Higher Order Log-Concavity in Euler's Difference Table

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Abstract. For $0 \leq k \leq n$, let e_n^k be the entries in Euler's difference table and let $d_n^k = e_n^k/k!$. Dumont and Randrianarivony showed e_n^k equals the number of permutations on $[n]$ whose fixed points are contained in $\{1, 2, \ldots, k\}$. Rakotondrajao found a combinatorial interpretation of the number d_n^k in terms of k-fixed-points-permutations of $[n]$. We show that for any $n \geq 1$, the sequence $\{d_n^k\}_{0 \leq k \leq n}$ is both 2-log-concave and reverse ultra log-concave.

Keywords: log-concavity, 2-log-concavity, reverse ultra log-concavity, Euler's difference table

Classification: 05A20; 05A10

1 Introduction

Euler's difference table $(e_n^k)_{0 \leq k \leq n}$ is defined by $e_n^n = n!$ and

$$e_n^{k-1} = e_n^k - e_{n-1}^{k-1}, \tag{1.1}$$

for $1 \leq k \leq n$. Dumont and Randrianarivony [5] showed that e_n^k equals the number of permutations on $[n]$ whose fixed points are contained in $\{1, 2, \ldots, k\}$. Clarke, Han and Zeng [4] gave a combinatorial interpretation of a q-analogue of Euler's difference table. This combinatorial interpretation was further extended by Faliharimalala and Zeng [7, 8] to the wreath product $C_\ell \wr S_n$ of the cyclic group and the symmetric group.

It is easily seen from the recurrence relation (1.1) that $k!$ divides e_n^k. Thus the number $d_n^k = e_n^k/k!$ is always an integer. Rakotondrajao [13] has shown that d_n^k equals the number of k-fixed-points-permutations of $[n]$, where a permutation $\pi \in \mathfrak{S}_n$ is called a k-fixed-points-permutation if there are no fixed points in the last $n-k$ positions and the first k elements are in different cycles. Based on this combinatorial explanation, Rakotondrajao [14] has found bijective proofs for the following recurrence relations

$$d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k, \tag{1.2}$$

and

$$d_n^k = nd_{n-1}^k - d_{n-2}^{k-1}, \tag{1.3}$$

where $0 \le k \le n-1$ and $d_n^n = 1$.

Recently, Eriksen, Freij and Wästlund [6] generalized the above recurrence relations to λ-colored permutations. By equating the right hand sides of (1.2) and (1.3), and changing the index from $n-1$ to n, we obtain the following relation for $1 \le k \le n-1$,

$$d_n^k = d_{n-1}^{k-1} + (n-k)d_{n-1}^k. \tag{1.4}$$

Applying the above relations (1.2) (1.3) and (1.4), we shall prove that for any $n \ge 1$, the sequence $\{d_n^k\}_{0\le k\le n}$ is 2-log-concave and reverse ultra log-concave.

2 The 2-log-concavity

In this section, we show that the sequence $\{d_n^k\}_{0\le k\le n}$ is 2-log-concave for any $n \ge 1$. Recall that a sequence $\{a_k\}_{k\ge 0}$ of real numbers is said to be log-concave if $a_k^2 \ge a_{k+1}a_{k-1}$ for all $k \ge 1$; see Stanley [15] and Brenti [2]. From the recurrence relation (1.4), it is easy to prove by induction that the sequence $\{d_n^k\}_{0\le k\le n}$ is log-concave.

Theorem 2.1 *The sequence $\{d_n^k\}_{0\le k\le n}$ is log-concave.*

The notion of high order log-concavity was introduced by Moll [12]; see also, [9]. Given a sequence $\{a_k\}_{k\ge 0}$, define the operator $\mathfrak{L}$ as $\mathfrak{L}\{a_k\} = \{b_k\}$, where

$$b_k = a_k^2 - a_{k-1}a_{k+1}.$$

The log-concavity of $\{a_k\}$ becomes non-negativity of $\mathfrak{L}\{a_k\}$. If the sequence $\mathfrak{L}\{a_k\}$ is not only nonnegative but also log-concave, then we say that $\{a_k\}$ is 2-log-concave. In general, we say that $\{a_k\}$ is l-log-concave if $\mathfrak{L}^l\{a_k\}$ is nonnegative, and that $\{a_k\}$ is infinite log-concave if $\mathfrak{L}^l\{a_k\}$ is nonnegative for any $l \ge 1$. From numerical evidence, we conjecture that the sequence $\{d_n^k\}_{0\le k\le n}$ is infinitely log-concave.

Recently, Brändén [1] has proved that if a polynomial has only real and nonpositive zeros, then its coefficients form an infinite log-concave sequence. However, this is not the case for the polynomials $\sum d_n^k x^k$, since not all polynomials $\sum d_n^k x^k$ have only real zeros, for example, when $n = 2$, the polynomial $x^2 + x + 1$ does not have any real root. Nevertheless, we shall show that the sequence $\{d_n^k\}$ is 2-log-concave in support of the general conjecture.

Theorem 2.2 *The sequence $\{d_n^k\}_{0\le k\le n}$ is 2-log-concave. In other words, for $n \ge 4$ and $2 \le k \le n-2$, we have*

$$\left((d_n^k)^2 - d_n^{k-1}d_n^{k+1}\right)^2 - \left((d_n^{k-1})^2 - d_n^{k-2}d_n^k\right)\left((d_n^{k+1})^2 - d_n^k d_n^{k+2}\right) \ge 0. \tag{2.1}$$

The idea to prove Theorem 2.2 can be outlined as follows.

1. As the first step, we reformulate the left hand side of inequality (2.1) as a cubic function f in $\frac{d_{n+1}^k}{d_n^k}$ by applying the recurrence relations (1.2), (1.3), (1.4) and a recurrence relation presented in Lemma 2.3.

2. We show that Theorem 2.2 follows from the assertion that $f \geq 0$ in the interval

$$I = \left[n + \frac{n-k}{n}, n + \frac{n-k}{n} + \frac{n-k}{n^2} \right],$$

since it can be verified that for $n \geq 4$ and $2 \leq k \leq n-2$,

$$n + \frac{n-k}{n} \leq \frac{d_{n+1}^k}{d_n^k} \leq n + \frac{n-k}{n} + \frac{n-k}{n^2}. \tag{2.2}$$

3. In order to prove $f > 0$, we consider f as a continuous function in x. It can be shown that $f'(x) < 0$ for $x \in I$ and

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2} \right) \geq 0.$$

Hence we deduce that $f \geq 0$ in the interval I. This proves Theorem 2.2.

Lemma 2.3 *For $1 \leq k \leq n$, we have*

$$d_n^{k-1} = (k+1)(n-k)d_n^{k+1} - (n-2k+1)d_n^k. \tag{2.3}$$

Proof. First, from (1.1) it is easy to establish the following recurrence relation for $1 \leq k \leq n$,

$$d_n^{k-1} = k d_n^k - d_{n-1}^{k-1}. \tag{2.4}$$

For $1 \leq k \leq n$, we find

$$\begin{aligned}
d_n^k &= (k+1)d_n^{k+1} - d_{n-1}^k \\
&= (k+1)d_n^{k+1} - \left(\frac{1}{n-k} d_n^k - \frac{1}{n-k} d_{n-1}^{k-1} \right) \quad \text{(by (1.4))} \\
&= (k+1)d_n^{k+1} - \frac{1}{n-k} d_n^k + \frac{1}{n-k} \left(k d_n^k - d_n^{k-1} \right) \quad \text{(by (2.4))} \\
&= (k+1)d_n^{k+1} + \frac{k-1}{n-k} d_n^k - \frac{1}{n-k} d_n^{k-1}.
\end{aligned}$$

Consequently,

$$d_n^{k-1} = (k+1)(n-k)d_n^{k+1} - (n-2k+1)d_n^k,$$

as desired. ∎

To prove (2.2), we need a lower bound on d_{n+1}^k / d_n^k.

Lemma 2.4 *For $n \geq 1$ and $1 \leq k \leq n-1$, we have*

$$\frac{d_{n+1}^k}{d_n^k} \geq n + \frac{n-k}{n}. \tag{2.5}$$

Proof. First we consider the case $1 \leq k \leq n-2$. We proceed by induction on n. It is clear that (2.5) holds for $n = 1$ and $n = 2$. We now assume that (2.5) holds for $n-2$, that is,

$$\frac{d_{n-1}^k}{d_{n-2}^k} \geq n - 2 + \frac{n-k-2}{n-2}. \tag{2.6}$$

By recurrence (1.2), we have

$$\begin{aligned}
\frac{d_{n+1}^k}{d_n^k} &= \frac{n d_n^k + (n-k) d_{n-1}^k}{d_n^k} \\
&= n + (n-k)\frac{d_{n-1}^k}{d_n^k} \\
&= n + (n-k)\frac{d_{n-1}^k}{(n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k}.
\end{aligned}$$

Thus (2.5) can be recast as

$$(n-1) + (n-k-1)\frac{d_{n-2}^k}{d_{n-1}^k} \leq n.$$

So it suffices to check that

$$\frac{d_{n-1}^k}{d_{n-2}^k} \geq n-k-1.$$

Since $n \geq 3$, by the induction hypothesis, we have

$$\begin{aligned}
\frac{d_{n-1}^k}{d_{n-2}^k} &\geq n - 2 + \frac{n-2-k}{n-2} \\
&= n - 1 - \frac{k}{n-2} \\
&\geq n-k-1.
\end{aligned}$$

as required.

We now turn to the case $k = n-1$. By (1.3), we get

$$d_n^{n-1} = (n-1) d_{n-1}^{n-1}.$$

By definition, we have $d_{n-1}^{n-1} = 1$. Moreover, it is easy to see that $d_n^{n-1} = n-1$. Hence, by (1.4), we have

$$\frac{d_{n+1}^{n-1}}{d_n^{n-1}} = \frac{nd_n^{n-1} + d_{n-1}^{n-1}}{d_n^{n-1}} = n + \frac{1}{n-1} > n + \frac{1}{n}.$$

This completes this proof. ∎

Next we give an upper bound on d_{n+1}^k/d_n^k.

Lemma 2.5 *For $n \geq 4$ and $2 \leq k \leq n-2$, we have*

$$\frac{d_{n+1}^k}{d_n^k} \leq n + \frac{n-k}{n} + \frac{n-k}{n^2}. \tag{2.7}$$

Proof. From (1.2) it follows that

$$\begin{aligned}\frac{d_{n+1}^k}{d_n^k} &= n + (n-k)\frac{d_{n-1}^k}{d_n^k} \\ &= n + (n-k)\frac{d_{n-1}^k}{(n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k}.\end{aligned}$$

Thus (2.7) can be rewritten as

$$(n-1) + (n-k-1)\frac{d_{n-2}^k}{d_{n-1}^k} \geq \frac{n^2}{n+1},$$

that is,

$$\frac{d_{n-1}^k}{d_{n-2}^k} \leq (n+1)(n-k-1). \tag{2.8}$$

By recurrence (1.3) for $2 \leq k \leq n-2$, we see that

$$\frac{d_{n-1}^k}{d_{n-2}^k} \leq n-1,$$

which implies (2.8). This completes the proof. ∎

We are now ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2 . It is easy to check that the theorem holds for $n = 4, 5, 6$. So we may assume that $n \geq 7$.

We claim that the left hand side of (2.1) can be expressed as a cubic function f in $\frac{d_{n+1}^k}{d_n^k}$. By the recurrences (1.2), (1.3), (1.4) and (2.3), we can derive the following relations,

$$\begin{aligned}
d_n^{k-2} &= (n-k+1)(n-k+3)d_n^k - (n-2k+3)d_{n+1}^k,\\
d_n^{k-1} &= d_{n+1}^k - (n-k+1)d_n^k,\\
d_n^{k+1} &= \frac{1}{(k+1)(n-k)}\left(d_{n+1}^k - kd_n^k\right),\\
d_n^{k+2} &= \frac{1}{(k+1)(k+2)(n-k-1)(n-k)}\left((n-2k-1)d_{n+1}^k + (n+k^2)d_n^k\right).
\end{aligned}$$

It follows that (2.1) can be rewritten as

$$A\cdot\Big(C_3(n,k)\left(d_{n+1}^k\right)^3 + C_2(n,k)\left(d_{n+1}^k\right)^2\left(d_n^k\right) + C_1(n,k)\left(d_{n+1}^k\right)\left(d_n^k\right)^2 + C_0(n,k)\left(d_n^k\right)^3\Big) \geq 0,$$

where

$$\begin{aligned}
A &= \frac{d_n^k}{(k+1)^2(n-k)^2(k+2)(n-k-1)},\\
C_3(n,k) &= -n^2 - 5n + 6k + 6,\\
C_2(n,k) &= n^3 + n^2k + 5n^2 + 3nk - 10k^2 + n - 16k - 6,\\
C_1(n,k) &= n^2 - 2n + 14k + 14k^2 + n^3 + 10nk^2 - 10n^2k - n^3k - 3nk,\\
C_0(n,k) &= -4n^2 - 12k^2 - 12k^3 + 10nk + 18nk^2 - 9n^2k + n^2k^2 - n^3k.
\end{aligned}$$

Since d_n^k are positive, it suffices to show that

$$C_3(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right)^3 + C_2(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right)^2 + C_1(n,k)\left(\frac{d_{n+1}^k}{d_n^k}\right) + C_0(n,k) \geq 0. \tag{2.9}$$

We now consider the function

$$f(x) = C_3(n,k)x^3 + C_2(n,k)x^2 + C_1(n,k)x + C_0(n,k),$$

with

$$f'(x) = 3C_3(n,k)x^2 + 2C_2(n,k)x + C_1(n,k). \tag{2.10}$$

We aim to show that $f'(x) < 0$, for $2 \leq k \leq n-1$ and $x \in I$.

It can be shown that $f'(-1) < 0$, $f'(k) > 0$, $f'(n) > 0$ and $C_3(n,k) < 0$. The proofs will be given later. Using the facts $f'(-1) < 0$, $f'(k) > 0$ and $f'(n) > 0$, we deduce that $f'(x)$ has a zero in the interval $[-1,k]$ and a zero in the interval $[k,n]$. This implies that $f'(x)$ has no zeros in the interval I since $f'(x)$ is a quadratic function. Since $f'(n) > 0$

and $C_3(n,k) < 0$, we see that $f'(x) < 0$ in the interval I. In other words, $f(x)$ is strictly decreasing in the interval I.

It will be also shown that

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2}\right) > 0. \tag{2.11}$$

Combining with the fact that $f(x)$ is strictly decreasing in I, we obtain that $f(x) > 0$ in I, as desired.

We now finish the proofs of the above claims. First, we show that $f'(-1) < 0$. Clearly, we have

$$f'(-1) = -(k+1)(n^3 + 12n^2 - 10nk + 19n - 34k - 30).$$

For $n \geq 7$ and $2 \leq k \leq n-2$, we find

$$\begin{aligned} n^3 &+ 12n^2 - 10nk + 19n - 34k - 30 \\ &\geq n^3 + 12n(k+2) + 19n - 30 - 10nk - 34k \\ &\geq (n^3 - 30) + 2nk + (43n - 34k) > 0. \end{aligned}$$

This implies that $f'(-1) < 0$.

Next we shall verify that $f'(k) > 0$ and $f'(n) > 0$. For $x = k$, we have

$$f'(k) = (k+1)(n-k)(n^2 + n + 2k - 2).$$

Since $n > k$ and $k > 1$, we see that $f'(k) > 0$.

For $x = n$, we have

$$f'(n) = -(n-k)(n^3 + 4n^2 - 10nk + 14k - 21n + 14). \tag{2.12}$$

To prove $f'(n) < 0$, it suffices to show that for $2 \leq k \leq n-2$,

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 > 0.$$

We consider two cases. For $2 \leq k \leq n-3$, we have

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n\left((n-3)^2 + 10(n-k-3)\right) + 14k + 14 > 0,$$

On the other hand, for $k = n-2$, we have

$$n^3 + 4n^2 - 10nk + 14k - 21n + 14 = n(n-3)^2 + 4n - 14 > 0.$$

Thus $f'(n) < 0$ holds for $2 \leq k \leq n-2$.

To prove $f'(x) > 0$, we need to verify that $C_3(n,k) < 0$. Since $n \geq k+2$, it is easily seen that

$$\begin{aligned} C_3(n,k) &= -(n^2 + 5n - 6k - 6) \\ &\leq -\big((k+2)^2 + 5(k+2) - 6k - 6\big) \\ &\leq -(k^2 + 3k + 8) < 0. \end{aligned}$$

Till now, we have proved the facts $f'(-1) < 0$, $f'(k) > 0$, $f'(n) > 0$ and $C_3(n,k) < 0$. Finally, we finish the proof of (2.11). It is easily checked that

$$f\left(n + \frac{n-k}{n} + \frac{n-k}{n^2}\right) = \frac{h(k)(n-k)^2}{n^6},$$

where

$$\begin{aligned} h(k) = {} & (-10n^4 - 26n^3 - 28n^2 - 18n - 6)k^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)k \\ & + (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n). \end{aligned}$$

We continue to show that $h(k) \geq 0$ for $n \geq 7$ and $2 \leq k \leq n-2$. We now consider $h(x)$ as a continuous function in x, that is,

$$\begin{aligned} h(x) = {} & (-10n^4 - 26n^3 - 28n^2 - 18n - 6)x^2 + (-n^6 + 20n^5 + 27n^4 + 19n^3 - 7n - 6)x \\ & + (n^7 - 10n^6 - 4n^5 - 4n^4 + 9n^3 + 7n^2 + 6n). \end{aligned}$$

Since the leading coefficient of $h(x)$ is negative, we only need to prove that $h(2) > 0$ and $h(n-1) > 0$. For $n \geq 7$, we have

$$\begin{aligned} h(n-1) &= n(n^5 - 3n^4 + 2n^3 + 2n^2 + 2n + 1) \\ &= n\left(n^3(n-1)(n-2) + 2n^2 + 2n + 1\right) > 0, \end{aligned}$$

and

$$\begin{aligned} h(2) &= n^7 - 12n^6 + 36n^5 + 10n^4 - 57n^3 - 105n^2 - 80n - 36 \\ &= n^5(n-5)(n-7) + n^4(n-6) + 16n^3(n-7) + 55n^2(n-7) \\ &\quad + 80n(n-1) + 200n^2 - 36 > 0. \end{aligned}$$

Thus we reach the conclusion that $h(k) > 0$ for $n \geq 7$ and $2 \leq k \leq n-2$. This completes the proof. ∎

3 The reverse ultra log-concavity

In this section, we show that for any $n \geq 1$, the sequence $\{d_n^k\}_{0\leq k\leq n}$ is reverse ultra log-concave. Recall that a sequence $\{a_k\}_{0\leq k\leq n}$ is called ultra log-concave if $\left\{a_k/\binom{n}{k}\right\}$ is log-concave. This condition can be restated as

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \geq 0. \tag{3.1}$$

It is well known that if a polynomial has only real zeros, then its coefficients form an ultra log-concave sequence. If a sequence $\{a_k\}_{0\leq k\leq n}$ is ultra log-concave, then the sequence $\{k!a_k\}_{0\leq k\leq n}$ is log-concave, see Liggett [11].

In comparison with ultra log-concavity, a sequence is said to be reverse ultra log-concave if it satisfies the reverse relation of (3.1), that is,

$$k(n-k)a_k^2 - (n-k+1)(k+1)a_{k-1}a_{k+1} \leq 0. \tag{3.2}$$

Chen and Gu [3] have shown the Boros-Moll polynomials are reverse ultra log-concave. The following theorem states that the sequence $\{d_n^k\}_{0\leq k\leq n}$ is reverse ultra log-concave.

Theorem 3.1 *For $n \geq 1$ and $1 \leq k \leq n-1$, we have*

$$\frac{d_n^{k-1}}{\binom{n}{k-1}} \cdot \frac{d_n^{k+1}}{\binom{n}{k+1}} \geq \left(\frac{d_n^k}{\binom{n}{k}}\right)^2,$$

or equivalently,

$$(n-k+1)(k+1)d_n^{k-1}d_n^{k+1} \geq k(n-k)\left(d_n^k\right)^2. \tag{3.3}$$

Proof. According to the recurrence relations (1.4) and (2.3), we find that (3.3) can be reformulated as

$$(n-k+1)\left(\frac{d_{n+1}^k}{d_n^k}\right)^2 - (n-k+1)(n+1)\left(\frac{d_{n+1}^k}{d_n^k}\right) + k(2n-2k+1) \geq 0. \tag{3.4}$$

The discriminant of the quadratic polynomial in d_{n+1}^k/d_n^k on the left hand side of (3.4) equals

$$\Delta = ((n-k+1)(n+1))^2 - 4k(n-k+1)(2n-2k+1).$$

We aim to show that $\Delta > 0$ for $1 \leq k \leq n-1$. We can rewrite Δ as follows

$$\Delta = (n-k+1)[(n-k-1)((n+1)^2-8k) + 2((n+1)^2-6k)].$$

Since $(n+1)^2-6k \geq (n+1)^2-8k = (n-3)^2 \geq 0$, it follows that $\Delta > 0$ for $1 \leq k \leq n-1$, as desired.

Therefore, the above quadratic function has two distinct real zeros. If we can prove that for $1 \leq k \leq n-1$, d_{n+1}^k/d_n^k is larger than the large zero, then (3.4) holds since $n-k+1>0$. Thus we still have to show that

$$\frac{d_{n+1}^k}{d_n^k} > \frac{(n-k+1)(n+1)+\sqrt{\Delta}}{2(n-k+1)} = \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)} \tag{3.5}$$

In view of (2.5), we see that (3.5) can be deduced from the following inequality

$$n + \frac{n-k}{n} \geq \frac{n+1}{2} + \frac{\sqrt{\Delta}}{2(n-k+1)},$$

which is equivalent to

$$(n-k+1)(n^2+n-2k) \geq n\sqrt{\Delta}.$$

Evidently,

$$\begin{aligned} &\big((n-k+1)(n^2+n-2k)\big)^2 - n^2\Delta \\ &= 4k(n-k+1)(n-k)(n^2-n+k-1), \end{aligned}$$

which is nonnegative for $1 \leq k \leq n-1$. This completes the proof. ∎

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References

[1] P. Brändén, Iterated sequences and the geometry of zeros, J. Reine Angew. Math., to appear.

[2] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc., 413 (1989), 1–106.

[3] W.Y.C. Chen and C.C.Y. Gu, The reverse ultra log-concavity of the Boros-Moll polynomials, Proc. Amer. Math. Soc., 137 (2009), 3991–3998.

[4] R.J. Clarke, G.N. Han and J. Zeng, A combinatorial interpretation of the Seidel generation of q-derangement numbers, Ann. Combin., 1 (1997), 313–327.

[5] D. Dumont and A. Randrianarivony, Dérangements et ombres de Genocchi, Discrete Math., 132 (1994), 37–49.

[6] N. Eriksen, R. Freij and J. Wästlund, Enumeration of derangements with descents in prescribed positions, Electron. J. Combin., 16 (2009), #R32.

[7] H.L.M. Faliharimalala and J. Zeng, Derangements and Euler's difference table for $C_\ell \wr S_n$, Electron. J. Combin., 15 (2008), #R65.

[8] H.L.M. Faliharimalala and J. Zeng, Fix-Euler-Mahonian statistics on wreath products, arXiv:math.CO/0810.2731.

[9] M. Kauers and P. Paule, A computer proof of Moll's log-concavity conjecture, Proc. Amer. Math. Soc. 135 (2007) 3847–3856.

[10] D.C. Kurtz, A note on concavity properties of triangular arrays of numbers, J. Combin. Theory Ser. A, 13 (1972), 135–139.

[11] T.M. Liggett, Ultra logconcave sequences and negative dependence, J. Combin. Theory Ser. A, 79 (1997), 315–325.

[12] V.H. Moll, Combinatorial sequences arising from a rational integral, Online Journal of Analytic Combin., 2 (2007), #4.

[13] F. Rakotondrajao, On Euler's difference table, 19th International Conference on Formal Power Series and Algebraic Combinatorics, Nankai University, Tianjin, 2007.

[14] F. Rakotondrajao, k-fixed-points-permutations, Integers, 7 (2007), #A36.

[15] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in Graph Theory and Its Applications: East and West, Ann. New York Acad. Sci., 576 (1989), 500–535.