Partitions and Partial Matchings Avoiding Neighbor Patterns

William Y. C. Chen[1], Neil J. Y. Fan[1] and Alina F. Y. Zhao[2]

[1] *Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, PR China*

[2] *School of Mathematical Science, Nanjing Normal University, Nanjing 210046, PR China*

Abstract. We obtain the generating functions for partial matchings avoiding neighbor alignments and for partial matchings avoiding neighbor alignments and left nestings. We show that there is a bijection between partial matchings avoiding the three neighbor patterns (neighbor alignments, left nestings and right nestings) and set partitions avoiding right nestings via an intermediate structure of upper triangular matrices. Combining our bijection and the bijection given by Dukes and Parviainen between upper triangular matrices and self-modified ascent sequences, we get a bijection between partial matchings avoiding the three neighbor patterns and self-modified ascent sequences.

Keywords: set partition, partial matching, neighbor alignment, left nesting, right nesting, self-modified ascent sequence.

AMS Subject Classification: 05A15, 05A19

1 Introduction

This paper is concerned with the enumeration of partial matchings and set partitions that avoid certain neighbor patterns. Recall that a partition π of $[n] = \{1, 2, \ldots, n\}$ can be represented by a diagram with vertices drawn on a horizontal line in increasing order. For a block B of π, we write the elements of B in increasing order. Suppose that $B = \{i_1, i_2, \ldots, i_k\}$. Then we draw an arc from i_1 to i_2, an arc from i_2 to i_3, and so on. Such a diagram is called the linear representation of π.

A partial matching is a partition for which each block contains at most two elements. A partial matching is also called a poor partition by Klazar [10], see also [2]. It can be viewed as an involution on a set. A partition for which each block contains exactly two elements is called a perfect matching.

A nesting of a partition π is formed by two arcs (i_1, j_1) and (i_2, j_2) in the linear representation such that $i_1 < i_2 < j_2 < j_1$. If we further require that $i_1 + 1 = i_2$, then such a nesting is called a left nesting. Similarly, one can define right nestings. A crossing is formed by two arcs (i_1, j_1) and (i_2, j_2) such that $i_1 < i_2 < j_1 < j_2$. A left crossing is a crossing formed by two arcs (i_1, j_1) and (i_2, j_2) subject to a further condition $i_1 + 1 = i_2$. Right crossings can

E-mail addresses: chen@nankai.edu.cn (W.Y.C. Chen), fjy@cfc.nankai.edu.cn (N.J.Y. Fan), zfeiyan@cfc.nankai.edu.cn (A.F.Y. Zhao).

be defined in the same way. Moreover, we say that k arcs $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ form a k-crossing if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. An alignment of a partition π is formed by two arcs (i_1, j_1) and (i_2, j_2) such that $i_1 < j_1 < i_2 < j_2$.

Perfect matchings avoiding certain patterns have been studied in [3, 4, 5, 8, 9, 11, 12, 15]. Left nestings and right nestings were introduced by Stoimenow [16] in the study of regular linearized chord diagrams, and were further investigated in [1, 5, 7]. In particular, Bousquet-Mélou, Claesson, Dukes and Kitaev [1] considered perfect matchings avoiding left nestings and right nestings, and found bijections with other combinatorial objects such as $(2+2)$-free posets.

In this paper, we define a neighbor alignment as an alignment consisting of two arcs (i_1, j_1) and (i_2, j_2) such that $j_1 + 1 = i_2$. The aforementioned patterns with neighbor constraints are called neighbor patterns. An illustration of neighbor patterns is given in Figure 1.1.

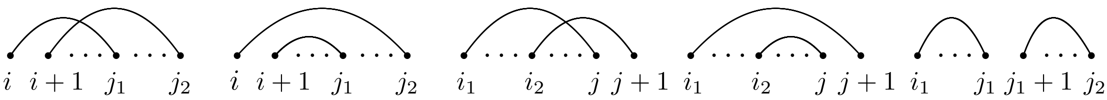

Figure 1.1: Left crossing, left nesting, right crossing, right nesting and neighbor alignment.

Our main results are the generating functions for three classes of partial matchings avoiding neighbor patterns. Denote the set of partial matchings of $[n]$ by $\mathcal{M}(n)$. The set of partial matchings in $\mathcal{M}(n)$ with no neighbor alignments is denoted by $\mathcal{P}(n)$, and the set of partial matchings in $\mathcal{P}(n)$ with k arcs is denoted by $\mathcal{P}(n,k)$. The set of partial matchings in $\mathcal{P}(n)$ with no left nestings is denoted by $\mathcal{Q}(n)$, and the set of partial matchings in $\mathcal{Q}(n)$ with k arcs is denoted by $\mathcal{Q}(n,k)$. Moreover, the set of partial matchings in $\mathcal{Q}(n)$ with no right nestings is denoted by $\mathcal{R}(n)$, and the set of partial matchings in $\mathcal{R}(n)$ with k arcs is denoted by $\mathcal{R}(n,k)$. For $0 \le k \le \lfloor n/2 \rfloor$, we set

$$P(n,k) = |\mathcal{P}(n,k)|, \quad Q(n,k) = |\mathcal{Q}(n,k)|, \quad R(n,k) = |\mathcal{R}(n,k)|.$$

Denote the set of partitions of $[n]$ by $\Pi(n)$ and denote the set of partitions in $\Pi(n)$ with k blocks by $\Pi(n,k)$. Note that $S(n,k) = |\Pi(n,k)|$ is the Stirling number of the second kind. The set of partitions in $\Pi(n)$ with no right nestings is denoted by $\mathcal{T}(n)$, and the set of partitions in $\mathcal{T}(n)$ with k arcs is denoted by $\mathcal{T}(n,k)$. For $0 \le k \le n-1$, we set $T(n,k) = |\mathcal{T}(n,k)|$.

We obtain the following generating function formulas for the numbers $P(n,k)$ and $Q(n,k)$.

Theorem 1.1 *We have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P(n,k) x^n y^k = \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 + kxy) x^n. \tag{1.1}$$

Theorem 1.2 *We have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} Q(n-1,k) x^n y^k = \sum_{n=0}^{\infty} \frac{x^n}{\prod_{k=1}^{n} (1 - kx^2 y)}. \tag{1.2}$$

It is clear that when $y = 1$, the right-hand side of (1.1) reduces to

$$\sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 + kx)x^n,$$

which is the generating function of the sequence $A124380$ in OEIS [13], whose first few entries are

$$1, 1, 2, 4, 9, 22, 57, 157, 453, 1368, 4290, \ldots.$$

Thus Theorem 1.1 can be considered as a combinatorial interpretation of the above generating function.

Meanwhile, when $y = 1$ the right-hand side of (1.2) reduces to

$$\sum_{n=0}^{\infty} \frac{x^n}{\prod_{k=1}^{n}(1 - kx^2)},$$

which is the generating function of the sequence $A024428$ in OEIS [13], whose first few entries are

$$1, 1, 2, 4, 8, 18, 42, 102, 260, 684, 1860, \ldots.$$

This sequence can be expressed in terms of Stirling numbers of the second kind. So Theorem 1.2 can be considered as another combinatorial interpretation of the above generating function.

Denote by $\mathcal{M}^{m\times m}(n)$ the set of $m \times m$ upper triangular matrices with nonnegative integer entries which sum to n. We derive the generating function of the numbers $R(n+k-1, k)$ by establishing a bijection between $\mathcal{R}(n+k-1, k)$ and $\mathcal{M}^{(n-k)\times(n-k)}(k)$. Moreover, by constructing a bijection between $\mathcal{M}^{(n-k)\times(n-k)}(k)$ and $\mathcal{T}(n, k)$, we show that there is a correspondence between $\mathcal{T}(n, k)$ and $\mathcal{R}(n+k-1, k)$. Hence by Theorem 1.3 we obtain the generating function formula for the numbers $T(n, k)$ as stated in Theorem 1.4. Furthermore, it turns out that this generating function coincides with the generating function for the number of self-modified ascent sequences of length n with largest element $n - k - 1$ or the number of $3\bar{1}52\bar{4}$-avoiding permutations on $[n]$ having $n - k$ right-to-left minima, as derived by Bousquet-Mélou, Claesson, Dukes and Kitaev [1].

Theorem 1.3 *We have*

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} R(n+k-1, k)x^n y^k = \sum_{n=1}^{\infty} \frac{x^n}{(1 - xy)^{\binom{n+1}{2}}}. \tag{1.3}$$

Theorem 1.4 *We have*

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} T(n, k)x^n y^k = \sum_{n=1}^{\infty} \frac{x^n}{(1 - xy)^{\binom{n+1}{2}}}. \tag{1.4}$$

This paper is structured as follows. In Section 2, we give a proof of Theorem 1.1 by deriving a recurrence relation of $P(n, k)$. In Section 3 we prove Theorem 1.2 by establishing a correspondence between $\Pi(n-k, n-2k)$ and $\mathcal{Q}(n-1, k)$. In Section 4, we present a bijection between $\mathcal{R}(n+k-1, k)$ and $\mathcal{M}^{(n-k)\times(n-k)}(k)$, which leads to the generating function formula in Theorem 1.3. In Section 5 we give a proof of Theorem 1.4 by constructing a correspondence between $\mathcal{R}(n+k-1, k)$ and $\mathcal{T}(n, k)$.

2 Neighbor alignments

In this section, we give a proof of the generating function formula for the number of partial matchings avoiding neighbor alignments. If (i, j) is an arc in the diagram of π, we call i a left-hand endpoint, and call j a right-hand endpoint. A singleton of a partial matching or a set partition is the only element in a block, which corresponds to an isolated vertex in its diagram representation. For a block with at least two elements, the minimum element is called an origin, and the maximum element is called a destination, and an element in between, if any, is called a transient vertex or simply a transient. An origin and a destination are also called an opener and a closer by some authors. We first give a recurrence relation of $P(n, k)$.

Theorem 2.1 *For $n \geq 3$, and $1 \leq k \leq n/2$, we have*

$$P(n, k) = P(n-1, k) + (n-k)P(n-2, k-1), \tag{2.1}$$

with initial values $P(1, 0) = 1, P(2, 0) = 1, P(2, 1) = 1$.

Proof. It is clear that the number of partial matchings in $\mathcal{P}(n, k)$ such that the element 1 is a singleton equals $P(n-1, k)$. So it suffices to show that the number of partial matchings in which 1 is not a singleton equals $(n-k)P(n-2, k-1)$. If no confusion arises, we do not distinguish a partial matching from its diagram representation. For a partial matching $M \in \mathcal{P}(n, k)$ in which 1 is not a singleton, we assume that $(1, i)$ is an arc of M. Deleting the arc $(1, i)$ and the two vertices 1 and i, we are led to a partial matching in $\mathcal{P}(n-2, k-1)$.

Conversely, given a partial matching $M \in \mathcal{P}(n-2, k-1)$ with $n-2$ vertices, in order to get a partial matching with k arcs, we can add an arc into M by placing the left-hand endpoint before the first vertex of M and inserting the right-hand endpoint at some position of M. Clearly, there are $n-1$ possible positions to insert the right-hand endpoint of the new arc. To ensure that the insertion will not cause any neighbor alignments, we should not allow the right-hand endpoint of the inserted arc to be placed before any origin of M. Since there are $k-1$ arcs in M, thus there are $k-1$ positions that are forbidden. Hence there are $(n-1)-(k-1) = n-k$ choices to insert the new arc. After relabeling, we get a partial matching in $\mathcal{P}(n, k)$. This completes the proof. ∎

As an example, let us consider a partial matching $M = \{\{1, 4\}, \{2\}, \{3, 5\}, \{6\}\} \in \mathcal{P}(6, 2)$. The possible positions for inserting an arc are marked by the symbol $*$ in Figure 2.1. In the left diagram, the positions before the vertices 1 and 3 are forbidden. If we choose the

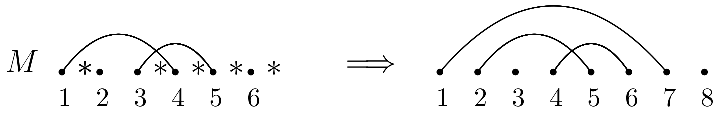

Figure 2.1: Possible positions for inserting an arc.

position between the vertices 5 and 6 to insert the right-hand endpoint of the new arc, then the diagram on the right represents the resulting partial matching.

Proof of Theorem 1.1. Let $f(n,k)$ denote the coefficient of $x^n y^k$ in the expansion of

$$\sum_{m=1}^{\infty}\prod_{i=1}^{m}(1+ixy)x^m.$$

It is not hard to see that $f(n,k)$ equals the coefficient of $x^k y^k$ in the expansion of

$$\prod_{i=1}^{n-k}(1+ixy).$$

It follows that

$$f(n,k)=f(n-1,k)+(n-k)f(n-2,k-1)$$

for $n\geq 3$ and $k\geq 1$, $f(n,0)=1$ for $n\geq 1$, and $f(2,1)=1$. Thus $P(n,k)$ and $f(n,k)$ satisfy the same recurrence relation with the same initial values. This completes the proof. ∎

To conclude this section, we give a recurrence relation of the generating function of $P(n,k)$. Let

$$f_n(y)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}P(n,k)y^k.$$

Corollary 2.2 *For $n\geq 3$, we have*

$$f_n(y)=f_{n-1}(y)+(n-1)yf_{n-2}(y)-y^2f'_{n-2}(y), \tag{2.2}$$

where $f_1(y)=1, f_2(y)=1+y$.

3 Neighbor alignments and left nestings

This section is concerned with the generating function for partial matchings avoiding neighbor alignments and left nestings. More precisely, we establish a bijection between set partitions and partial matchings avoiding neighbor alignments and left nestings. As a consequence, we obtain the generating function in Theorem 1.2.

Theorem 3.1 *For $0 \le k \le \lfloor \frac{n-1}{2} \rfloor$, there exists a bijection between the set $\Pi(n-k, n-2k)$ and the set $\mathcal{Q}(n-1, k)$. Moreover, this bijection transforms the number of transients of a partition to the number of left crossings of a partial matching.*

Proof. For $k = 0$ the theorem is obvious. We only consider the case $k \ge 1$. Let $\pi \in \Pi(n-k, n-2k)$ be a partition of $[n-k]$ with k arcs, we wish to add $k-1$ vertices to π in order to form a partial matching $\alpha(\pi)$ in $\mathcal{Q}(n-1, k)$, that is, a partial matching on $[n-1]$ avoiding neighbor alignments and left nestings. First, we add a vertex before each origin, except for the first origin, and relabel the vertices from left to right by using $1, 2, \ldots$. Let the resulting partition be denoted by σ.

To transform the partition σ to a partial matching in $\mathcal{Q}(n-1, k)$, we need the operation of changing a 2-path to a left crossing, see Figure 3.1 for an illustration. Such a transformation is called a splitting of a 2-path, where a 2-path means two arcs (i, j) and (j, k) with $i < j < k$. We shall order the 2-paths of a partition in terms of their transient vertices.

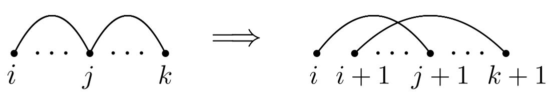

Figure 3.1: Changing a 2-path to a left crossing.

Assume that there are m 2-paths in σ, that is, there are m transient vertices in σ. We shall apply the splitting operation m times to get a sequence of partitions $\sigma = \sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(m)}$ such that for $1 \le i \le m$, the partition $\sigma^{(i)}$ is obtained from $\sigma^{(i-1)}$ by splitting the smallest 2-path of $\sigma^{(i-1)}$ and by relabeling the vertices afterwards.

Let $\alpha(\pi) = \sigma^{(m)}$ denote the resulting partition. It is easy to see that $\alpha(\pi)$ no longer contains any transient vertex, in other words, $\alpha(\pi)$ is a partial matching. It is clear that the splitting operation generates a new origin and a new destination. From the construction of σ, we see that for $1 \le i \le m$, there is a singleton before each origin of $\sigma^{(i)}$, except the first origin and the origins caused by the splitting operation. So we deduce that in the process of constructing $\sigma^{(i)}$ from $\sigma^{(i-1)}$ we do not get any new left nestings or new neighbor alignments in $\sigma^{(i)}$.

We claim that there are no left nestings in $\alpha(\pi)$. To prove this claim, we introduce a linear order on the set of left nestings of a partition. For a left nesting N consisting of two arcs (i_1, j_1) and (i_1+1, j_2) and a left nesting N' consisting of two arcs (i_1', j_1') and $(i_1'+1, j_2')$, we say that N is smaller than N' if $i_1 < i_1'$. To see that all the left nestings of σ will disappear in $\alpha(\pi) = \sigma^{(m)}$, we start with the smallest left nesting of $\sigma = \sigma^{(0)}$. By the construction of σ, if the smallest left nesting of σ consists of two arcs (i, j) and $(i+1, k)$, then $i+1$ is a transient vertex of σ. Clearly, the vertex i is either a transient vertex or an origin of σ.

If the vertex i is a transient vertex of σ, then we may assume that the 2-path V_i containing i as the transient vertex is the t-th ($t \ge 1$) 2-path of σ. Note that we always split the smallest 2-path and the splitting operation does not cause any new left nestings. We see that after applying the splitting operation $t-1$ times, the 2-path V_i (after relabeling) becomes the smallest 2-path of $\sigma^{(t-1)}$. One can check that after we split the smallest 2-path of $\sigma^{(t-1)}$, the smallest left nesting in σ disappears in $\sigma^{(t)}$.

If the vertex i is an origin of σ, then we may assume that the 2-path V_{i+1} of σ containing $i+1$ as the transient vertex is the t'-th ($t' \geq 1$) 2-path of σ. After applying the splitting operation $t'-1$ times, the 2-path V_{i+1} (after relabeling) becomes the smallest 2-path of $\sigma^{(t'-1)}$. Splitting the smallest 2-path of $\sigma^{(t'-1)}$, the smallest left nesting of σ disappears in $\sigma^{(t')}$. See Figure 3.2 as an illustration for the above two cases.

Therefore, the smallest left nesting of σ disappears in some $\sigma^{(t)}$ ($t \geq 1$). If there still exist left nestings in $\sigma^{(t)}$, we can repeat the above process for the smallest left nesting of $\sigma^{(t)}$. Since the splitting operation does not cause any new left nestings, the left nestings will disappear at last. Thus the proof of the claim is completed.

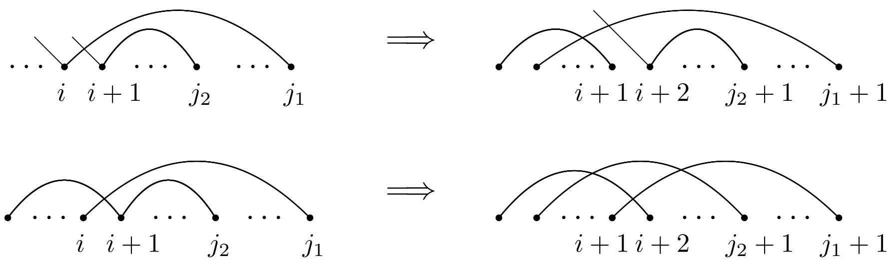

Figure 3.2: The disappearing of left nestings.

We still need to show that there are no neighbor alignments in $\alpha(\pi)$. To this end, we define a linear order on neighbor alignments of a partition τ. For a neighbor alignment A consisting of two arcs (i_1, j_1) and (j_1+1, j_2) and a neighbor alignment A' consisting of two arcs (i'_1, j'_1) and (j'_1+1, j'_2), we say that A is smaller than A' if $i_1 < i'_1$. It is easily checked that this is a linear order on the set of neighbor alignments of τ.

To see that all the neighbor alignments of σ will disappear in $\alpha(\pi) = \sigma^{(m)}$, we start with the smallest neighbor alignment of $\sigma = \sigma^{(0)}$. By the construction of σ, if the smallest neighbor alignment of σ consists of two arcs (i, j) and $(j+1, k)$, then $j+1$ is a transient vertex of σ. Obviously, the vertex j is either a transient vertex or a destination of σ.

If the vertex j is a transient vertex of σ, then we assume that the 2-path V_j containing j is the r-th ($r \geq 1$) 2-path of σ. Applying the splitting operation $r-1$ times to σ, we get $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(r-1)}$. Now the 2-path V_j (after relabeling) becomes the smallest 2-path of $\sigma^{(r-1)}$. Moreover, after splitting the smallest 2-path of $\sigma^{(r-1)}$, the 2-path V_{j+1} of σ (after relabeling) becomes the smallest 2-path of $\sigma^{(r)}$. It can be verified that the smallest neighbor alignment of σ disappears in $\sigma^{(r+1)}$.

If the vertex j is a destination of σ, then we assume that the 2-path V_{j+1} is the r'-th ($r' \geq 1$) 2-path of σ. After applying the splitting operation $r'-1$ times to σ, we obtain $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(r'-1)}$, and the 2-path V_{j+1} (after relabeling) becomes the smallest 2-path of $\sigma^{(r'-1)}$. It is easily seen that the smallest neighbor alignment of σ disappears in $\sigma^{(r')}$.

So in either case the smallest neighbor alignment of σ disappears in some $\sigma^{(t)}$ $(t \geq 1)$. If $\sigma^{(t)}$ still contains any neighbor alignments, then there exist 2-paths in $\sigma^{(t)}$. We may repeat the above process with respect to the smallest neighbor alignment in $\sigma^{(t)}$. Since there is neither increase of neighbor alignments nor increase of 2-paths at any step, all the neighbor alignments will disappear eventually. We conclude that there are no neighbor alignments in $\alpha(\pi)$.

It remains to show that $\alpha(\pi) \in \mathcal{Q}(n-1,k)$ and the number of transients of π equals the number of left crossings of $\alpha(\pi)$. It is clear that $\alpha(\pi)$ has k $(k \geq 1)$ arcs. It suffices to show that to construct $\alpha(\pi)$ from π we get $k-1$ more vertices. Recall that π has $n-k$ vertices and k arcs. Suppose that (i,j) is an arc of π such that i is not the first origin of π. Clearly, there are $k-1$ such arcs. Observe that i is either an origin or a transient vertex of π. If i is an origin, then a singleton is added before the vertex i in the construction of σ. If i is a transient vertex, then there is a 2-path V_i containing the vertex i. After changing V_i to a left crossing, we get one more vertex. Combining the above two cases, we see that in the construction of $\alpha(\pi)$, there are $k-1$ vertices added. Thus $\alpha(\pi) \in \mathcal{Q}(n-1,k)$. It is easily verified that the number of transient vertices of π equals the number of left crossings of $\alpha(\pi)$.

Conversely, given a partial matching M in $\mathcal{Q}(n-1,k)$, we can recover a partition $\alpha'(M)$ in $\Pi(n-k, n-2k)$. As the first step, we change all the left crossings of M to 2-paths. Suppose that there are m left crossings in M. We aim to construct a sequence of partitions $M = M^{(0)}, M^{(1)}, \ldots, M^{(m)}$ such that for $1 \leq i \leq m$, $M^{(i)}$ is obtained from $M^{(i-1)}$ by changing a unique left crossing to a 2-path.

Let us define a linear order on the set of left crossings of a given partition. For a left crossing C consisting of two arcs (i_1, j_1) and (i_1+1, j_2) and a left crossing C' consisting of two arcs (i_1', j_1') and $(i_1'+1, j_2')$, we say that C is smaller than C' if $i_1 < i_1'$. It is obvious to see that this is a linear order. Assume that the largest left crossing of $M^{(i-1)}$ is formed by two arcs (i, j_1) and $(i+1, j_2)$. Set $M^{(i)}$ to be the partition obtained from $M^{(i-1)}$ by deleting the vertex $i+1$, and transforming the left crossing formed by (i, j_1) and $(i+1, j_2)$ to a 2-path formed by (i, j_1-1) and (j_1-1, j_2-1). Then we relabel the vertices from left to right with $1, 2, \ldots$.

From the above procedure, it can be seen that $M^{(m)}$ is a partition without left crossings. Now we delete the singleton immediately before each origin of $M^{(m)}$, if there is any, except for the singleton immediately before the first origin. Finally we relabel the vertices from left to right with $1, 2, \ldots$. Denote the resulting partition by $\alpha'(M)$.

We continue to show that $\alpha'(M) \in \Pi(n-k, n-2k)$ and the number of left crossings of M equals the number of transient vertices of $\alpha'(M)$. Apparently there are k arcs in $\alpha'(M)$. So it suffices to show that there are $n-k$ vertices in $\alpha'(M)$, that is, in the construction of $\alpha'(M)$, we need to delete $k-1$ vertices of M. Suppose that (i,j) is an arc of M such that i is not the first origin of M. Obviously, there are $k-1$ such arcs. By the assumption, we see that i is not the first vertex of M, hence $i-1$ is also a vertex of M.

We claim that either the vertex $i-1$ or the vertex i, but not both, will be deleted in the construction of $\alpha'(M)$. There are two cases. Case 1. The vertex $i-1$ is a singleton.

It can be seen that in the construction of $\alpha'(M)$, the vertex $i-1$ is deleted. Case 2. The vertex $i-1$ is not a singleton. Since there are neither neighbor alignments nor left nestings in M, there exists an arc $(i-1,k)$ of M such that the two arcs $(i-1,k)$ and (i,j) form a left crossing of M. In this case, in the construction of $\alpha'(M)$, the left crossing formed by the arcs $(i-1,k)$ and (i,j) is transformed into a 2-path by deleting the vertex i. So the claim is proved.

Thus in the construction of $\alpha'(M)$, there are $k-1$ vertices deleted from M. This implies that $\alpha'(M) \in \Pi(n-k, n-2k)$. It is easily seen that the number of left crossings of M equals the number of transient vertices of $\alpha'(M)$. Moreover, one can check that the map α' is the inverse of the map α. Thus the map α is a bijection. This completes the proof. ∎

For example, for $n = 13$ and $k = 5$, let $\pi = \{\{1,5\},\{2,3,4,7\},\{6,8\}\} \in \Pi(8,3)$ be a partition with 8 vertices and 5 arcs. We need to add 4 vertices to π in order to get a partial matching in $\mathcal{Q}(12,5)$. We first add a singleton before the arc $(2,3)$ and a singleton before the arc $(6,8)$. Then we change the two 2-paths to left crossings from left to right. An illustration of the above procedure is given in Figure 3.3.

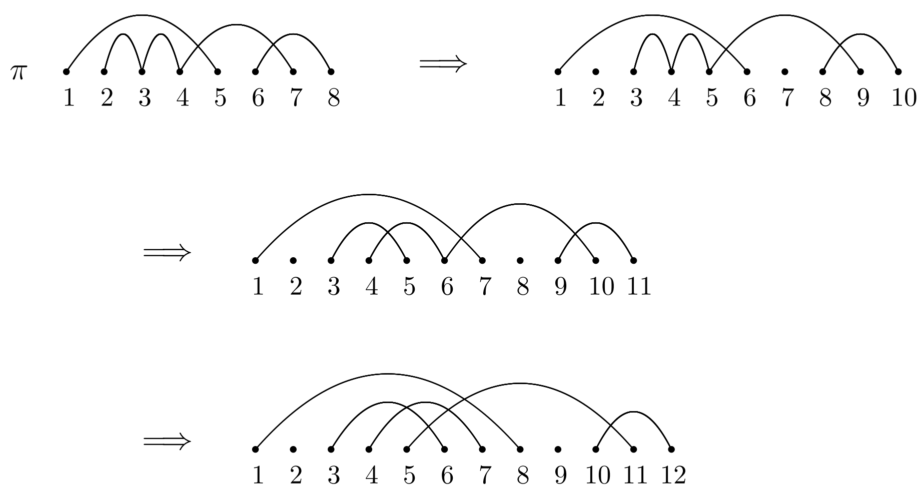

Figure 3.3: An illustration of the bijection α.

We are now ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2: Let $g_n(y)$ be the generating function for the numbers $Q(n-1,k)$. From Theorem 3.1 we see that

$$g_n(y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} Q(n-1,k) y^k = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} S(n-k, n-2k) y^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind. Using the generating function of the Stirling numbers of the second kind [14]

$$\sum_{n \geq k} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)},$$

we obtain that

$$\sum_{n=1}^{\infty} g_n(y)x^n = \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n}(1-kx^2y)}.$$

This completes the proof. ∎

It should be mentioned that the generating function of the numbers

$$g_n(1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} S(n-k, n-2k)$$

can be found in OEIS [13], that is,

$$\sum_{n=1}^{\infty} g_n(1)x^n = \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n}(1-kx^2)}.$$

Thus Theorem 1.2 can be viewed as another combinatorial interpretation for the numbers $g_n(1)$.

To conclude this section, we give a recurrence relation of $g_n(y)$. By Theorem 3.1 and the recurrence relation of the Stirling numbers of the second kind

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),$$

we have

$$Q(n,k) = Q(n-1,k) + (n-2k-1)Q(n-2,k-1). \tag{3.1}$$

Note that we can also give a direct combinatorial proof of (3.1) which is similar to the proof of (2.1) in Theorem 2.1. In view of (3.1), we are led to the following recurrence relation of $g_n(y)$.

Corollary 3.2 *For $n \geq 3$, we have*

$$g_n(y) = g_{n-1}(y) + (n-2)y \cdot g_{n-2}(y) - 2y^2 \cdot g'_{n-2}(y), \tag{3.2}$$

where $g_1(y) = 1, g_2(y) = 1$.

4 Neighbor alignments and left, right nestings

In this section, we prove the formula (1.3) for the bivariate generating function of the number of partial matchings of $[n+k-1]$ with k arcs that avoid neighbor alignments, left nestings and right nestings. This generating function turns out to be equal to the generating function for the number of self-modified ascent sequences of length n with largest element $n-k-1$ or the number of $3\bar{1}52\bar{4}$-avoiding permutations on $[n]$ that have $n-k$ right-to-left minima, as found by Bousquet-Mélou, Claesson, Dukes and Kitaev [1].

Recall that $\mathcal{R}(n,k)$ denotes the set of partial matchings of $[n]$ with k arcs that avoid neighbor alignments and both left and right nestings, and $\mathcal{M}^{m\times m}(n)$ denotes the set of $m\times m$ upper triangular matrices with nonnegative integer entries which sum to n. We shall give a bijection between the set $\mathcal{R}(n+k-1,k)$ and the set $\mathcal{M}^{(n-k)\times(n-k)}(k)$, from which we can deduce the generating function formula for the numbers $R(n+k-1,k)$.

Theorem 4.1 *For $0 \le k \le n-1$, there is a bijection between the set $\mathcal{R}(n+k-1,k)$ and the set $\mathcal{M}^{(n-k)\times(n-k)}(k)$.*

Proof. Let $M \in \mathcal{R}(n+k-1,k)$ be a partial matching with $n+k-1$ vertices and k arcs avoiding left nestings, right nestings and neighbor alignments. We wish to construct an upper triangular matrix $\beta(M)$ in $\mathcal{M}^{(n-k)\times(n-k)}(k)$. Clearly, there are $n-k-1$ singletons in M. These singletons divide the vertices of M into $n-k$ intervals, the first interval is the interval before the first singleton and the $(i+1)$-st interval is the interval between the i-th and $(i+1)$-st singletons, the $(n-k)$-th interval is the interval after the last singleton.

From these $n-k$ intervals, we can construct an $(n-k)\times(n-k)$ upper triangular matrix $\beta(M)$. For $1 \le i \le j \le n-k$, define the (i,j)-entry of $\beta(M)$ to be the number of arcs of M starting with a vertex in the i-th interval and ending with a vertex in the j-th interval. Clearly, for an origin in the i-th interval, the corresponding destination is in some j-th interval, where $j \ge i$. Since there are k arcs in M, we see that $\beta(M) \in \mathcal{M}^{(n-k)\times(n-k)}(k)$.

Conversely, given an upper triangular matrix $T \in \mathcal{M}^{(n-k)\times(n-k)}(k)$, we can recover the linear representation of a partial matching $\beta'(T)$ in $\mathcal{R}(n+k-1,k)$. For $1 \le i,j \le n-k$, let $t_{i,j}$ denote the (i,j)-entry of T. Let r_i and s_j denote the i-th row sum and the j-th column sum of T respectively.

The partial matching $\beta'(T)$ is constructed as follows. First, we draw $n-k-1$ singletons on a line to form $n-k$ intervals. Then we need to determine the origins and the destinations in each interval. For $1 \le i \le n-k$, we put r_i origins and s_i destinations in the i-th interval, where all the destinations are placed after all the origins. So there are $(n-k-1)+2k = n+k-1$ vertices. Next, we label the vertices from left to right by $1,2,\ldots,n+k-1$.

Finally, we should match the k origins and the k destinations to form k arcs. For $1 \le i \le n-k$, for the r_i origins in the i-th interval, their corresponding destinations are determined as follows. As the initial step, for each j $(i \le j \le n-k)$, we choose the first $t_{i,j}$ available destinations (i.e., the destinations that have not been matched) in the j-th interval. It is easy to check that there are $t_{i,i}+t_{i,i+1}+\cdots+t_{i,n-k}=r_i$ destinations that have been chosen so far. Then we match these r_i destinations with the r_i origins in the i-th interval to form an r_i-crossing. This construction ensures that there are neither left nestings nor right nestings in $\beta'(T)$. Furthermore, the positions of singletons guarantee that there are no neighbor alignments in $\beta'(T)$. Therefore $\beta'(T)$ is a partial matching in $\mathcal{R}(n+k-1,k)$. Moreover, it is easy to see that the map β' is the inverse of the map β. Thus β is a bijection. This completes the proof. ∎

For example, for $n=10$ and $k=6$, let

$$M = \{\{1,6\},\{2,7\},\{3\},\{4,8\},\{5,14\},\{9\},\{10,11\},\{12\},\{13,15\}\},$$

which belongs to $\mathcal{R}(15,6)$. The three singletons $3, 9, 12$ divide the vertices into 4 intervals, namely, $\{1,2\}$, $\{4,5,6,7,8\}$, $\{10,11\}$, $\{13,14,15\}$. According to the construction of $\beta(M)$, we have

$$\beta(M) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Conversely, given the upper triangular matrix in (4.1), we can recover a partial matching. First, use three singletons to form four intervals. Assign two origins in the first interval, all their corresponding destinations are in the second interval. Similarly, assign two origins in the second interval, their corresponding destinations are in the second and the fourth interval, and so on. The construction of the corresponding partial matching is illustrated in Figure 4.1.

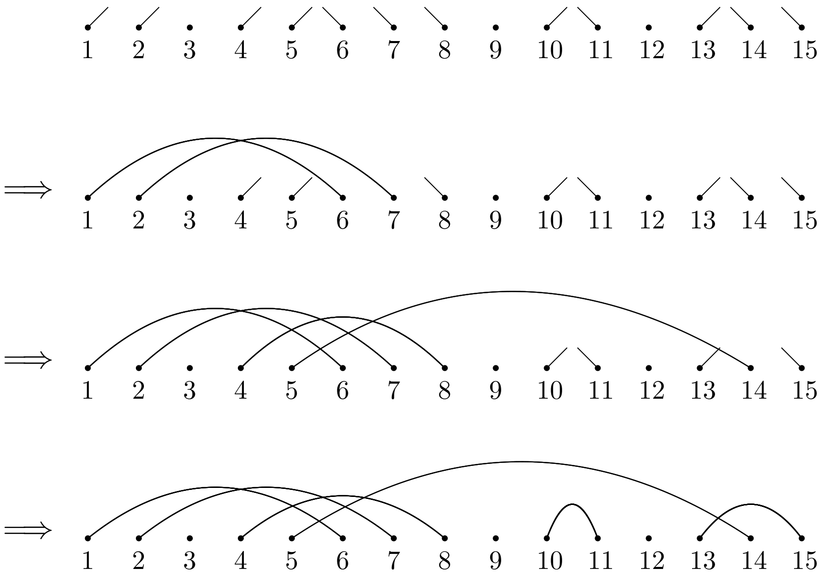

Figure 4.1: An illustration of the bijection β.

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Denote by $\mathcal{C}(k, \binom{n-k+1}{2})$ the set of compositions of k into $\binom{n-k+1}{2}$ parts. Since there are $\binom{n-k+1}{2}$ positions (i,j) of an $(n-k)\times(n-k)$ matrix such that $i \leq j$, there is a one-to-one correspondence between the set $\mathcal{M}^{(n-k)\times(n-k)}(k)$ and the set $\mathcal{C}(k, \binom{n-k+1}{2})$. Note that

$$|\mathcal{C}(k, \binom{n-k+1}{2})| = \binom{k + \binom{n-k+1}{2} - 1}{k} = \binom{\binom{n-k}{2} + n - 1}{k}.$$

By Theorem 4.1, we obtain

$$R(n+k-1,k) = |\mathcal{R}(n+k-1,k)| = |\mathcal{M}^{(n-k)\times(n-k)}(k)| = \binom{\binom{n-k}{2}+n-1}{k}.$$

On the other hand, the coefficient of x^n in

$$\sum_{k=1}^{\infty} \frac{x^k}{(1-xy)^{\binom{k+1}{2}}}$$

equals

$$\sum_{k=1}^{n} \binom{\binom{k}{2}+n-1}{n-k} y^{n-k} = \sum_{k=0}^{n-1} \binom{\binom{n-k}{2}+n-1}{k} y^k.$$

Therefore we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} R(n+k-1,k)x^n y^k &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \binom{\binom{n-k}{2}+n-1}{k} x^n y^k \\ &= \sum_{n=1}^{\infty} \frac{x^n}{(1-xy)^{\binom{n+1}{2}}}. \end{aligned} \tag{4.2}$$

This completes the proof. ∎

The above generating function formula (4.2) suggests a connection between self-modified ascent sequences and partial matchings avoiding neighbor alignments, left and right nestings. Dukes and Parviainen [6] give a bijection from ascent sequences to a special type of upper triangular matrices, which specializes to a correspondence between the set of self-modified ascent sequences of length n with largest element $k-1$ and the set of $k \times k$ upper triangular matrices with nonnegative integer entries which sum to n such that there are no zeros on the diagonal. It is clear that such matrices correspond to general $k \times k$ upper triangular matrices with nonnegative integer entries which sum to $n-k$. In view of the bijection β in Theorem 4.1, we see that there is a one-to-one correspondence between the set $\mathcal{M}^{k\times k}(n-k)$ and the set $\mathcal{R}(2n-k-1, n-k)$. Thus we have the following theorem.

Theorem 4.2 *There is a bijection between the set of self-modified ascent sequences of length n with largest element $k-1$ and the set $\mathcal{R}(2n-k-1, n-k)$ of partial matchings of $[2n-k-1]$ with $n-k$ arcs avoiding left nestings, right nestings and neighbor alignments.*

5 Partitions with no right nestings

In this section, we give a bijection between partitions avoiding right nestings and partial matching avoiding neighbor alignments, left nestings and right nestings. More precisely, we shall construct a bijection between the set $\mathcal{T}(n,k)$ of partitions of $[n]$ with k arcs but with no right nestings and the set $\mathcal{R}(n+k-1,k)$ of partial matchings of $[n+k-1]$ with k arcs that avoid neighbor alignments, left and right nestings.

In fact, we only need to establish a correspondence between the set $\mathcal{T}(n,k)$ and the set $\mathcal{M}^{(n-k)\times(n-k)}(k)$. Combining the bijection β in Section 4 between upper triangular matrices and partial matchings without left, right nestings and neighbor alignments, we obtain a bijection between the set $\mathcal{T}(n,k)$ and the set $\mathcal{R}(n+k-1,k)$. In the previous section we have computed the generating function for the numbers $R(n+k-1,k)$. So we are led to the same generating function formula for $T(n,k)$ as stated in Theorem 1.4.

Theorem 5.1 *For $0 \le k \le n-1$, there exists a bijection between the set $\mathcal{R}(n+k-1,k)$ and the set $\mathcal{T}(n,k)$. Moreover, this bijection transforms the number of left crossings of a partial matching to the number of transient vertices of a partition.*

Proof. It is clear that the theorem holds for $k=0$. We only consider the case $k \ge 1$. Let $M \in \mathcal{R}(n+k-1,k)$, namely, M is a partial matching of $[n+k-1]$ with k arcs but without left nestings, right nestings and neighbor alignments. We wish to construct a partition $\gamma(M) \in \mathcal{T}(n,k)$. We first use the bijection β in Section 4 to transform M to an upper triangular matrix $T=\beta(M)$ which is in $\mathcal{M}^{(n-k)\times(n-k)}(k)$. Using the matrix T we can construct a partition $\gamma(M) \in \mathcal{T}(n,k)$.

The construction of a partition $\gamma(M)$ from T can be described as follows. We start with $n-k$ empty intervals by putting down $n-k-1$ singletons on a line. Then we determine the left-hand and right-hand endpoints in each interval so that all the arcs can be determined by the endpoints.

To achieve this goal, we define a k-path as a sequence of k arcs of the form $(v_1,v_2),(v_2,v_3)$, $\ldots,(v_k,v_{k+1})$, where $v_1<v_2<\cdots<v_{k+1}$. Let r_i denote the i-th row sum of T, and let $t_{i,j}$ denote the (i,j)-entry of T. For $1 \le i \le n-k$, we shall construct an r_i-path P_i consists of $(v_1,v_2),(v_2,v_3),\ldots,(v_{r_i},v_{r_i+1})$ such that the origin v_1 of P_i is in the i-th interval and the right-hand endpoints $v_2,\ldots,v_{r_i+1}$ of P_i are determined by the entries in the i-th row of T.

First, we put the origin v_1 of P_i immediately before the leftmost right-hand endpoint that has been constructed in the i-th interval. If there are no right-hand endpoints in the i-th interval, we place v_1 before the i-th singleton. For $i \le j \le n-k$, we assign $t_{i,j}$ right-hand endpoints to the j-th interval.

After the origin v_1 is determined, we continue to determine the position of the right-hand endpoint v_2 of P_i. We observe that v_2 must be in the m-th interval, where $m=\min\{j : i \le j \le n-k, t_{i,j} \ne 0\}$. Furthermore, we claim that there is a unique position for v_2 in the m-th interval such that the insertion of the arc (v_1,v_2) does not cause any right nestings. We consider two cases.

Case 1. There are no right-hand endpoints to the right of v_1 in the m-th interval. In this case, we put the right-hand endpoint v_2 of P_i immediately before the m-th singleton. Then we relabel the resulting partition.

Case 2. There are t ($t \ge 1$) right-hand endpoints $u_1,u_2,\ldots,u_t$ to the right of v_1 in the m-th interval. The strategy of finding the position of v_2 can be described as follows. We begin with the position immediately to the left of u_1. If v_2 can be placed in this position without causing any right nestings, then this is the position we are looking for. Otherwise, we consider the position immediately before u_2 as the second candidate.

Like the case for u_1, if putting v_2 immediately before u_2 does not cause any right nestings, then it is the desired choice. Otherwise, we consider the position immediately before u_3 as the third candidate. Repeating this process until we find the position of v_2 such that inserting (v_1, v_2) creates no right nestings.

To see that the above process will terminate at some point, we assume that v_2 cannot be placed immediately before u_r $(1 \leq r \leq t)$, and we assume that putting v_2 immediately after u_r also yields a right nesting. Then this right nesting caused by putting v_2 immediately after u_r must be formed by the arc (v_1, v_2) and the arc whose right-hand endpoint is immediately after u_r. This means that there is a right-hand endpoint after u_r. Since the number of right-hand endpoints in every interval is finite, we conclude that there always exists a position such that inserting the arc (v_1, v_2) does not cause any right nestings in the construction of the r_i-path P_i based on the matrix T.

Once the origin v_1 of P_i is determined, we need to show that there exists a position to put the right-hand endpoint v_2 of P_i such that the insertion of the arc (v_1, v_2) does not cause any right nestings. Furthermore, we also need the choice for the position of v_2 is unique. Assume that we have found a position immediately before the vertex u_{k_1} $(1 \leq k_1 \leq t)$ such that no right nestings will be formed after the insertion of the arc (v_1, v_2). It can be checked that all the positions to the right of u_{k_1} cannot be chosen for the insertion of (v_1, v_2).

To the contrary, assume that the position immediately after the vertex u_{k_2} is a possible choice, where $1 \leq u_{k_1} < u_{k_2} \leq t$. We now proceed to find a right nesting that leads to a contradiction. If v_2 can be put immediately before u_{k_1}, then the arc (v_1, v_2) and the arc $e_1 = (l_1, u_{k_1})$ form a crossing, that is, $v_1 < l_1$; On the other hand, if v_2 can be placed immediately after u_{k_2}, then (v_1, v_2) and $e_2 = (l_2, u_{k_2})$ form a crossing as well, that is, $l_2 < v_1$. This implies that $l_2 < l_1$. So the arcs e_1 and e_2 form a nesting.

In fact, based on the nesting formed by e_1 and e_2 it can be seen that there exists a right nesting formed by two arcs with right-hand endpoints between u_{k_1} and u_{k_2} in the construction of the r_i-path P_i. To this end, let us consider the distance between u_{k_1} and u_{k_2}. If $u_{k_1} + 1 = u_{k_2}$, then e_1 and e_2 form a right nesting. If $u_{k_1} + 2 = u_{k_2}$, namely, there is a vertex u_{k_1+1} between u_{k_1} and u_{k_2}, then the arc with right-hand endpoint u_{k_1+1} forms a right nesting with e_1 or e_2. We now turn to the case that there are at least two vertices between u_{k_1} and u_{k_2}. Assume that $e_3 = (l_3, u_{k_1+1})$ is the arc with right-hand endpoint u_{k_1+1}, and $e_4 = (l_4, u_{k_2-1})$ is the arc with right-hand endpoint u_{k_2-1}. Since at each step of the insertion process no right nestings are formed, e_3 and $e_1 = (l_1, u_{k_1})$ form a crossing, that is, $l_1 < l_3$. Moreover, e_4 and $e_2 = (l_2, u_{k_2})$ form a crossing, namely, $l_4 < l_2$. Thus we deduce that e_3 and e_4 form a nesting as well, and the distance between the right-hand endpoints of e_3 and e_4 is shorter than the distance between the right-hand endpoints of e_1 and e_2. See Figure 5.1 for an illustration.

Iterating the above process to reduce the distance between the vertex u_{k_1+1} and the vertex u_{k_2-1}, we eventually find a right nesting, which is a contradiction. This implies that

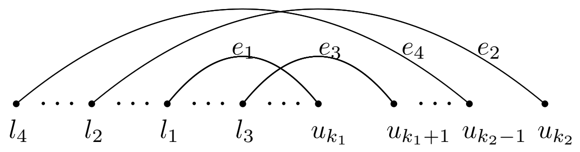

Figure 5.1: The uniqueness of inserting an arc.

there is a unique position for v_2 such that the insertion of the arc (v_1, v_2) causes no right nestings in the construction of the r_i-path P_i based on the matrix T.

Using the same process for the arc (v_2, v_3), we see that there is a unique position for v_3 such that the insertion (v_2, v_3) does not cause any right nestings. By iteration, we find that we can construct a unique r_i-path P_i by inserting the arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{r_i}, v_{r_i+1})$.

Since P_i has $r_i + 1$ vertices, after we construct $n - k$ paths $P_1, P_2, \ldots, P_{n-k}$ based on the matrix T, we obtain a partition τ with each P_i representing a block. Clearly τ has $(r_1 + 1) + \cdots + (r_{n-k} + 1) + (n - k - 1) = 2n - k - 1$ vertices. Moreover, since no right nestings are formed in the construction of P_i for each i, we see that τ has no right nestings. Finally, delete the $n - k - 1$ singletons immediately before each origin of τ, except for the first origin. Denote the resulting partition by $\gamma(M)$. Thus $\gamma(M) \in \mathcal{T}(n, k)$ is a partition of $[n]$ with k arcs but without right nestings. Furthermore, it is easily seen that the number of left crossings of M equals

$$\sum_{r_i : r_i > 0} (r_i - 1),$$

which is also the number of transient vertices of $\gamma(M)$.

Conversely, for $k \geq 1$, given a partition $\pi \in \mathcal{T}(n, k)$ with k arcs that has no right nestings, we can recover a partial matching $\gamma'(\pi)$ in $\mathcal{R}(n + k - 1, k)$. It is clear that we should add $k - 1$ vertices to π. The construction can be described as follows.

First, we add a singleton before each origin of π except the first origin. Let π' denote the resulting partition. Since the partition π has $n - k$ blocks and the number of singletons added to π equals the number of non-singleton blocks of π minus one, we deduce that π' has $n - k - 1$ singletons which divide the vertices of π' into $n - k$ intervals.

From these $n - k$ intervals and the arcs of π', it is easy to construct an upper triangular matrix T' in $\mathcal{M}^{(n-k)\times(n-k)}(k)$. Note that in each interval, there is at most one origin of π'. The entries of the upper triangular matrix T' can be determined as follows. For $1 \leq i \leq j \leq n - k$, if there is an r-path P corresponding to a block of π' whose origin is in the i-th interval, then we set the (i, j)-entry of T' to be the number of right-hand endpoints of P which are in the j-th interval. Otherwise, set the (i, j)-entry of T' to be zero. It is evident that $T' \in \mathcal{M}^{(n-k)\times(n-k)}(k)$.

Finally, set $\gamma'(\pi) = \beta(T')$. Recall that the map β in Section 4 is a bijection between the set $\mathcal{M}^{(n-k)\times(n-k)}(k)$ and the set $\mathcal{R}(n + k - 1, k)$. Thus $\gamma'(\pi)$ is a partial matching in $\mathcal{R}(n + k - 1, k)$. Furthermore, it can be checked that the number of transient vertices of π

equals to

$$\sum_{r_i':\, r_i'>0} (r_i' - 1), \tag{5.1}$$

where r_i' is the i-th row sum of T'. It can be verified that (5.1) is also the number of left crossings in $\gamma'(\pi)$. Thus the number of transient vertices of π is equal to the number of left crossings in $\gamma'(\pi)$.

It is routine to check that the map γ' is the inverse of the map γ. Hence γ is a bijection. This completes the proof. ∎

Figure 5.2 gives an example of a partial matching M without left, right nestings and neighbor alignments. It also demonstrates the procedure to construct a partition $\gamma(M)$ without right nestings. There are two singletons in M which create three intervals. For the origins in the first interval, their corresponding destinations are in the second and the third interval, and so on. The upper triangular matrix T corresponding to M is

$$T = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the upper triangular matrix T, one can construct $\gamma(M)$. The first row of T corresponds to a 3-path of $\gamma(M)$, whose origin is in the first interval and the three right-hand endpoints are in the second and the third interval, and so on.

To conclude, we remark that in general the number of partitions of $[n]$ avoiding right crossings is not equal to the number of partitions of $[n]$ avoiding right nestings. It would be interesting to find the generating function for the number of partitions of $[n]$ avoiding right crossings.

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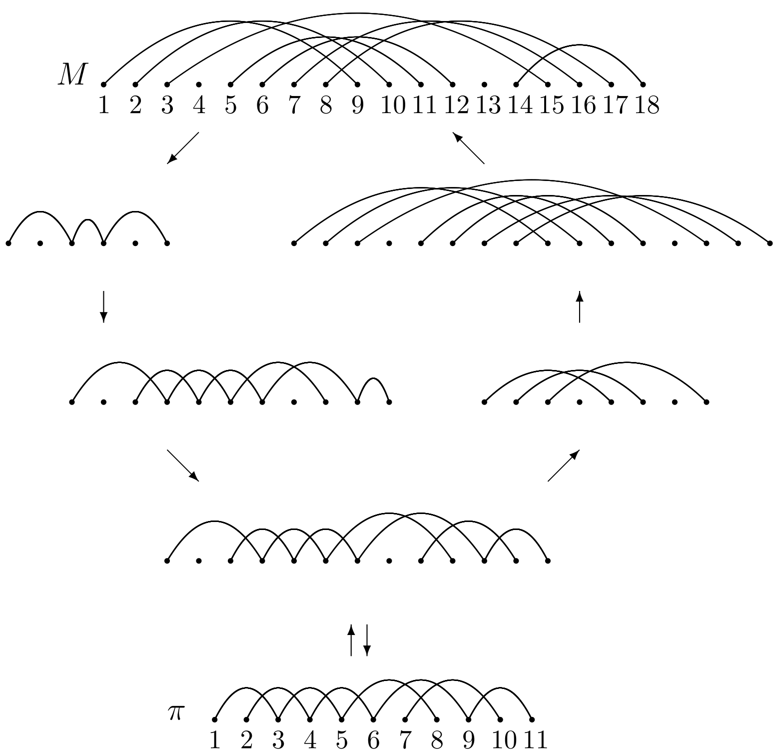

Figure 5.2: An illustration of the bijection γ.

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