutation on $[n]$. In other words, r -signed permutations are permutations on $[n]$ in which each element is assigned one of r signs.

To define the descent set of r -signed permutations, let Λ be the set of such labels of the edges on the Hasse diagram of the r -cubical lattice. As is easily seen,

$$\Lambda = ([r] \times [n]) \cup \{G\}.$$

Let p be an integer such that $0 \leq p \leq r$. For fixed positive integers r , n and p , we can define a linear order on Λ which satisfies the following conditions

$$(i, j) <_{\Lambda} G \Rightarrow i \leq r - p, \quad (4.31)$$

and

$$(i, j) >_{\Lambda} G \Rightarrow i > r - p, \quad (4.32)$$

where (i, j) is the label of the edge corresponding to the cover relation $A < B$ such that A covers $\hat{0}$, and G is the special element. For the remaining labels, we may arrange them in the lexicographic order. The descent set of an augmented r -signed permutation (g_0, g_1, \dots, g_n) is defined as the set $\{k: g_{k-1} >_{\Lambda} g_k\}$, where $g_0 = G$ and $g_k = (i_k, j_k)$ for $1 \leq k \leq n$. Therefore, for an Λ -alternating augmented r -signed permutation, that is, permutation having descent set $\{2, 4, 6, \dots\}$, it is necessary to have the condition $(i_1, j_1) >_{\Lambda} G$, or $i_1 > r - p$. In other words, the labels above G in this ordering are those whose first coordinate may take p possible values from the set $\{r - p + 1, r - p + 2, \dots, r\}$.

Recall that $\Lambda_{r,p,n}$ denotes the number of Λ -alternating augmented r -signed permutations. Since there is a one-to-one correspondence between the set of maximal chains in the r -cubical lattice and the set of augmented r -signed permutations, $\Lambda_{r,p,n}$ also equals the number of maximal chains with descent set $\{2, 4, 6, \dots\}$ in the r -cubical lattice for which the first non-special edge has label (i_1, j_1) with $r - p + 1 \leq i_1 \leq r$. Here the first non-special edge is defined as the edge corresponding to the cover relation $A < B$ such that A covers $\hat{0}$. Based on this observation, Ehrenborg and Readdy derived the following generating function for $\Lambda_{r,p,n}$

$$\Lambda_{r,p}(x) = \sum_{n \geq 0} \Lambda_{r,p,n} \frac{x^n}{n!} = \frac{\cos((r-p)x) + \sin(px)}{\cos(rx)}. \quad (4.33)$$

In view of the above formula (4.33) and the generating functions for the generalized Euler and class numbers $s_{m,n}$ in the previous sections, for any square-free m we can give a combinatorial interpretation of $s_{m,n}$ in terms of the augmented m -signed permutations, or equivalently, in terms of the maximal chains in the Hasse diagram of the m -cubical lattice.

When m is square-free, as is shown in Section 2, $s_{m,n}$ is a linear combination of the numbers $\Lambda_{m,p,n}$ with integer coefficients being 1 or -1 . For example, when $m = 7$, since

$$s_7(x) = \frac{\cos 3x + \sin 4x}{\cos 7x} + \frac{\cos x + \sin 6x}{\cos 7x} - \frac{\cos 5x + \sin 2x}{\cos 7x},$$

we obtain that

$$s_{7,n} = \Lambda_{7,6,n} - \Lambda_{7,2,n} + \Lambda_{7,4,n}.$$

When m is not square-free, write $m = bu^2$ and $u > 1$. By Theorem (3.1), $K_b^{-1}s_{m,n}$ can be written as a linear combination of the numbers $\Lambda_{bt,pt,n}$ with integer coefficients. Recall that $K_1^{-1} = 2$ and $K_b^{-1} = 1$ for $b \neq 1$. For instance, we have

$$2s_9(x) = 3(\sec(9x) + \tan(9x)) + (\sec(3x) - \tan(3x)),$$

which implies

$$2s_{9,n} = 3\Lambda_{9,9,n} + (-1)^n \Lambda_{3,3,n}.$$

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