On the strong rainbow connection of a graph*

Xueliang Li, Yuefang Sun
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
E-mails: lxl@nankai.edu.cn, bruceseun@gmail.com

Abstract

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. For any two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number of G, denoted $src(G)$, is the minimum number of colors that are needed in order to make G strong rainbow connected. In this paper, we first give a sharp upper bound for $src(G)$ in terms of the number of edge-disjoint triangles in graph G, and give a necessary and sufficient condition for the equality. We next investigate the graphs with large strong rainbow connection numbers. Chartrand et al. obtained that G is a tree if and only if $src(G) = m$, we will show that $src(G) \neq m-1$, and characterize the graphs G with $src(G) = m-2$ where m is the number of edges of G.

Keywords: edge-colored graph, rainbow path, rainbow geodesic, strong rainbow connection number, edge-disjoint triangle.

AMS Subject Classification 2000: 05C15, 05C40

1 Introduction

All graphs in this paper are finite, undirected and simple. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-coloring graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph G, denoted $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. Let c be a rainbow coloring

*Supported by NSFC No. 11071130.

of a connected graph G. For any two vertices u and v of G, a *rainbow* $u - v$ *geodesic* in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u - v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected. A strong rainbow coloring of G using $src(G)$ colors is called a *minimum strong rainbow coloring* of G. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$ where $diam(G)$ denotes the diameter of G and m is the number of edges of G.

The topic of rainbow connection is fairly interesting and recently a series papers have been written about it. The reader can see [8] for a survey of this topic. The strong rainbow connection is also interesting and, by definition, the investigation of it is more challenging than that of rainbow connection. However, there are very few papers that have been written about it. In this paper, we do research on it. In [3], Chartrand et al. determined the precise strong rainbow connection numbers for some special graph classes including trees, complete graphs, wheels, complete bipartite (multipartite) graphs.

Recently, P. Ananth and M. Nasre [1] derived the following hardness result about strong rainbow connection number.

Theorem 1.1 *[1] For every $k \geq 3$, deciding whether $src(G) \leq k$, is NP-hard even when G is bipartite.*

So, for a general graph G, it is almost impossible to give the precise value for $src(G)$. And we aim to give upper bounds for it according to some graph parameters. In this paper, we will derive a sharp upper bound for $src(G)$ in terms of the number of edge-disjoint triangles (if exist) in graph G, and give a necessary and sufficient condition for the equality (Theorem 3.1).

In [4], the authors investigated the graphs with small rainbow connection numbers, they determined a sufficient condition that guarantee $rc(G) = 2$ and give the threshold function for a random graph $G = G(n, p)$ to have $rc(G(n, p)) \leq 2$..

Theorem 1.2 *([4]) Any non-complete graph with $\delta(G) \geq n/2 + \log n$ has $rc(G) = 2$.* ∎

Theorem 1.3 *([4]) $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G(n, p)) \leq 2$.* ∎

In [3], the authors derived that the problem of considering graphs with $rc(G) = 2$ is equivalent to that of considering graphs with $src(G) = 2$.

Proposition 1.4 *([3]) $rc(G) = 2$ if and only if $src(G) = 2$.* ∎

In [7], Li and Sun did research on graphs with large rainbow connection numbers, and showed that $rc(G) \neq m - 1$ and characterize the graphs with $rc(G) = m - 2$. In this paper, we aim to investigate the graphs with large strong rainbow connection numbers. In [3], Chartrand et al. obtained that $src(G) = m$ if and only if G is a tree. We will show that $src(G) \neq m - 1$ and characterize the graphs with $src(G) = m - 2$ by showing that $src(G) = m - 2$ if and only if G is a 5-cycle or belongs to one of two graph classes (Theorem 4.1).

We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ be the subgraph induced by X, and $E[X]$ the edge set of $G[X]$; similarly, for any subset E_1 of $E(G)$, let $G[E_1]$ be the subgraph induced by E_1. Let $\mathcal{G}$ be a set of graphs, then $V(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} V(G)$, $E(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} E(G)$. A *rooted tree* $T(x)$ is a tree T with a specified vertex x, called the *root* of T. The path xTv is the only $x - v$ path in T, each vertex on the path xTv, including the vertex v itself, is called an *ancestor* of v, an ancestor of a vertex is *proper* if it is not the vertex itself, the immediate proper ancestor of a vertex v other than the root is its *parent* and the vertices whose parent is v are its *children* or *son*. We let P_n and C_n be the path and cycle with n vertices, respectively. $P : u_1, u_2, \cdots, u_t$ is a path, then the $u_i - u_j$ *section* of P, denoted by u_iPu_j, is the path: $u_i, u_{i+1}, \cdots, u_j$. Similarly, for a cycle $C : v_1, \cdots, v_t, v_1$; we define the $v_i - v_j$ section, denoted by v_iCv_j of C, and C contains two $v_i - v_j$ sections. Note the fact that if P is a $u_1 - u_t$ geodesic, then u_iPu_j is also a $u_i - u_j$ geodesic where $1 \leq i, j \leq t$. We use $l(P)$ to denote the length of path P. For a set S, $|S|$ denote the cardinality of S. In a graph G which has at least one cycle, the length of a shortest cycle is called its *girth*, denoted $g(G)$. In an edge-colored graph G, we use $c(e)$ to denote the color of edge e, then for a subgraph G_1 of G, $c(G_1)$ denotes the set of colors of edges in G_1. We follow the notation and terminology of [2].

2 Basic results

We first give a necessary condition for an edge-colored graph to be strong rainbow connected. If G contains at least two cut edges, then for any two cut edges $e_1 = u_1u_2$, $e_1 = v_1v_2$, there must exist some $1 \leq i_0, j_0 \leq 2$, such that any $u_{i_0} - v_{j_0}$ path must contain edge e_1, e_2. So we have:

Observation 2.1 *If G is strongly rainbow connected under some edge coloring and e_1 and e_2 are two cut edges, then $c(e_1) \neq c(e_2)$.* ∎

The following lemma will be useful in our discussion.

Lemma 2.2 *If $src(G) = m - 1$ or $m - 2$, then $3 \leq g(G) \leq 5$.*

Proof. Let $C : v_1, \cdots, v_k, v_{k+1} = v_1$ be a minimum cycle of G with $k = g(G)$, and $e_i = v_iv_{i+1}$ for each $1 \leq i \leq k$, we suppose that $k \geq 6$. We give the cycle C a strong rainbow coloring the same as [3]: If k is even, let $k = 2\ell$ for some integer $\ell \geq 3$, $c(e_i) = i$ for $1 \leq i \leq \ell$ and $c(e_i) = i - \ell$ for $\ell + 1 \leq i \leq k$; If k is odd, let $k = 2\ell + 1$ for some integer $\ell \geq 3$, $c(e_i) = i$ for $1 \leq i \leq \ell + 1$ and $c(e_i) = i - \ell - 1$ for $\ell + 2 \leq i \leq k$. We color each other edge with a fresh color. This procedure costs $\lceil \frac{k}{2} \rceil + (m - k) = m - (k - \lceil \frac{k}{2} \rceil) \leq m - 3$ colors totally.

We only consider the case $k = 2\ell(\ell \geq 3)$, since the case that $k = 2\ell + 1(\ell \geq 3)$ is similar. Let $P : u = u_1, \cdots, v = u_t$ be a $u - v$ geodesic of G. If there are two edges of P, say e'_1, e'_2, with the same color, then they must be in C. Without loss of generality, let $e'_1 = v_1v_2$, we first consider the case that $e'_1 = v_1v_2$, and $v_1 = u_{i_1}, v_2 = u_{i_1+1}$ for some $1 \leq i_1 \leq t$, then we must have $e'_2 = v_{\ell+1}v_{\ell+2}$ where $v_{\ell+1} = u_{j_1}$, $v_{\ell+2} = u_{j_1+1}$ for some $i_1+1 \leq j_1 \leq t$ or $v_{\ell+2} = u_{j_2}$, $v_{\ell+1} = u_{j_2+1}$ for some $i_1 + 1 \leq j_2 \leq t$. If $v_{\ell+1} = u_{j_1}$, $v_{\ell+2} = u_{j_1+1}$ for some $i_1 + 1 \leq j_1 \leq t$, then the section $v_2Pv_{\ell+1}$ of P is a $v_2 - v_{\ell+1}$ geodesic, so it is not longer than the section $C' : v_2, v_3, \cdots, v_{\ell+1}$ of C, then the length of $v_2Pv_{\ell+1}$, $l(v_2Pv_{\ell+1}) \leq \ell - 1$, is smaller than the length of the section $C'' : v_2, v_1, v_k, \cdots, v_{\ell+1}$ of C. So the sections $v_2Pv_{\ell+1}$ and C' will produce a smaller cycle than C(this produces a contradiction), or $v_2Pv_{\ell+1}$ is the same as C'(but in this case, the section $C''' : v_1, v_k, \cdots, v_{\ell+2}$ of C is shorter than $v_1Pv_{\ell+2}$ which now is a $v_1 - v_{\ell+2}$ geodesic, this also produces a contradiction). If $v_{\ell+2} = u_{j_2}$, $v_{\ell+1} = u_{j_2+1}$ for some $i_1 + 1 \leq j_2 \leq t$, then the section $v_1Pv_{\ell+2}$ of P is a $v_1 - v_{\ell+2}$ geodesic, so it is not longer than the length of the section $\overline{C'} : v_1, v_k, v_{k-1}, \cdots, v_{\ell+2}$ of C and its length, $l(v_1Pv_{\ell+2}) \leq \ell - 1$, is smaller than that of the section $\overline{C''} : v_1, v_2, \cdots, v_{\ell+2}$ of C. So the sections $v_1Pv_{\ell+2}$ and $\overline{C'}$ will produce a smaller cycle than C, this also produces a contradiction. So P is rainbow. The remaining two subcases correspond to the case that $v_1 = u_{i_1+1}$, $v_2 = u_{i_1}$, and with a similar argument, a contradiction will be produced. Then the conclusion holds. ∎

Note that we prove the above lemma by contradiction: we first choose a smallest cycle C of a graph G, then give it a strong rainbow coloring the same as [3], and give a fresh color to any other edge. Then for any $u - v$ geodesic P, we derive that either one section of P is the same as one section of C and then find a shorter path than the geodesic, or one section of P and one section of C produce a smaller cycle than C, each of these two cases will produce a contradiction. This technique will be useful in the sequel.

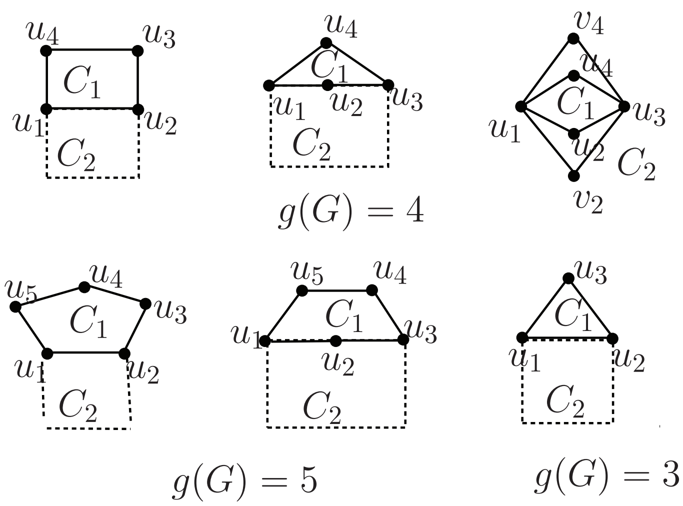

Figure 2.1 Figure for Observation 2.3.

The following observation is obvious and we omit the proof.

Observation 2.3 *G is a connected graph with at least one cycle, and $3 \leq g(G) \leq 5$. Let C_1 be the smallest cycle of G, and C_2 be the smallest cycle among all remaining cycles (if exist) of G. If C_1 and C_2 have at least two common vertices, then we have:*

1. *If $g(G) = 3$, then C_1 and C_2 have one common edge as shown in Figure 2.1;*

***2.** If $g(G) = 4$, then C_1 and C_2 have one common edge, or two common adjacent edges, or C_1 and C_2 are two edge-disjoint 4-cycles, as shown in Figure 2.1;*

***3.** If $g(G) = 5$, then C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 2.1.*

The following observation is very useful and can be proved by contradiction.

Observation 2.4 *For any two vertices u, $v \in G$, we have the following.*

***1.** If T is a triangle in graph G, then any $u - v$ geodesic P contains at most one edge of T;*

***2.** If $g(G) = 4$ and C_1 is the smallest cycle of G, then any $u - v$ geodesic P contains at most one edge or two adjacent edges of C_1;*

***3.** If $g(G) = 5$ and C_1 is the smallest cycle of G, then any $u - v$ geodesic P contains at most one edge or two adjacent edges of C_1.*

3 A sharp upper bound for $src(G)$ in terms of edge-disjoint triangles

In this section, we give an upper bound for $src(G)$ in terms of their edge-disjoint triangles (if exist) in graph G, and give a necessary and sufficient condition for the equality.

Recall that a *block* of a connected graph G is a maximal connected subgraph without a cut vertex. Thus, every block of graph G is either a maximal 2-connected subgraph or a bridge (cut edge). We now introduce a new graph class. For a connected graph G, we say $G \in \overline{\mathcal{G}}_t$, if it satisfies the following conditions:

C_1**.** Each block of G is a bridge or a triangle;

C_2**.** G contains exactly t triangles;

C_3**.** Each triangle contains at least one vertex of degree two in G.

By the definition, each graph $G \in \overline{\mathcal{G}}_t$ is formed by (edge-disjoint) triangles and paths (may be trivial), these triangles and paths fit together in a treelike structure, and G contains no cycles but the t (edge-disjoint) triangles. For example, see Figure 3.1, here $t = 2$, u_1, u_2, u_6 are vertices of degree 2 in G. If a tree is obtained from a graph $G \in \overline{\mathcal{G}}_t$ by deleting one vertex of degree 2 for each triangle, then we call this tree is a *D_2-tree* of G, denoted T_G. For example, in Figure 3.1, T_G is a D_2-tree of G. Clearly, the D_2-tree is not unique, since in this example, we can obtain another D_2-tree by deleting vertex u_1 instead of u_2. On the other hand, we can say any element of $\overline{\mathcal{G}}_t$ can be obtained from a tree by adding t new vertices of degree 2. It is easy to show that number of edges of T_G is $m - 2t$ where m is the number of edges of G.

Theorem 3.1 *G is a graph with m edges and t edge-disjoint triangles, then*

$$src(G) \leq m - 2t,$$

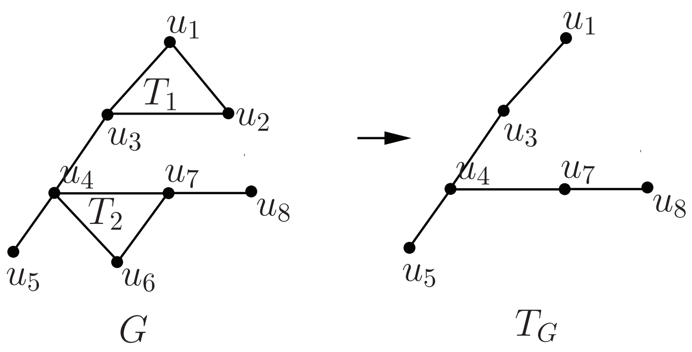

Figure 3.1 An example of $G \in \overline{\mathcal{G}}_t$ with $t = 2$.

the equality holds if and only if $G \in \overline{\mathcal{G}}_t$.

Proof. Let $\mathcal{T} = \{T_i : 1 \leq i \leq t\}$ be a set of t edge-disjoint triangles in G.

We color each triangle with a fresh color, that is, the three edges of each triangle receive the same color, then we give each other edge a fresh color. For any two vertices u, v of G, let P be any $u-v$ geodesic, then P contains at most one edge from each triangle by Observation 2.4, so P is rainbow under the above coloring. As this procedure costs $m - 2t$ colors totally, we have $src(G) \leq m - 2t$.

Claim 1. If the equality holds, then for any set $\mathcal{T}$ of edge-disjoint triangles of G, we have $|\mathcal{T}| \leq t$.

Proof of Claim 1. We suppose there is a set $\mathcal{T}'$ of t' edge-disjoint triangles in G with $t' > t$, then with a similar procedure, we have $src(G) \leq m - 2t' < m - 2t$, a contradiction.

Claim 2. If the equality holds, then G contains no cycle but the above t (edge-disjoint) triangles.

Proof of Claim 2. We suppose that there are at least one cycles distinct with the above t triangles. Let $\mathcal{C}$ be the set of these cycles and C_1 be the smallest element of $\mathcal{C}$ with $|C_1| = k$. We will consider two cases:

Case 1. $E(C_1) \cap E(\mathcal{T}) = \emptyset$, that is, C_1 is edge-disjoint with each of the above t triangles. Clearly, C_1 has at most one common vertex with each of the above t triangles. In this case $k \geq 4$ by **Claim 1**. We give G an edge coloring as follows: we first color edges of cycle C_1 the same as [3] (this is shown in the proof of Lemma 2.2); then we color each triangle with a fresh color; for the remaining edges, we give each one a fresh color. Recall the fact that any geodesic contains at most one edge from each triangle and with a similar procedure to the proof of Lemma 2.2, we know the above coloring is strong rainbow, as this procedure costs $\lceil \frac{k}{2} \rceil + t + (m - k - 3t) = (m - 2t) + (\lceil \frac{k}{2} \rceil - k) < m - 2t$, we have $src(G) < m - 2t$, this produces a contradiction.

Case 2. $E(C_1) \cap E(\mathcal{T}) \neq \emptyset$, that is, C_1 has common edges with the above t triangles, in this case $k \geq 3$. By the choice of C_1, we know $|E(C_1) \cap E(T_i)| \leq 1$ for each $1 \leq i \leq t$. We will consider two subcases according to the parity of k.

Subcase 2.1. $k = 2\ell$ for some $\ell \geq 2$. For example, see graph (α) of Figure 3.2, here $\mathcal{T} =$

$\{T_1, T_2, T_3\}$, $V(C_1) = \{u_i : 1 \leq i \leq 6\}$, $E(C_1) \cap E(T_1) = \{u_1u_2\}$, $E(C_1) \cap E(T_2) = \{u_4u_5\}$. Without loss of generality, we assume that there exists a triangle, say T_1, which contains edge u_1u_2 and let $V(T_1) = \{u_1, u_2, w_1\}$, $G' = G \backslash E(T_1)$. If there exists some triangle, say T_2, which contains edge $u_{\ell+1}u_{\ell+2}$, we let $V(T_2) = \{u_{\ell+1}, u_{\ell+2}, w_2\}$.

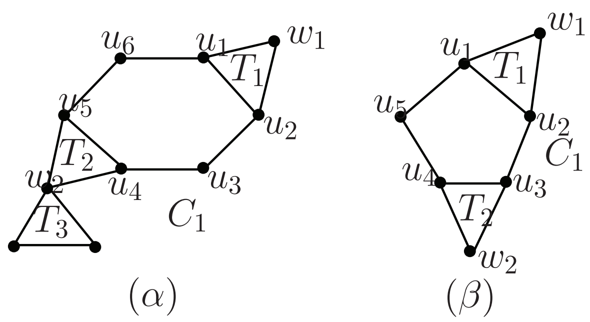

Figure 3.2 Graphs of two examples in Theorem 3.1.

We first consider the case that $\ell = 2$, see Figure 3.3, we first give each triangle of G' a fresh color; for the remaining edges of G', we give each of them a fresh color; for edges of T_1, let $c(u_1w_1) = c(u_2u_3)$, $c(u_2w_1) = c(u_1u_4)$, $c(u_1u_2) = c(u_3u_4)$. Then it is easy to prove that there is a $u - v$ geodesic which contains at most one edge from any two edges with the same color for $u, v \in G$, so the above coloring is strong rainbow. As this procedure costs $m - 2t - 1 < m - 2t$ colors totally, we have $src(G) < m - 2t$, a contradiction.

We next consider the case that $\ell \geq 3$. Let $G'' = G \backslash (E(T_1) \cup E(T_2))$. We give G an edge-coloring as follows: We first give each triangle of G'' a fresh color; then give a fresh color to each of the remaining edges of G''; for the edges of T_1 and T_2, let $c(u_1w_1) = c(u_2u_3) = a$, $c(u_2w_1) = c(u_1u_k) = b$, $c(u_1u_2) = c(u_{\ell+1}u_{\ell+2}) = c$, $c(w_2u_{\ell+1}) = c(u_{\ell+2}u_{\ell+3}) = d$, $c(w_2u_{\ell+2}) = c(u_\ell u_{\ell+1}) = e$ where a, b, c, d, e are five new colors. Then it is easy to show that there is a $u - v$ geodesic which contains at most one edge from any two edges with the same color for $u, v \in G$, so the above coloring is strong rainbow. As this procedure costs $m - 2t - 1 < m - 2t$ colors totally, we have $src(G) < m - 2t$, a contradiction.

Subcase 2.2. $k = 2\ell + 1$ for some $\ell \geq 1$.

We first consider the case that $\ell \geq 2$. For example, see graph (β) of Figure 3.2, here $\mathcal{T} = \{T_1, T_2\}$, $V(C_1) = \{u_i : 1 \leq i \leq 5\}$, $E(C_1) \cap E(T_1) = \{u_1u_2\}$, $E(C_1) \cap E(T_2) = \{u_3u_4\}$. Without loss of generality, we assume that there exists a triangle, say T_1, which contains edge u_1u_2 and let $V(T_1) = \{u_1, u_2, w_1\}$. If there exists some triangle, say T_2, which contains edge $u_{\ell+1}u_{\ell+2}$, we let $V(T_2) = \{u_{\ell+1}, u_{\ell+2}, w_2\}$ and $G' = G \backslash (E(T_1) \cup E(T_2))$.

We give G an edge-coloring as follows: We first give each triangle of G' a fresh color; then give a fresh color to each of the remaining edges of G'; for the edges of T_1 and T_2, let $c(u_1w_1) = c(u_2u_3)$, $c(u_2w_1) = c(u_1u_k)$, $c(u_{\ell+1}w_2) = c(u_{\ell+2}u_{\ell+3})$ and let $c(u_1u_2) = c(u_{\ell+1}u_{\ell+2}) = c(w_2u_{\ell+2})$ be a fresh color. With a similar procedure to the proof of Lemma 2.2, we can show that G is strong rainbow connected, and so $src(G) \leq (t-1) + (m - 3t) = (m - 2t) - 1 < m - 2t$, this produces a contradiction.

For the case that $\ell = 1$, that is, C_1 is a triangle. See Figure 3.3, we color the three edges (if exist) with color 1, these edges are shown in the figure; the remaining edges of these three triangles (if exist) all receive color 2; each of other triangles receive a fresh color; for the remaining edges, we give each one a fresh color. It is easy to show that the above coloring is strong rainbow, so we have $src(G) < m - 2t$ in this case, a contradiction. So the claim holds.

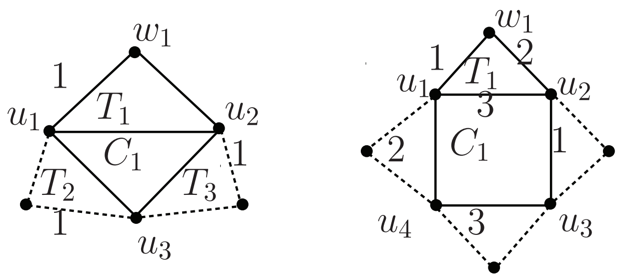

Figure 3.3 Edge coloring for the case that C_1 is a triangle and the case that C_1 a 4-cycle in Theorem 3.1.

Claim 3. If the equality holds, then $G \in \overline{\mathcal{G}}_t$.

Proof of Claim 3. If the equality holds, to prove that $G \in \overline{\mathcal{G}}_t$, it suffices to show that each triangle contains at least one vertex of degree 2 in G. Suppose it doesn't holds, without loss of generality, let T_1 be the triangle with $deg_G(v_i) \geq 3$, where $V(T_1) = \{v_i : 1 \leq i \leq 3\}$. By **Claim 2**, it is easy to show that $E(T_1)$ is an edge-cut of G, let H_i be the subgraph of $G \backslash E(T_1)$ containing vertex v_i $(1 \leq i \leq 3)$, by the assumption of T_1, we know each H_i is nontrivial. We now give G an edge-coloring: for the $t-1$ (edge-disjoint) triangles of $G \backslash E(T_1)$, we give each of them a fresh color; for the remaining edges of $G \backslash E(T_1)$ (by **Claim 2**, each of them must be a cut edge), we give each of them a fresh color; for the edges of $E(T_1)$, let $c(v_1v_3) \in c(H_2)$, $c(v_1v_2) \in c(H_3)$, $c(v_2v_3) \in c(H_1)$. It is easy to show, with the above coloring, G is strong rainbow connected, and we have $src(G) < m - 2t$, a contradiction, so the claim holds.

Claim 4. If $G \in \overline{\mathcal{G}}_t$, then the equality holds.

Proof of Claim 4. Let T_G be a D_2-tree of G, the result clearly holds for the case $|E(T_G)| = 1$. So now we assume $|E(T_G)| \geq 2$. We will show, for any strong rainbow coloring of G, $c(e_1) \neq c_(e_2)$ where $e_1, e_2 \in T_G$, that is, each edge of T_G receive a distinct color, so edges of T_G cost $m - 2t$ colors totally, recall that $|E(T_G)| = m - 2t$, then $src(G) \geq m - 2t$, by the above claim, **Claim 4** holds.

For any two edges, say e_1, e_2, of T_G, let $e_1 = u_1u_2$, $e_2 = v_1v_2$. Without loss of generality, we assume $d_{T_G}(u_1, v_2) = \max\{d_{T_G}(u_i, v_j) : 1 \leq i, j \leq 2\}$ where $d_{T_G}(u, v)$ denote the distance between u and v in T_G. As T_G is a tree, the (unique) $u_1 - v_2$ geodesic, say P, in T_G must contains edges e_1, e_2. Moreover, it is easy to show P is also an unique $u_1 - v_2$ geodesic in G, so $c(e_1) \neq c_(e_2)$ under any strong rainbow coloring.

By **Claim 3** and **Claim 4**, the equality holds if and only if $G \in \overline{\mathcal{G}}_t$. Then our result holds. ∎

In [5, 6], Li and Sun investigated rainbow connection numbers of line graphs. As an application to Theorem 3.1, we consider the strong rainbow connection numbers of line graphs of connected cubic graphs. Recall that the *line graph* of a graph G is the graph

$L(G)$ whose vertex set $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G. The star, denoted $S(v)$, at a vertex v of graph G, is the set of all edges incident to v. Let $\langle S(v)\rangle$ be the subgraph of $L(G)$ induced by $S(v)$, clearly, it is a clique of $L(G)$. A *clique decomposition* of G is a collection $\mathscr{C}$ of cliques such that each edge of G occurs in exactly one clique in $\mathscr{C}$. An *inner vertex* of a graph is a vertex with degree at least two. For a graph G, we use $\overline{V_2}$ to denote the set of all inner vertices of G. Let $\mathscr{K}_0 = \{\langle S(v)\rangle : v \in V(G)\}$, $\mathscr{K} = \{\langle S(v)\rangle : v \in \overline{V_2}\}$. It is easy to show that $\mathscr{K}_0$ is a clique decomposition of $L(G)$ and each vertex of the line graph belongs to at most two elements of $\mathscr{K}_0$. We know that each element $\langle S(v)\rangle$ of $\mathscr{K}_0 \setminus \mathscr{K}$, a single vertex of $L(G)$, is contained in the clique induced by u that is adjacent to v in G. So $\mathscr{K}$ is a clique decomposition of $L(G)$.

Corollary 3.2 *Let $L(G)$ be the line graph of a connected cubic graph with n vertices, then $src(L(G)) \leq n$.*

Proof. Since G is a connected cubic graph, each vertex is an inner vertex and the clique $\langle S(v)\rangle$ in $L(G)$ corresponding to each vertex v is a triangle. We know that $\mathscr{K} = \{\langle S(v)\rangle : v \in \overline{V_2}\} = \{\langle S(v)\rangle : v \in V\}$ is a clique decomposition of $L(G)$. Let $\mathcal{T} = \mathscr{K}$. Then $\mathcal{T}$ is a set of n edge-disjoint triangles that cover all edges of $L(G)$. As there are $3n$ edges in $L(G)$, by Theorem 3.1, we have $src(L(G)) \leq 3n - 2n = n$. ∎

4 Graphs with large strong rainbow connection numbers

In this section, we will give our result on graphs with large strong rainbow connection numbers. We first introduce two graph classes. Let C be the cycle of a unicyclic graph G, $V(C) = \{v_1, \cdots, v_k\}$ and $\mathcal{T}_G = \{T_i : 1 \leq i \leq k\}$ where T_i is the unique tree containing vertex v_i in subgraph $G\backslash E(C)$. We say T_i and T_j are *adjacent*(*nonadjacent*) if v_i and v_j are adjacent(nonadjacent) in cycle C. Then let
$\mathcal{G}_1 = \{G : G$ is a unicyclic graph, $k = 3$, $\mathcal{T}_G$ contains at most two nontrivial elements$\}$,
$\mathcal{G}_2 = \{G : G$ is a unicyclic graph, $k = 4$, $\mathcal{T}_G$ contains two nonadjacent trivial elements and the other two (nonadjacent) elements are paths.$\}$.

Theorem 4.1 *G is a connected graph with m edges, then we have:*
(i) $src(G) \neq m - 1$,
(ii) $src(G) = m - 2$ if and only if G is a 5-cycle or belongs to $\mathcal{G}_1$ or $\mathcal{G}_2$.

Proof. In [3], the authors obtained that $src(G) = m$ if and only if G is a tree, so $src(G) \leq m - 1$ if and only if G is not a tree. In order to derive our conclusion, we need the following claim.

Claim 1. If $src(G) = m - 1$ or $m - 2$, then G is a unicyclic graph.

Proof of Claim 1. Suppose G contains at least two cycles, let C_1 be the smallest cycle of G and C_2 be the smallest one among all the remaining cycles in G. Let $|C_i| = k_i (i = 1, 2)$, so by Lemma 2.2, we have $3 \leq k_1 \leq 5$ and $k_2 \geq k_1$. We will consider two cases according to the value of $|E(C_1) \cap E(C_2)|$.

Case 1.$|E(C_1) \cap E(C_2)| = 0$, that is, C_1 and C_2 have no common edge. There are three subcases:

Subcase 1.1. $k_1 = 3$, that is, C_1 is a triangle.

By Observation 2.3, we must have $|V(C_1) \cap V(C_2)| \leq 1$. We first give cycle C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as [3]; then give a fresh color to C_1, that is, edges of C_1 receive the same color; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.2 and by Observation 2.4, we can show that the above coloring is strong rainbow, as this costs $1 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 3)$ colors totally, we have $src(G) \leq 1 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 3) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3$, a contradiction.

Subcase 1.2. $k_1 = 4$, that is, C_1 is a 4-cycle.

If $|V(C_1) \cap V(C_2)| \leq 1$. We first give cycle C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as [3]; then give two fresh colors to C_1 in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.2 and by Observation 2.4, we can show that the above coloring is strong rainbow, as this costs $2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4)$ colors totally, we have $src(G) \leq 2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3$, a contradiction.

Otherwise, by Observation 2.3, it must be the right graph of the three graphs with $g(G) = 4$ in Figure 2.1. We let $c(u_1u_2) = c(u_3u_4) = a, c(u_2u_3) = c(u_1u_4) = b, c(u_1v_2) = c(u_3v_4) = c, c(v_2u_3) = c(u_1v_4) = d$, where a, b, c, d are four distinct colors; for the remaining edges, we give each of them a fresh color. This procedure costs $m - 4$ colors totally. As now both C_1 and C_2 are the smallest cycle of G, by Observation 2.4, any geodesic contains at most one of the two edges with the same color, so $src(G) \leq m - 4$. A contradiction.

Subcase 1.3. $k_1 = 5$, that is, C_1 is a 5-cycle.

By Observation 2.3, we must have $|V(C_1) \cap V(C_2)| \leq 1$. We first give cycle C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as [3]; then give three fresh colors to C_1 in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.2 and by Observation 2.4, we can show that the above coloring is strong rainbow, as this costs $3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5)$ colors totally, we have $src(G) \leq 3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3$, a contradiction.

Note that for each above subcase, by Observation 2.4, the cycle produced during the procedure while we use the similar technique of Lemma 2.2 cannot be the cycle C_1 and must be smaller than C_2, then a contradiction will be produced.

Case 2.$|E(C_1) \cap E(C_2)| \geq 1$, that is, C_1 and C_2 have at least one common edge. And so C_1 and C_2 have at least two common vertices. There are also three subcases:

Subcase 2.1. $k_1 = 3$, that is, C_1 is a triangle. By Observation 2.3, C_1 and C_2 have one common edge as shown in Figure 2.1. Let $V(C_1) = \{u_i : 1 \leq i \leq 3\}$ and $V(C_2) = \{v_i : 1 \leq i \leq k_2\}$ and $v_{k_2+1} = v_1$, where $v_1 = u_1, v_2 = u_2$. Let P' be the subpath of C_2 that doesn't contain edge v_1v_2. We now give G an edge-coloring as follows:

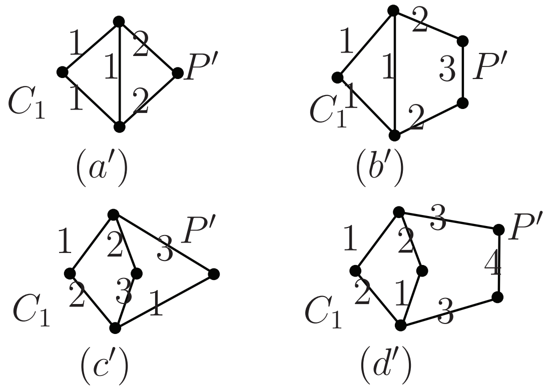

Figure 4.1 Graphs for Case 2 of Claim 1.

For the case $l(P') = 2, 3$, we first color edges of $C_1 \cup C_2$ as shown in Figure 4.1 (graphs a' and b'); then give each other edge of G a fresh color. This procedure costs $m - 3$ colors totally. Then it is easy to show show that any geodesic cannot contain two edges with the same color, so $src(G) \leq m - 3$. This produces a contradiction.

For the remaining case, that is, $l(P') \geq 4$ and $k_2 \geq 5$. We first give cycle C_1 a color, say a, that is, three edges of C_1 receive the same color. Then in C_2, if $k_2 = 2\ell$ for some $\ell \geq 2$, then let $c(v_2v_3) = c(v_{\ell+2}v_{\ell+3})$ be a new color, say b; if $k_2 = 2\ell + 1$ for some $\ell \geq 2$, then let $c(v_2v_3) = c(v_{\ell+3}v_{\ell+4})$ be a new color, say b. For the remaining edges, we give each of them a fresh color. This procedure costs $m - 3$ colors totally. For any two vertices u, v, if P is a $u - v$ geodesic, by Observation 2.4, P cannot contain two edges with color a; for the two edges with color b, with a similar argument to that of Lemma 2.2 (Note that now, by Observation 2.4, the cycle produced during the procedure cannot be C_1 and must be shorter than C_2, then a contradiction will be produced), we can show P contains at most one of them. So P is strong rainbow and $src(G) \leq m - 3$. This produces a contradiction.

Subcase 2.2. $k_1 = 4$, that is, C_1 is a 4-cycle. By Observation 2.3, C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 2.1.

If C_1 and C_2 have one common edge, say u_1u_2 (see the left one of the three graphs with $g(G) = 4$ in Figure 2.1). We let $V(C_2) = \{v_i : 1 \leq i \leq k_2\}$, where $v_1 = u_1, v_2 = u_2$. We let $c(v_2v_3) = c(u_4v_1) = a$, $c(v_2u_3) = c(v_1v_{k_2}) = b$, $c(v_1v_2) = c(u_3u_4) = c$. For the remaining edges, we give each of them a fresh color. This procedure costs $m - 3$ colors totally. For any two vertices u, v, P is a $u - v$ geodesic, then by Observation 2.4, P contains at most one of the two edges with color c; for the two edges with color $a(b)$, it is easy to show that there exists one $u - v$ geodesic which contains at most one of them. So we have $src(G) \leq m - 3$. This produces a contradiction.

Otherwise, then C_1 and C_2 have two common adjacent edges, say u_1u_2, u_2u_3 (see the middle one of the three graphs with $g(G) = 4$ in Figure 2.1). We let $V(C_2) = \{v_i : 1 \leq i \leq k_2\}$, where $v_1 = u_1, v_2 = u_2, v_3 = u_3$. Let P' be the subpath of C_2 which doesn't contain edges u_1u_2, u_2u_3.

For the case $l(P') = 2, 3$, we first color edges of $C_1 \cup C_2$ as shown in Figure 4.1 (graphs c' and d'); then give each other edge of G a fresh color. This procedure costs $m - 3$ colors totally. Then it is easy to show that any geodesic cannot contain two edges with the same color, so we have $src(G) \leq m-3$. This produces a contradiction.

For the case $l(P') \geq 4$, that is $k_2 \geq 6$. We let $c(u_4v_1) = c(v_3v_4) = a$, $c(v_1v_2) = c(v_3u_4) = b$; for edge v_2v_3, we give a similar treatment to that of **Subcase 2.1** and let $c(v_2v_3) = c$; we then give each other edge of G a fresh color. This procedure costs $m-3$ colors totally. For any two vertices u, v, P is a $u-v$ geodesic, then by Observation 2.4, P contains at most one of the two edges with color b; for the two edges with color a, it is easy to show that there exists one $u-v$ geodesic which contains at most one of them. With a similar argument to that of Lemma 2.2(Note that now, by Observation 2.4, the cycle produced during the procedure cannot be C_1 and must be shorter than C_2, then a contradiction will be produced), we can show any geodesic contains at most one edge with color c. So we have $src(G) \leq m-3$. This produces a contradiction.

Subcase 2.3. $k_1 = 5$, that is, C_1 is a 5-cycle. By Observation 2.3, C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 2.1. The following discussion will use Observation 2.4.

If C_1 and C_2 have one common edge, say u_1u_2 (see the left one of the two graphs with $g(G) = 5$ in Figure 2.1). We let $V(C_2) = \{v_i : 1 \leq i \leq k_2\}$, where $v_1 = u_1, v_2 = u_2$. We let $c(u_4u_5) = c(v_2v_3) = a$, $c(v_1u_5) = c(v_2u_3) = b$, and $c(v_1v_2) = c(u_3u_4) = c$; for the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. With a similar argument to above, we can show that $src(G) \leq m-3$. This produces a contradiction.

Otherwise, then C_1 and C_2 have two common adjacent edges, say u_1u_2, u_2u_3 (see the right one of the two graphs with $g(G) = 5$ in Figure 2.1). We let $c(v_1u_5) = c(v_3v_4) = a$, $c(v_1v_2) = c(v_3u_4) = b$, and $c(v_2v_3) = c(u_4u_5) = c$; for the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. With a similar argument to above, we can show that $src(G) \leq m-3$. This produces a contradiction.

With the above discussion, **Claim 1** holds.

Let G be a unicyclic graph and C be its cycle, $|C| = k$ where $3 \leq k \leq 5$. We now investigate the strong rainbow connection number of G.

Case 1. $k = 3$.

Subcase 1.1. All T_is are nontrivial. We first give each edge of $G \backslash E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_3)$, $c(v_2v_3) \in c(T_1)$, $c(v_1v_3) \in c(T_2)$, it is easy to show, with this coloring, G is strong rainbow connected, so $src(G) \leq m-3$ in this case.

Subcase 1.2. At most two T_is are nontrivial, that is, $G \in \mathcal{G}_1$. We first consider the case that there are exactly two T_is which are nontrivial, say T_1 and T_2. We first give each edge of $G \backslash E(C)$ a fresh color, then let $c(v_1v_2) = c(v_2v_3) = c(v_1v_3)$, it is easy to show, with this coloring, G is strong rainbow connected, so now $src(G) \leq m-2$. On the other hand, by Observation 2.1 and the definition of rainbow geodesic, we know that in a strong rainbow coloring $c(T_1) \cap c(T_2) = \emptyset$ and $c(v_1v_2)$ doesn't belong to $c(T_1) \cup c(T_2)$. So we have $src(G) = m-2$ in this case. With a similar argument, we can derive $src(G) = m-2$ for the case that at most one T_i is nontrivial. So $src(G) = m-2$ if $G \in \mathcal{G}_1$.

Case 2. $k = 4$.

Subcase 2.1. There are at least three nontrivial T_is, say T_1, T_3, T_4. We first give each edge of $G \backslash E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_4)$, $c(v_3v_4) \in c(T_1)$, $c(v_1v_4) \in c(T_3)$ and we give edge v_2v_3 a fresh color. It is easy to show, with this coloring, G is strong rainbow connected, so $src(G) \leq m-3$ in this case.

Subcase 2.2. There are exactly two nontrivial T_is, say T_{i_1} and T_{i_2}.

Subsubcase 2.2.1. T_{i_1} and T_{i_2} are adjacent, say T_1 and T_2. We first give each edge of $G \backslash E(C)$ a fresh color, then let $c(v_2v_3) \in c(T_1)$, $c(v_1v_4) \in c(T_2)$ and we color edges v_1v_2 and v_3v_4 with the same new color. It is easy to show, with this coloring, G is strong rainbow connected, so $src(G) \leq m-3$ in this case.

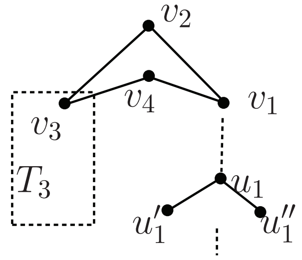

Figure 4.2 Graph for Subsubcase 2.2.2.

Subsubcase 2.2.2. T_{i_1} and T_{i_2} are nonadjacent, say T_1 and T_3. We can consider T_i as rooted tree with root v_i $(i = 1, 3)$. If there exists some T_i, say T_1, that contains a vertex, say u_1, with at least two sons, say u_1', u_1''(see Figure 4.2). We first color each edge of $\bigcup_{i=1,3} T_i \cup \{v_1v_2\}$ with a distinct color, this costs $m-3$ colors, then we let $c(v_1v_4) = c(v_1v_2), c(v_2v_3) = c(u_1u_1'), c(v_3v_4) = c(u_1u_1'')$. It is easy to show that this coloring is strong rainbow and we have $src(G) \leq m-3$ in this case. If G also belongs to $\mathcal{G}_2$, we first give each edge of $G \backslash E(C)$ a fresh color, then let $c(v_1v_2) = c(v_3v_4) = a$ and $c(v_2v_3) = c(v_1v_4) = b$ where a and b are two new colors. It is easy to show, with this coloring, G is strong rainbow connected, so $src(G) \leq m-2$ in this case. On the other hand, $src(G) \geq m-2 = diam(G)$. So $src(G) = m-2$ in this case.

Subcase 2.3. There are at most one nontrivial T_i. Then with a similar argument to **Subsubcase 2.2.2**, we can derive that $src(G) = m-2$ if G also belongs to $\mathcal{G}_2$.

By the discussions of **Subsubcase 2.2.2** and **Subcase 2.3**, we derive that $src(G) = m-2$ if $G \in \mathcal{G}_2$.

Case 3. $k = 5$.

If there are at least one nontrivial T_i, say T_1, then we give each edge of $G \backslash E(C)$ a fresh color, let $v_3v_4 \in c(T_1)$, $c(v_1v_2) = c(v_4v_5) = a$ and $c(v_2v_3) = c(v_1v_5) = b$ where a and b are two new colors. It is easy to show, with this coloring, G is strong rainbow connected, so now we have $src(G) \leq m-3$. On the other hand, we know $src(G) = m-2 = 3$ if $G \cong C_5$ from [3].

By Lemma 2.2 and **Claim 1**, we derive that if $src(G) = m-1$ or $m-2$, then G is a unicyclic graph with the cycle of length at most 5. By the discussion from the above **Case 1** to **Case 3**, we know that if G is a unicyclic graph with the cycle of length at most 5, then

$src(G) \neq m-1$. So $src(G) \neq m-1$ for any graph G. Furthermore, we have $src(G) = m-2$ if and only if G is a 5-cycle or belongs to one of $\mathcal{G}_i$s$(1 \leq i \leq 2)$. So the theorem holds. ∎

References

[1] P. Ananth, M. Nasre, *New hardness results in rainbow connectivity*, Arxiv preprint arXiv:1104.2074v1 [cs.CC].

[2] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[3] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133**(2008) 85-98.

[4] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, *On rainbow connection*, Electron. J. Combin. **15** (2008), R57.

[5] X. Li, Y. Sun, *Rainbow connection numbers of line graphs*, Ars Combin. 100(2011), 449–463.

[6] X. Li, Y. Sun, *Upper bounds for the rainbow connection numbers of line graphs*, Graphs & Combin., in press.

[7] X. Li, Y. Sun, *Characterize graphs with rainbow connection number $m-2$ and rainbow connection numbers of some graph operations*, submitted, June, 2010.

[8] X. Li, Y. Sun, Rainbow connections of graphs- A survey, *Arxiv preprint arXiv:1101.5747v2 [math.CO] (2011).*